Singularities, Monodromy and Zeta functions
Blatt 2

Exercises for discussion in Übung on 26.10.2018

Aufgabe 1:

Suppose $X \subset \mathbb{Q}_p^m$ and $Y \subset \mathbb{Q}_p^n$ are semi-algebraic sets.
Let $f : X \to Y$ be a semi-algebraic map, meaning its graph $\{(x, f(x))|x \in X\} \subset \mathbb{Q}_p^{m+n}$ is semi-algebraic.

(a) Prove that, if $V \subset X$ is semi-algebraic, then $f(V)$ is semi-algebraic.

(b) Assuming $W \subset Y$ is semi-algebraic, prove that $f^{-1}(W)$ is semi-algebraic.

(c) Let $g : Y \to \mathbb{Q}_p^k$ be semi-algebraic. Prove that $g \circ f$ is semi-algebraic.

Aufgabe 2:

The *angular component* map on $\mathbb{Q}_p$ is the map $\mathrm{ac}_1 : \mathbb{Q}_p \to \mathbb{F}_p$ such that,

- $\mathrm{ac}_1(0) = 0$;
- for $a \in \mathbb{Q}_p^\times$ writing $a = \sum a_i p^i$ with $a_i \in \{1, ..., p-1\}$, $\mathrm{ac}_1(a) = a_{v(a)}$.

That is, the angular component of a non-zero element is the coefficient of the leading term in its p-adic expansion.

Let μ be the Haar measure on $\mathbb{Q}_p$.
Compute the measures of the following sets, which we will later see are semi-algebraic:

(a) $\mu(\{x \in \mathbb{Z}_p | \, \mathrm{ac}(x) = 1\})$;

(b) $\mu(\{x \in \mathbb{Z}_p | \; 2 \mid v(x)\})$.

Aufgabe 3:

Prove that for $p \geq 3$, an element $x \in \mathbb{Q}_p$ is a square, if and only if $2 \mid v(x)$ and $\mathrm{ac}_1(x)$ is a square.

Hint: Use Hensel's Lemma.