Self–dual modules over local rings of curve singularities

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Abstract

Let C be a reduced curve singularity. C is called of finite self–dual type if there exist only a finitely many isomorphism classes of indecomposable, self–dual, torsion–free modules over the local ring of C. In this paper it is shown that the singularities of finite self–dual type are those which dominate a simple plane singularity.

Let R be a local ring of a reduced curve singularity C, i.e., R is a one–dimensional, reduced quotient ring of a formal or convergent power series ring. We will always assume that the residue field $k = R/m$ has characteristic 0. Let $\tilde{R}$ be the normalization of R in the total quotient ring K. The local ring R' of another reduced curve singularity C' dominates R iff $R \subseteq R' \subseteq \tilde{R}$ or equivalently iff there is a birational morphism $C' \to C$. We will consider only finitely generated modules over R. Such a module M is called torsion–free, iff for each non–zero divisor $r \in R$ the left multiplication map $\lambda_r : M \to M$ is injective. For these rings the torsion–free modules are precisely the maximal Cohen–Macaulay modules.

The above assumptions on R imply that R is a Cohen–Macaulay ring, thus there exists a dualizing module ω. Setting $M^* = \mathrm{Hom}_R(M, \omega)$ for any R–module M, the characterizing property of ω is $M^{**} \cong M$ for all modules M. A module M is called self–dual iff $M^* \cong M$.

This paper is devoted to proving the following

Theorem. *Let R be the local ring of a reduced curve singularity C, then the following statements are equivalent:*

1. *There are only finitely many indecomposable, self–dual, torsion–free module over R (up to isomorphism.)*
2. *For any $n \geq 1$ there are only finitely many indecomposable, self–dual, torsion–free module over R of rank n over R.*
3. *For some $n \geq 1$ there are only finitely many indecomposable, self–dual, torsion–free module over R of rank n over R.*
4. *R dominates a plane simple curve singularity, i.e., plane ADE–singularity.*

5. *C is either a plane ADE–singularity or a space D_l^-, E_6^-, E_7^-, E_8^- singularity. (See [AGV] for a description of the plane ADE–singularities and [Co, 2.4] for the space singularities.)*

Greuel and Knörrer proved the same theorem without the restriction to self–dual modules [GK], the equivalence of 4) and 5) is only implicitely there and was made explicit by Cook [Co, 2.4]. Due to their Theorem it is enough to construct families of indecomposable, self–dual, torsion–free modules of rank n over the local ring of any singularity not listed in 5). The corresponding task was also the difficult part in the proof of Greuel and Knörrer. While their main construction idea still works in our case, the families have to be selected more carefully and explicitely, because they must be self–dual and we must be able to prove it. In fact, for a fixed ring R a family of pairwise non–isomorphic M_λ, $\lambda \in k$, is selected such that M_λ^* and partially ω can be computed at the same time, based on the fact that the M_λ^* must be pairwise non–isomorphic as well.

The author's interest in self–dual modules was raised by the question how many theta–characteristics a singular curve posses. A theta–characteristic on C is a torsion–free sheaf $\mathcal{F}$ of rank 1 with $\mathcal{F} \cong \mathcal{H}om(\mathcal{F}, \omega)$. Theta–characteristics have been extensively studied, for example by Riemann, Atiyah, Mumford, and Harris [A, M, H]. Their uses include finding contact curves and representations of the equation of a plane curve as the determinants of symmetric matrices with polynomial entries or classifying nets of quadrics [B, C, D, He, W]. Our theorem implies that there are only finitely many theta–characteristics on a singular curve iff all its singularities are of the types listed in 5), see [P]. This answers a question of Sorger [S].

The Proof

By the Theorem of Greuel and Knörrer we know that for the singularities listed in 5) there are only finitely many isomorphism classes of indecomposable, torsion–free modules, thus in particular there are only finitely many self–duals ones. Therefore, it is enough to construct for the local ring of any other singularity infinitely many pairwise non–isomorphic modules of rank n which are torsion–free and self–dual. We give the proof in detail only for $n = 1$ — this being the most important case — and indicate at the end the changes necessary to construct modules of higher rank.

Let us recall a few facts about torsion–free modules [GK, p. 414]. A torsion–free module of rank 1 can always be embedded into $\tilde{R}$ as an R–module. From now on we will consider every torsion–free module to be embedded in $\tilde{R}$ in some fixed way. Then the homomorphisms between two torsion–free modules of rank 1 are given by

$$\mathrm{Hom}_R(M, N) = \{u \in K \mid uM \subseteq N\}.$$

The dualizing module ω of R is also a torsion–free module of rank 1, hence $M^* = \{u \in K \mid uM \subseteq \omega\}$. For an embedded module $M \subseteq \tilde{R}$ we define the conductor to be

$$C(M) = \mathrm{Ann}_R(\tilde{R} \cdot M/M) = \mathrm{Hom}_R(\tilde{R} \cdot M, M) = \{r \in K \mid r\tilde{R}M \subseteq M\}.$$

Obviously, $C(M)$ is independent of the chosen embedding $M \subseteq \tilde{R}$ and not only a R–module, but also a $\tilde{R}$–module. For notational convenience we will always embed $\omega \subseteq \tilde{R}$ such that $\tilde{R}\omega = \tilde{R}$.

We proceed by considering rings which describe curve singularities which are not listed in 5). For those we have the following list, where for each entry we assume that the ring is not of a type considered before [GK, Lemma 2]:

1. unibranched singularities, $\tilde{R} = k[[t]]$
   (a) $R \subseteq k + t^4\tilde{R}$, unibranched singularities of multiplicity ≥ 4
   (b) $R \subseteq k + kt^3 + t^6\tilde{R}$, unibranched singularities of multiplicity 3, not E_6, E_8, E_6^-, or E_8^-
2. bibranched singularities, $\tilde{R} = k[[t]]^2$
   (a) $R \subseteq k + t^2\tilde{R}$, two branches of multiplicity ≥ 2
   (b) $R \subseteq k + k(t^3, t)\tilde{R}$, one branch of multiplicity 3 and a smooth branch
   (c) $R \subseteq k + k(t^2, t) + (t^4, t^2)\tilde{R}$, a $A_{2\delta}$, $\delta \geq 2$, singularity with a tangential smooth branch
3. tribranched singularities, $\tilde{R} = k[[t]]^3$
   (a) $R \subseteq k + (t^2, t, t)\tilde{R}$, three branches, at least one of which is singular
   (b) $R \subseteq k + k(t, t, t) + t^2\tilde{R}$, three smooth branches with a common tangent
4. singularities with four or more branches, $R \subseteq \tilde{R} = k[[t]]^n$ for $n \geq 4$

Now the proof has to be done case by case. While the arguments are similar in each case, there seems to be no way to unify some cases because we need very specific knowledge about the dual module in each case.

Case 1a. Here $R = k[[\varphi_i]] \subset k[[t]] = \tilde{R}$ with $\operatorname{ord}\varphi_i \geq 4$. We show that

$$M_\lambda = \langle 1, t^2 + \lambda t^3 \rangle + t^4\tilde{R} \subset \tilde{R}$$

is a family of pairwise non–isomorphic modules, which are self–dual for nearly all $\lambda \in k$.

Assume $M_\lambda \cong M_\mu$, i.e., $uM_\lambda = M_\mu$ for some $u \in K$. Because 1 is an element of minimal order in M_λ as well as M_μ, we get $u \in \tilde{R}^*$. From $u = u1 \in M_\mu$ we get $u = \alpha 1 + \beta(t^2 + \mu t^3) + \ldots$ with $\alpha \neq 0$. Finally, $u(t^2 + \lambda t^3) = \alpha(t^2 + \lambda t^3) + \ldots \in M_\mu$ implies $\lambda = \mu$.

We compute the dual of M_λ,

$$M_\lambda^* = \{u \in K \mid uM_\lambda \subseteq \omega\} = \{u \in \omega \mid u(t^2 + \lambda t^3) \in \omega,\ ut^4\tilde{R} \subseteq \omega\}.$$

Since we do not know ω, we have to partially determine it at the same time! Choose $c \in \mathbb{N}$ such that $C(\omega) = t^c\tilde{R}$. Let $u \in M_\lambda^*$. From $ut^4\tilde{R} \subseteq \omega$, we get $u \in t^{-4}C(\omega) = t^{c-4}\tilde{R}$. As obviously $C(\omega) \subseteq M_\lambda^*$, we may compute M_λ^* modulo $C(\omega)$ and assume that $u = \sum_{i=1}^{4} u_i t^{c-i}$.

We claim that ω contains an element $t^{c-2}+\vartheta t^{c-1}$ for some $\vartheta \in k$. We cannot have $M_\lambda^* \subseteq t^{c-3}\tilde{R}$ for nearly all λ, because otherwise the condition $(t^2+\lambda t^3)u \in \omega$ — and thus M_λ^* — is independent of λ. Hence, we can find a $z \in M_\lambda^* \setminus t^{c-3}\tilde{R} \subseteq t^{c-4}\tilde{R}$ for general λ and $z(t^2+\lambda t^3) \in \omega$ will be the desired element after multiplication by a suitable element of k^*.

Using the element $t^{c-2}+\vartheta t^{c-1} \in \omega$ and $t^{c-1} \notin \omega$, the condition $u(t^2+\lambda t^3) = u_4 t^{c-2} + (u_3+\lambda u_4)t^{c-1} \in \omega$ is equivalent to $u_3+\lambda u_4 = \vartheta u_4$ or $u_3 = (\vartheta-\lambda)u_4$, i.e., $u = u_4 t^{c-4} + u_4(\vartheta-\lambda)t^{c-3} + \ldots$. Now we know that the above element z can be taken to be $z = t^{c-4} + (\vartheta-\lambda)t^{c-3} + \ldots$. Since obviously $M_\lambda^* \cap t^{c-2}\tilde{R} = \omega \cap t^{c-2}\tilde{R}$, we find

$$M_\lambda^* = \left\langle z, t^{c-2}+\vartheta t^{c-1} \right\rangle + t^c \tilde{R}.$$

Multiplying M_λ^* by z^{-1},

$$\begin{aligned} z^{-1}M_\lambda^* &= \left\langle 1, (t^{4-c} - (\vartheta-\lambda)t^{5-c})(t^{c-2}+\vartheta t^{c-1}) \right\rangle + t^4\tilde{R} \\ &= \left\langle 1, t^2+\lambda t^3 \right\rangle + t^4 \tilde{R} = M_\lambda, \end{aligned}$$

proves that M_λ is self–dual for nearly all λ.

Case 1b. This time $R = k[[t^3, \varphi_i]] \subset k[[t]] = \tilde{R}$ with $\operatorname{ord}\varphi_i \geq 7$. We claim that

$$M_\lambda = \left\langle 1, t^3, t^4+\lambda t^5 \right\rangle + t^6\tilde{R} \subset \tilde{R}$$

is a family of pairwise non–isomorphic modules, which are self–dual for nearly all $\lambda \in k$.

Assume that $M_\lambda \cong M_\mu$, i.e., there exist an $u \in K$ such that $uM_\lambda = M_\mu$. As $1 \in M_\lambda, M_\mu \subseteq \tilde{R}$, we get $u \in \tilde{R}^*$. Since $u = u1 \in M_\mu$ we find $u = \alpha 1 + \beta t^3 + \gamma(t^4+\mu t^5) + \ldots$ with $\alpha, \beta, \gamma \in k$ and $\alpha \neq 0$. Finally, $u(t^4+\lambda t^5) = \alpha(t^4+\lambda t^5) + \ldots \in M_\mu$ implies $\lambda = \mu$.

Now we compute the dual of M_λ,

$$\begin{aligned} M_\lambda^* &= \operatorname{Hom}_R(M_\lambda, \omega) = \{u \in K \mid uM_\lambda \subseteq \omega\} \\ &= \left\{ u \in \omega \mid ut^3, u(t^4+\lambda t^5) \in \omega,\ ut^6\tilde{R} \subseteq \omega \right\} \supseteq C(\omega). \end{aligned}$$

Choose $c \in \mathbb{N}$ such that $C(\omega) = t^c\tilde{R}$ and let $u \in M_\lambda^*$. From $ut^6\tilde{R} \subseteq \omega$, we find $u \in t^{c-6}\tilde{R}$. We compute M_λ^* modulo $C(\omega)$ and assume $u = \sum_{i=1}^6 u_i t^{c-i}$.

We claim that $\omega \cap t^{c-3}\tilde{R} = \left\langle t^{c-3}+\sigma t^{c-1}, t^{c-2}+\vartheta t^{c-1} \right\rangle$ for some $\sigma, \vartheta \in k$. As $t^{c-1} \notin \omega$, $\omega \cap t^{c-3}\tilde{R}$ modulo $C(\omega)$ must have dimension less than 3. Further, we must have $M_\lambda^* \not\subseteq t^{c-5}\tilde{R}$ for nearly all λ, because otherwise the restriction $u(t^4+\lambda t^5) \in \omega$ on u — and with it M_λ^* — does not depend on λ. Now if $z \in M_\lambda^* \setminus t^{c-5}\tilde{R} \subset t^{c-6}\tilde{R}$ then $t^3 z, (t^4+\lambda t^5)z \in \omega$ are the desired elements after some normalization.

Knowing a basis for $\omega \cap t^{c-3}\tilde{R}$ modulo $C(\omega)$, the computation of M_λ^* is easy. From

$$\begin{aligned} ut^3 &= u_6 t^{c-3} + u_5 t^{c-2} + u_4 t^{c-1} \in \omega \\ u(t^4+\lambda t^5) &= u_6 t^{c-2} + (u_5+\lambda u_6)t^{c-1} \in \omega \end{aligned}$$

we get $u_4 = \sigma u_6 + \vartheta u_5$ and $\vartheta u_6 = u_5 + \lambda u_6$, i.e., $u_5 = (\vartheta - \lambda)u_6$, $u_4 = (\sigma + \vartheta^2 - \lambda\vartheta)u_6$, and

$$u = u_6 t^{c-6} + u_6(\vartheta - \lambda)t^{c-5} + u_6(\sigma + \vartheta^2 - \lambda\vartheta)t^{c-4} + \dots .$$

We already argued above for the existence of an element $z \in M_\lambda^* \setminus t^{c-5}\tilde{R} \subset t^{c-6}\tilde{R}$ for nearly all λ. Now we know that it can be taken to be

$$z = t^{c-6} + (\vartheta - \lambda)t^{c-5} + (\sigma + \vartheta^2 - \lambda\vartheta)t^{c-4} + \dots$$

and all other elements of a basis of $M_\lambda^*/C(\omega)$ can be taken out of $t^{c-3}\tilde{R} \cap M_\lambda^*$. Using the obvious $M_\lambda^* \cap t^{c-3}\tilde{R} = \omega \cap t^{c-3}\tilde{R}$, we find

$$M_\lambda^* = \left\langle z, t^{c-3} + \sigma t^{c-1}, t^{c-2} + \vartheta t^{c-1} \right\rangle + t^c\tilde{R} = \left\langle z, t^3 z, t^{c-2} + \vartheta t^{c-1} \right\rangle + t^c\tilde{R}.$$

Multiplying M_λ^* with $z^{-1} \in K$, we get

$$\begin{aligned} z^{-1}M_\lambda^* &= \left\langle 1, t^3, (t^{6-c} - (\vartheta - \lambda)t^{7-c})(t^{c-2} + \vartheta t^{c-1}) \right\rangle + t^6\tilde{R} \\ &= \left\langle 1, t^3, t^4 + \lambda t^5 \right\rangle + t^6\tilde{R} = M_\lambda, \end{aligned}$$

showing that M_λ is self–dual for nearly all λ.

Case 2a. The local ring is $R = k[[(\varphi_i, \psi_i)]] \subseteq \tilde{R} = k[[t]]^2$ with $\operatorname{ord}(\varphi_i), \operatorname{ord}(\psi_i) \geq 2$. We will show that

$$M_\lambda = \langle 1 = (1,1),\, (t, \lambda t) \rangle + t^2\tilde{R} \subseteq \tilde{R}$$

is a family of pairwise non–isomorphic modules, whose general member is self–dual.

Assume $M_\lambda \cong M_\mu$ by multiplication by an element $u \in K$. By $1 \in M_\lambda, M_\mu \subset \tilde{R}$ we find $u \in \tilde{R}^*$. From $1 \in \mathrm{M}_\lambda$ we get $u = u1 \in M_\mu$, thus $u = \alpha 1 + \beta(t, \mu t) + \dots$, $\alpha \neq 0$. Finally, $u(t, \lambda t) = \alpha(t, \lambda t) + \dots \in M_\mu$ implies $\lambda = \mu$.

We start computing the dual module of M_λ

$$\begin{aligned} M_\lambda^* &= \{(u,v) \in K \subset k((t))^2 \mid (u,v)M_\lambda \subseteq \omega\} \\ &= \{(u,v) \in \omega \mid (ut, v\lambda t) \in \omega, (u,v)t^2\tilde{R} \subseteq \omega\} \supseteq C(\omega). \end{aligned}$$

Choose $c_1, c_2 \in \mathbb{N}$ with $C(\omega) = (t^{c_1}, t^{c_2})\tilde{R}$ and compute modulo $C(\omega)$. The condition $(u,v)t^2\tilde{R} \subseteq \omega$ is equivalent to $(u,v) \in t^{-2}C(\omega) = (t^{c_1-2}, t^{c_2-2})\tilde{R}$. We claim that ω contains an element $(t^{c_1-1}, \vartheta t^{c_2-1})$ with $\vartheta \neq 0$. We note that $M_\lambda^* \not\subseteq (t^{c_1-1}, t^{c_2-1})\tilde{R}$ for nearly all λ, because otherwise $(ut, v\lambda t) \in \omega$ imposes no restriction on (u,v) and M_λ^* would be independent of λ. Hence, for general λ we find an element $z \in M_\lambda^* \setminus (t^{c_1-1}, t^{c_2-1})\tilde{R} \subseteq (t^{c_1-2}, t^{c_2-2})\tilde{R}$, and multiplying it by $(t, \lambda t)$ we get the desired element, recalling that $(t^{c_1-1}, 0), (0, t^{c_2-1}) \notin \omega$ by the definition of c_1 and c_2.

Now it is easy to determine the elements

$$(u,v) = (u_2 t^{c_1-2} + u_1 t^{c_1-1}, v_2 t^{c_2-2} + v_1 t^{c_2-1}) \in M_\lambda^*.$$

$(ut, v\lambda t) = (u_2 t^{c_1-1}, v_2\lambda t^{c_2-1}) \in \omega$ is equivalent to $u_2\vartheta = v_2\lambda$, i.e.,

$$(u,v) = \left(v_2 \tfrac{\lambda}{\vartheta} t^{c_1-2} + u_1 t^{c_1-1}, v_2 t^{c_2-2} + v_1 t^{c_2-1}\right).$$

In particular, the above element z may be taken to be $z = (\lambda/\vartheta \cdot t^{c_1-2} + z_1 t^{c_1-1}, t^{c_2-2} + \bar{z}_1 t^{c_2-1})$ and all additional elements for a basis of $M_\lambda^*/C(\omega)$ can be found in $(t^{c_1-1}, t^{c_2-1})\tilde{R}$. Using the obvious $\omega \cap (t^{c_1-1}, t^{c_2-1})\tilde{R} = M_\lambda^* \cap (t^{c_1-1}, t^{c_2-1})\tilde{R}$, we obtain

$$M_\lambda^* = \left\langle z, (t^{c_1-1}, \vartheta t^{c_2-1})\right\rangle + (t^{c_1}, t^{c_2})\tilde{R}.$$

Clearly, $z^{-1}M_\lambda^* = M_\lambda$ for nearly all λ, thus M_λ is self–dual.

Case 2b. Here the ring is $R = k[[(\varphi_i, \psi_i)]] \subset \tilde{R} = k[[t]]^2$ with $\operatorname{ord}\varphi_i \geq 3$, $\operatorname{ord}\psi_i \geq 1$. We show that

$$M_\lambda = \left\langle 1, (t+\lambda t^2, 1)\right\rangle + (t^3, t)\tilde{R}$$

is a family of pairwise non–isomorphic, self–dual modules.

Let $M_\lambda \cong M_\mu$, i.e., $uM_\lambda = M_\mu$ for an element $u \in K$. By $1 \in M_\lambda, M_\mu \subseteq \tilde{R}$ we find $u \in \tilde{R}^*$. From $u1 \in M_\mu$ we know $u = \alpha 1 + \beta(t + \mu t^2, 1) + \ldots$ with $\alpha \neq 0$. At last, from $u(t+\lambda t^2, 1) = (\alpha t + (\alpha\lambda + \beta)t^2, \alpha + \beta) + \ldots \in M_\mu$ we get $\alpha = \alpha + \beta$ and $\alpha\lambda + \beta = \alpha\mu$, in particular $\lambda = \mu$.

Let us compute the dual of M_λ,

$$\begin{aligned} M_\lambda^* &= \{(u,v) \in K \subset k((t))^2 \mid (u,v)M_\lambda \subseteq \omega\} \\ &= \{(u,v) \in \omega \mid (u(t+\lambda t^2), v) \in \omega,\ (ut^3, vt)\tilde{R} \subseteq \omega\} \supseteq C(\omega). \end{aligned}$$

Choose again $c_1, c_2 \in \mathbb{N}$ such that $C(\omega) = (t^{c_1}, t^{c_2})\tilde{R}$, and compute modulo $C(\omega)$. From $(t^3u, tv)\tilde{R} \subseteq \omega$ we get $(u,v) \in (t^{c_1-3}, t^{c_2-1})\tilde{R}$.

Our first task is — as always — to determine the canonical module partially. We claim that

$$(t^{c_1-3}, \sigma t^{c_2-1}),\ (t^{c_1-2}, \vartheta t^{c_2-1}),\ (t^{c_1-1}, \varrho t^{c_2-1}) \quad \text{for some } \sigma, \vartheta, \varrho \in k,\ \varrho \neq 0$$

is a basis for $(\omega \cap (t^{c_1-3}, t^{c_2-1})\tilde{R})/C(\omega)$. First, we note that for nearly all λ, $M_\lambda^* \not\subseteq (t^{c_1-2}, t^{c_2-1})\tilde{R}$, because otherwise the condition $(u(t+\lambda t^2), v) \in \omega$ — and hence M_λ^* — does not depend on λ. Therefore, for some λ we can find an element

$$\xi = (t^{c_1-3} + \xi_2 t^{c_1-2} + \xi_1 t^{c_1-1}, \bar{\xi}_1 t^{c_2-1}) \in M_\lambda^* \subseteq \omega.$$

Multiplying by $(t+\lambda t^2, 1) \in M_\lambda$, we find the following element in ω

$$\zeta = (t^{c_1-2} + (\xi_1 + \lambda)t^{c_1-1}, \bar{\xi}_1 t^{c_2-1}) \in \omega.$$

Next, we note that M_λ^* modulo $C(\omega)$ cannot be a one–dimensional vector space, because otherwise multiplication of M_λ^* with the inverse of a basis element of $M_\lambda^*/C(\omega)$ that is chosen with non–zero components of the smallest possible order shows that the M_λ^* are all isomorphic to a finite collection of modules, which is impossible. From $\dim M_\lambda^*/C(\omega) \geq 2$ we deduce the existence of an element

$$\varrho = (t^{c_1-1}, \bar{\varrho} t^{c_2-1}) \in \omega,$$

because either $M_\lambda^* \subseteq \omega$ contains such an element ϱ or M_λ^* contains $(t^{c_1-2}+\ldots,\ldots)$ and we obtain ϱ as the product of this element with $(t+\lambda t^2,1)$. Because $(t^{c_1-1},0)\notin\omega$, the triple (ξ,ζ,ϱ) is a basis of $(\omega\cap(t^{c_1-3},t^{c_2-1})\tilde{R})/C(\omega)$. A base change proves the claim.

Now the computation of dual module M_λ^* is straight forward. Since $(u,v)\in\omega\cap(t^{c_1-3},t^{c_2-1})\tilde{R}$, it is a linear combination of the above basis elements, i.e.,

$$(u,v)=(\alpha t^{c_1-3}+\beta t^{c_1-2}+\gamma t^{c_1-1},(\alpha\sigma+\beta\vartheta+\gamma\varrho)t^{c_2-1}).$$

The condition

$$(u(t+\lambda t^2),v)=(\alpha t^{c_1-2}+(\alpha\lambda+\beta)t^{c_1-1},(\alpha\sigma+\beta\vartheta+\gamma\varrho)t^{c_2-1})\in\omega$$

is equivalent to $\alpha\sigma+\beta\vartheta+\gamma\varrho=\alpha\vartheta+(\alpha\lambda+\beta)\varrho$ or $\gamma=(\vartheta-\sigma+\lambda\varrho)/\varrho\cdot\alpha+(\varrho-\vartheta)/\varrho\cdot\beta$. Therefore,

$$\begin{aligned}M_\lambda^* &= \Big\langle (t^{c_1-3}+\tfrac{\vartheta-\sigma+\lambda\varrho}{\varrho}t^{c_1-1},(\vartheta+\lambda\varrho)t^{c_2-1}),\ (t^{c_1-2}+\tfrac{\varrho-\vartheta}{\varrho}t^{c_1-1},\varrho t^{c_2-1})\Big\rangle + C(\omega)\\ &= \Big\langle z:=(t^{c_1-3}+\tfrac{\varrho-\vartheta-\lambda\varrho}{\varrho}t^{c_1-2}+\ldots,\varrho t^{c_2-1}),\ (t^{c_1-2}+\tfrac{\varrho-\vartheta}{\varrho}t^{c_1-1},\varrho t^{c_2-1})\Big\rangle + C(\omega)\end{aligned}$$

and $z^{-1}M_\lambda^*=M_\lambda$ shows that M_λ is self–dual.

Case 2c. We may assume that the overring R is

$$R=k[[(t^2,t),(t^{2\delta+1},\varphi_1),(0,\varphi_i)]]\quad\text{with } \delta\geq 2,\ \operatorname{ord}\varphi_1,\operatorname{ord}\varphi_i\geq 2.$$

This time

$$M_\lambda=\big\langle 1,(t^2,t),(t^2+\lambda t^3,0)\big\rangle+(t^4,t^2)\tilde{R}$$

is a family of pairwise non–isomorphic modules, which are self–dual for nearly all $\lambda\in k$.

Assume $M_\lambda\cong M_\mu$ by multiplication by $u\in K$. The usual argument yields $u\in\tilde{R}^*$ and $M_\mu\ni u=u1=\alpha 1+\beta(t^2,t)+\ldots$ with $\alpha\neq 0$. Therfore, $u(t^2+\lambda t^3,0)=\alpha(t^2+\lambda t^3,0)+\ldots\in M_\mu$ yields $\lambda=\mu$.

We find the dual module as

$$\begin{aligned}M_\lambda^* &= \{(u,v)\in K\subset k((t))^2 \mid (u,v)M_\lambda\subseteq\omega\}\\ &= \{(u,v)\in\omega \mid (u(t^2+\lambda t^3),0)\in\omega,\ (ut^4,vt^2)\tilde{R}\subseteq\omega\}\supseteq C(\omega).\end{aligned}$$

Again, choose $c_1,c_2\in\mathbb{N}$ with $C(\omega)=(t^{c_1},t^{c_2})\tilde{R}$ and compute modulo $C(\omega)$. The condition $(ut^4,vt^2)\tilde{R}\subseteq\omega$ implies $M_\lambda^*\subseteq(t^{c_1-4},t^{c_2-2})\tilde{R}$.

We need to get a grip on $\omega\cap(t^{c_1-2},t^{c_2-1})\tilde{R}$. We claim that a basis of it modulo $C(\omega)$ is given by two elements

$$(t^{c_1-1},\sigma t^{c_2-1}),\ (t^{c_1-2}+\vartheta t^{c_1-1},0)\quad\text{for some } \sigma,\vartheta\in k,\ \sigma\neq 0.$$

We note that $M_\lambda^*\not\subseteq(t^{c_1-3},t^{c_2-2})\tilde{R}$ for nearly all λ, because otherwise the condition $(u(t^2+\lambda t^3),0)\in\omega$ — and thus M_λ^* — does not depend on λ. Therefore, we find an element $z=(t^{c_1-4}+z_3t^{c_1-3}+\ldots,\bar{z}_2t^{c_2-2}+\bar{z}_1t^{c_2-1})\in M_\lambda^*$ and

$$\begin{aligned}\zeta=(t^2,t)z &= (t^{c_1-2}+z_3t^{c_1-1},\bar{z}_2t^{c_2-1})\in M_\lambda^*\subseteq\omega\quad\text{as well as}\\ \varrho=(t^2+\lambda t^3,0)z &= (t^{c_1-2}+(z_3+\lambda)t^{c_1-1},0)\in\omega.\end{aligned}$$

As the vector space $(\omega\cap(t^{c_1-2},t^{c_2-1})\tilde{R})/C(\omega)$ has dimension at most three and $(t^{c_1-1},0)\notin\omega$, the elements ζ,ϱ are a basis of it. A base change proves the claim.

We proceed with the computation of the dual module. Let $(u,v) = (\sum_{i=1}^4 u_i t^{c_1-i}, \sum_{i=1}^2 v_i t^{c_2-i}) \in M_\lambda^*$. The requirements

$$\begin{aligned}(t^2,t)(u,v) &= (u_4 t^{c_1-2}+u_3 t^{c_1-1}, v_2 t^{c_2-1}) \in \mathrm{M}_\lambda^* \subseteq \omega\\ (t^2+\lambda t^3,0)(u,v) &= (u_4 t^{c_1-2}+(u_3+\lambda u_4)t^{c_1-1},0)\in\omega\end{aligned}$$

imply $v_2=\sigma(u_3-\vartheta u_4)$ and $u_3+\lambda u_4=\vartheta u_4$ or equivalently $v_2=-\sigma\lambda u_4$ and $u_3=(\vartheta-\lambda)u_4$, thus elements of M_λ^* look like

$$(u,v)=(u_4t^{c_1-4}+u_4(\vartheta-\lambda)t^{c_1-3}+\ldots,-u_4\sigma\lambda t^{c_2-1}+\ldots).$$

In particular, we may take the above mentioned element z as

$$z=(t^{c_1-4}+(\vartheta-\lambda)t^{c_1-3}+\ldots,-\sigma\lambda t^{c_2-1}+\ldots),$$

and z together with some elements of $(t^{c_1-1},t^{c_2-1})\tilde{R}$ form a basis of $M_\lambda^*/C(\omega)$. Now with the obvious $M_\lambda^*\cap(t^{c_1-2},t^{c_2-1})\tilde{R}=\omega\cap(t^{c_1-2},t^{c_2-1})\tilde{R}$, we get for nearly all λ

$$\begin{aligned}M_\lambda^* &= \big\langle z,(t^{c_1-1},\sigma t^{c_2-1}),(t^{c_1-2}+\vartheta t^{c_1-1},0)\big\rangle + C(\omega)\\ &= \big\langle z,(t^2,t)z,\ (t^{c_1-2}+\vartheta t^{c_1-1},0)\big\rangle + C(\omega).\end{aligned}$$

Multiplication of M_λ^* with z^{-1},

$$z^{-1}M_\lambda^*=\big\langle 1,(t^2,t),(t^2+\lambda t^3,0)\big\rangle+(t^4,t^2)\tilde{R}=M_\lambda,$$

reveals that M_λ is self–dual for nearly all λ.

Case 3a. We assume that the first branch is singular, thus

$$R=k[[(\varphi_i,\psi_i,\varrho_i)]]\subset k[[t]]^3 \quad \text{with } \operatorname{ord}\varphi_i\geq 2,\ \operatorname{ord}\psi_i,\varrho_i\geq 1.$$

This time

$$M_\lambda=\langle 1,(\lambda t,-1,1)\rangle+(t^2,t,t)\tilde{R}$$

will be a family of pairwise non–isomorphic modules, which are self–dual for nearly all λ.

Let $M_\lambda\cong M_\mu$ by multiplication by an element $u\in K$. By the usual arguments $u=\alpha 1+\beta(\mu t,-1,1)+\ldots$ with $\alpha\neq 0$. From $u(\lambda t,-1,1)=\alpha(\lambda t,-1,1)+\ldots\in M_\mu$ we conclude $\lambda=\mu$.

The dual module of M_λ is

$$\begin{aligned}M_\lambda^* &= \{(u,v,w)\in K\subset k((t))^3 \mid (u,v,w)M_\lambda\subseteq\omega\}\\ &= \{(u,v,w)\in\omega \mid (\lambda tu,-v,w)\in\omega,\ (ut^2,vt,wt)\tilde{R}\subseteq\omega\}\supseteq C(\omega).\end{aligned}$$

Choose $c_1,c_2,c_3\in\mathbb{N}$ with $C(\omega)=(t^{c_1},t^{c_2},t^{c_3})\tilde{R}$ and compute modulo $C(\omega)$. From the condition $(ut^2,vt,wt)\tilde{R}\subset\omega$, i.e., $(ut^2,vt,wt)\in C(\omega)$, we obtain $(u,v,w)=(u_2t^{c_1-2}+u_1t^{c_1-1},v_1t^{c_2-1},w_1t^{c_3-1}) \bmod C(\omega)$.

We claim that $\omega \cap (t^{c_1-1}, t^{c_2-1}, t^{c_3-1})\tilde{R}$ has modulo $C(\omega)$ a basis

$$(t^{c_1-1}, 0, \sigma t^{c_3-1}), (0, t^{c_2-1}, \vartheta t^{c_3-1}) \quad \text{for some } \sigma, \vartheta \in k \setminus \{0\}.$$

To prove this, we note first that $\dim M_\lambda^*/C(\omega) \geq 2$ for general λ, because there are only finitely many non–isomorphic modules with $\dim M_\lambda^*/C(\omega) \leq 1$. Namely, $M_\lambda^* = C(\omega)$ can at most hold for one special λ. If $\dim M_\lambda^*/C(\omega) = 1$, we can choose a $z \in M_\lambda^* \setminus C(\omega) \subset k[[t]]^3$ where all components of z are non–zero and of the smallest possible order. Write $z = z^*\bar{z}$ with $z^* \in R^*$ and $\bar{z} \in \{t^{c_1-2}, t^{c_1-1}, t^{c_1}\} \times \{t^{c_2-1}, t^{c_2}\} \times \{t^{c_3-1}, t^{c_3}\}$, then $(z^*)^{-1}M_\lambda^* = k\bar{z} + (t^2, t, t)\tilde{R}$.

Multiplying the elements of $M_\lambda^*/C(\omega)$ by $(\lambda t, -1, 1)$ yields elements of $\omega \cap (t^{c_1-1}, t^{c_2-1}, t^{c_3-1})\tilde{R}/C(\omega)$. If one non–zero element $y \in M_\lambda^*/C(\omega)$ were mapped to zero by this multiplication, then y must be $y = (y_1 t^{c_1-1}, 0, 0) + C(\omega) \subseteq M_\lambda^* + C(\omega) \subseteq \omega$, which contradicts the definition of $C(\omega)$. This implies that the vector space $\omega \cap (t^{c_1-1}, t^{c_2-1}, t^{c_3-1})\tilde{R}/C(\omega)$ is at least two dimensional. However, since $(t^{c_1-1}, 0, 0)$, $(0, t^{c_2-1}, 0)$, $(0, 0, t^{c_3-1})$ are not contained in ω, this vector space must be of dimension two; in particular, we can find a basis like the above claimed one.

Now we attack the computation of the dual module M_λ^*. The above condition $(\lambda tu, -v, w) \in \omega$ is now seen to be equivalent to $w_1 = \sigma\lambda u_2 - \vartheta v_1$. We note that $M_\lambda^* \not\subseteq (t^{c_1-1}, t^{c_2-1}, t^{c_3-1})\tilde{R}$ for nearly all λ, because otherwise the condition $(\lambda tu, -v, w) \in \omega$ — and thus M_λ^* — does not depend on λ. Therfore, for a general λ we find a $\tilde{z} \in M_\lambda^* \setminus (t^{c_1-1}, t^{c_2-1}, t^{c_3-1})\tilde{R}$ of the form

$$\tilde{z} = (t^{c_2-1} + z_1 t^{c_1-1}, z_2 t^{c_2-1}, (\sigma\lambda - \vartheta z_2)t^{c_3-1}).$$

The remaining basis elements of $M_\lambda^*/C(\omega)$ can be found inside $M_\lambda^* \cap (t^{c_1-1}, t^{c_2-1}, t^{c_3-1})\tilde{R}/C(\omega)$. Due to $M_\lambda^* \subseteq \omega$ they are all of the form

$$(u_1 t^{c_1-1}, v_1 t^{c_2-1}, (u_1\sigma + v_1\vartheta)t^{c_3-1}).$$

The product of such an element with $(\lambda t, -1, 1)$,

$$(0, -v_1 t^{c_2-1}, (u_1\sigma + v_1\vartheta)t^{c_3-1})$$

must lie inside ω, i.e., $-v_1\vartheta = u_1\sigma + v_1\vartheta$ or equivalently $u_1 = -2\vartheta/\sigma \cdot v_1$. Therefore, $M_\lambda^* \cap (t^{c_1-1}, t^{c_2-1}, t^{c_3-1})\tilde{R}/C(\omega)$ is generated by

$$y := (-2\tfrac{\vartheta}{\sigma} t^{c_1-1}, t^{c_2-1}, -\vartheta t^{c_3-1}).$$

In the whole we find

$$\begin{aligned} M_\lambda^* &= \langle \tilde{z}, y \rangle + C(\omega) = \left\langle \tilde{z} + \left(-z_2 + \tfrac{\sigma\lambda}{2\vartheta}\right) y, -\tfrac{\sigma\lambda}{2\vartheta} y \right\rangle + C(\omega) \\ &= \left\langle z := \left(t^{c_1-2} + \ldots, \tfrac{\sigma\lambda}{2\vartheta} t^{c_2-1}, \tfrac{\sigma\lambda}{2} t^{c_3-1}\right), \left(\lambda t^{c_1-1}, -\tfrac{\sigma\lambda}{2\vartheta} t^{c_2-1}, \tfrac{\sigma\lambda}{2} t^{c_3-1}\right) \right\rangle + C(\omega). \end{aligned}$$

Multiplication by z^{-1},

$$z^{-1}M_\lambda^* = \left\langle 1, (\lambda t^{c_1-1}, -t^{c_2-1}, t^{c_3-1}) \right\rangle + C(\omega) = M_\lambda,$$

reveals that M_λ is self–dual for nearly all λ.

Case 3b. The local ring may be taken to be

$$R = k[[(t,t,t),(0,\varphi_i,\psi_i)]] \subset \tilde{R} = k[[t]]^3 \quad \text{with } \operatorname{ord}\varphi_i,\psi_i \geq 2.$$

Here

$$M_\lambda = \langle 1,(t,t,t),(0,t,\lambda t)\rangle + t^2\tilde{R}$$

will be a family of pairwise non–isomorphic modules, which are self–dual for nearly all λ.

Let $M_\lambda \cong M_\mu$ by multiplication by $u \in K$. By the usual argument $\mu = \alpha 1 + \beta(t,t,t) + \ldots$ for $\alpha \neq 0$. From $u(0,t,\lambda t) = \alpha(0,t,\lambda t) + \ldots \in M_\mu$, we get $\lambda = \mu$.

The dual module of M_λ^* is

$$\begin{aligned} M_\lambda^* &= \{(u,v,w) \in K \subset k((t))^3 \mid (u,v,w)M_\lambda \subseteq \omega\} \\ &= \{(u,v,w) \in \omega \mid (0,tv,\lambda tw) \in \omega,\ (u,v,w)t^2\tilde{R} \subset \omega\} \supseteq C(\omega). \end{aligned}$$

As always we choose $c_1,c_2,c_3 \in \mathbb{N}$ with $C(\omega) = (t^{c_1},t^{c_2},t^{c_3})\tilde{R}$ and compute modulo $C(\omega)$. The condition $(u,v,w)t^2\tilde{R} \subset \omega$ is equivalent to $(u,v,w) \in (t^{c_1-2},t^{c_2-2},t^{c_3-2})\tilde{R}$.

We claim that $(\omega \cap (t^{c_1-1},t^{c_2-1},t^{c_3-1})\tilde{R})/C(\omega)$ has a basis

$$(t^{c_1-1},0,\sigma t^{c_3-1}),(0,t^{c_2-1},\vartheta t^{c_3-1}) \quad \text{for some } \sigma,\vartheta \in k \setminus \{0\}.$$

M_λ^* cannot be contained in $(t^{c_1-2},t^{c_2-2},t^{c_3-1})\tilde{R}$ for nearly all λ, because otherwise the condition $(0,tv,\lambda tw) \in \omega$ — and thus M_λ^* — does not depend on λ. Therefore, for general λ we find a $z = (z_1t^{c_1-2}+\ldots,z_2t^{c_2-2}+\ldots,z_3t^{c_3-2}+\ldots) \in M_\lambda^* \setminus (t^{c_1-2},t^{c_2-2},t^{c_3-1})\tilde{R}$, i.e., $z_3 \neq 0$, and further

$$\begin{aligned} (t,t,t)z \quad &= (z_1t^{c_1-1},z_2t^{c_2-1},z_3t^{c_3-1}) \in M_\lambda^* \subseteq \omega \\ (0,t,\lambda t)z &= (0,z_2t^{c_2-1},z_3\lambda t^{c_3-1}) \in \omega. \end{aligned}$$

Since $(0,0,t^{c_3-1}) \notin \omega$, the above vectors must be linearly independent and z_1,z_2 must be non–zero — at least for $\lambda \notin \{0,1\}$. A coordinate change takes these vector to the above described vectors. By the definition of $C(\omega)$ the vector space $(\omega \cap t^{-1}C(\omega))/C(\omega)$ cannot be $t^{-1}C(\omega)/C(\omega)$ itself, hence it is at most two–dimensional and the two linear independent vectors in question form a basis.

Now the computation of M_λ^* is straight forward. Let $(u,v,w) \in M_\lambda^*$, it must satisfy the conditions

$$\begin{aligned} (tu,tv,tw) \ &= (u_2t^{c_1-1},v_2t^{c_2-1},w_2t^{c_3-1}) \in \omega \\ (0,tv,\lambda tw) &= (0,v_2t^{c_2-1},w_2\lambda t^{c_3-1}) \in \omega. \end{aligned}$$

Using the above basis, they are equivalent to $w_2 = u_2\sigma + v_2\vartheta$ and $w_2\lambda = v_2\vartheta$ or $u_2 = (1-\lambda)/\sigma \cdot w_2$ and $v_2 = \lambda/\vartheta \cdot w_2$, i.e.,

$$(u,v,w) = \left(w_2\tfrac{1-\lambda}{\sigma}t^{c_1-2}+\ldots,w_2\tfrac{\lambda}{\vartheta}t^{c_2-2}+\ldots,w_2t^{c_3-2}+\ldots\right).$$

Hence, the above mentioned vector z can be taken to be the above vector with $w_2 = 1$ and all further elements of a basis of $M_\lambda^*/C(\omega)$ can be found in $t^{-1}C(\omega)$. Using the obvious $M_\lambda^* \cap t^{-1}C(\omega) = \omega \cap t^{-1}C(\omega)$ we get

$$\begin{aligned} M_\lambda^* &= \left\langle z, (t^{c_1-1}, 0, \sigma t^{c_3-1}), (0, t^{c_2-1}, \vartheta t^{c_3-1}) \right\rangle + (t^{c_1}, t^{c_2}, t^{c_3})\tilde{R} \\ &= \left\langle z, (t,t,t)z, (0, t^{c_2-1}, \vartheta t^{c_3-1}) \right\rangle + (t^{c_1}, t^{c_2}, t^{c_3})\tilde{R}. \end{aligned}$$

Multiplication by z^{-1} yields $z^{-1}M_\lambda^* = M_\lambda$, thus M_λ is self–dual for nearly all λ.

Case 4. In the last case we only assume that the singularity has $b \geq 4$ branches, i.e., $R \subseteq k[[t]]^b$ and $R/t\tilde{R} = \langle 1 \rangle$. For notational convenience we treat only the case $b = 4$, the general case is the same — the occurring elements only need to be extended in the obvious way. We will show that

$$M_\lambda = \langle 1, (0,1,2,\lambda) \rangle + (t,t,t,t)\tilde{R}$$

is a family of pairwise non–isomorphic modules which are self–dual for nearly all λ.

Let $M_\lambda \cong M_\mu$ by multiplication by $u \in K$. The usual argument yields $u \in \tilde{R}^*$. From $u = u1 \in \mathrm{M}_\mu$ we get $u = \alpha 1 + \beta(0,1,2,\mu) + \ldots$. The condition $u(0,1,2,\lambda) = \alpha(0,1,2,\lambda) + \ldots \in M_\mu$ implies $\lambda = \mu$.

We start with the computation of the dual module of M_λ,

$$\begin{aligned} M_\lambda^* &= \{(u,v,w,x) \in K \subset k((t))^4 \mid (u,v,w,x)M_\lambda \subseteq \omega\} \\ &= \{(u,v,w,x) \in \omega \mid (0,v,2w,\lambda x) \in \omega,\ (ut,vt,wt,xt)\tilde{R} \subseteq \omega\} \supseteq C(\omega). \end{aligned}$$

Choose $c_i \in \mathbb{N}$ with $C(\omega) = (t^{c_1}, t^{c_2}, t^{c_3}, t^{c_4})\tilde{R}$ and compute modulo $C(\omega)$. The condition $(ut,vt,wt,xt)\tilde{R} \subseteq \omega$ is equivalent to $(u,v,w,x) \in t^{-1}C(\omega)$ or $(u,v,w,x) = (u_1 t^{c_1-1}, v_1 t^{c_2-1}, w_1 t^{c_3-1}, x_1 t^{c_4-1})$.

We claim that the vector space $(\omega \cap t^{-1}C(\omega))/C(\omega)$ is three–dimensional and thus possesses a basis

$$(t^{c_1-1}, 0, 0, \sigma t^{c_4-1}),\ (0, t^{c_2-1}, 0, \vartheta t^{c_4-1}),\ (0, 0, t^{c_3-1}, \varrho t^{c_4-1})$$

for some $\sigma, \vartheta, \varrho \in k \setminus \{0\}$. First the vector space cannot be four–dimensional, because in that case $t^{-1}C(\omega) \subseteq \omega$ which contradicts the definition of $C(\omega)$. If the vector space has dimension $d \leq 2$, the condition $(u,v,w,x) \in \omega$ imposes $4-d$ homogeneous linear relations on u_1, v_1, w_1, x_1. The condition $(0,v,2w,\lambda x) \in \omega$ must impose at least one further relation on u_1, v_1, w_1, x_1, because otherwise there would be no restriction on (u,v,w,x) depending on λ. Therefore, there would be at least three relations and $\dim M_\lambda^*/C(\omega) \leq 1$. An argument like in the case *3a* shows that this is impossible. Hence, the vector space $(\omega \cap t^{-1}C(\omega))/C(\omega)$ is three–dimensional and recalling that $(t^{c_1-1},0,0,0)$, $(0,t^{c_2-1},0,0)$, $(0,0,t^{c_3-1},0)$, $(0,0,0,t^{c_4-1}) \notin \omega$, we can obviously find a base like above.

Now we can proceed with the computation of M_λ^*. The conditions

$$\begin{aligned} (u,v,w,x) &= (u_1 t^{c_1-1}, v_1 t^{c_2-1}, w_1 t^{c_3-1}, x_1 t^{c_4-1}) \in \omega \\ (0,v,2w,\lambda x) &= (0, v_1 t^{c_2-1}, 2w_1 t^{c_3-1}, \lambda x_1 t^{c_4-1}) \in \omega \end{aligned}$$

are now seen to be equivalent to $x_1 = u_1\sigma + v_1\vartheta + w_1\varrho$ and $\lambda x_1 = v_1\vartheta + 2w_1\varrho$ or $u_1 = (\varrho w_1 + (1-\lambda)x_1)/\sigma$ and $v_1 = (-2\varrho w_1 + \lambda x_1)/\vartheta$. Plugging in $(w_1, x_1) = (\lambda(\lambda-1), 2\varrho)$ resp. $(2\lambda(\lambda-1), 2\varrho\lambda)$, we obtain a basis for M_λ^* modulo $C(\omega)$, namely

$$M_\lambda^* = \Big\langle z := \Big(\tfrac{\varrho(\lambda-2)(\lambda-1)}{\sigma}, \tfrac{2\varrho\lambda(2-\lambda)}{\vartheta}, \lambda(\lambda-1), 2\varrho\Big), \\ \Big(0, \tfrac{2\varrho\lambda(2-\lambda)}{\vartheta}, 2\lambda(\lambda-1), 2\varrho\lambda\Big)\Big\rangle + C(\omega).$$

Multiplying M_λ^* by z^{-1} yields M_λ, showing that M_λ is self–dual.

Higher Rank. We obtain modules of higher rank by using the ideas of Greuel and Knörrer. Let E be the identity matrix of size n and J_λ the Jordan matrix of size n consisting of only one block with eigenvalue $\lambda \in k$. The following table contains families of indecomposable, torsion–free, self–dual modules of rank n

Case	$M_\lambda \subseteq \tilde{R}^n$
1a	$R^n + (t^2E + t^3J_\lambda)R^n + t^4\tilde{R}^n$
1b	$R^n + (t^4E + t^5J_\lambda)R^n + t^6\tilde{R}^n$
2a	$R^n + ((t,0)E + (0,t)J_\lambda)R^n + t^2\tilde{R}^n$
2b	$R^n + ((t,1)E + (t^2,0)J_\lambda)R^n + (t^3,t)\tilde{R}^n$
2c	$R^n + ((t^2,0)E + (t^3,0)J_\lambda)R^n + (t^4,t^2)\tilde{R}^n$
3a	$R^n + ((0,-1,1)E + (t,0,0)J_\lambda)R^n + (t^2,t,t)\tilde{R}^n$
3b	$R^n + ((0,t,0)E + (0,0,t)J_\lambda)R^n + t^2\tilde{R}^n$
4	$R^n + ((0,1,2,0)E + (0,0,0,1)J_\lambda)R^n + t\tilde{R}^n$

The modules are of rank n and torsion–free, because they contain R^n and are contained in $\tilde{R}^n$. They are pairwise nonisomorphic and indecomposable by the same arguments as in the proof of [GK, Lemma 4]. It remains to show that they are self–dual. The computation of the dual module

$$M_\lambda^* = \{u \in K^n \mid u^t \cdot M_\lambda \subseteq \omega\}$$

is notationally more inconvenient as in the rank 1 case, but easier because we now know ω already — at least partially. Finding the isomorphism between M_λ and M_λ^* is more difficult as before, but not too hard, since it is given by a multiplication by an matrix of $\mathrm{GL}(n,K)$.

References

[A] Atiyah, M.: *Riemann surfaces and spin structures.* Ann. Sci. Éc. Norm. Supér., IV. Sér. **4** (1971), 47–62.

[AGV] Arnol'd, V.,S. Guseĭn-Zade, and A. Varchenko. *Singularities of differentiable maps. I.* Birkhuser Boston, 1985.

[B] Beauville, A.: *Determinantal hypersurfaces.* Mich. Math. J. **48** (2000), 39–64.

[C] Catanese, F.: *Homological algebra and algebraic surfaces.* Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., 62, Part 1, Amer. Math. Soc., Providence, RI, 1997, 3–56.

[Co] Cook, P.: *Local and Global Aspects of the Module Theory of Singular Curves.* Ph. D. Thesis, University of Liverpool 1993.

[D] Dixon, A.: *Note on the reduction of a ternary quartic to a symmetrical determinant.* Proc. Camb. Phil. Soc. **11** (1900–1902), 350–351.

[GK] Greuel, G.–M. and H. Knörrer: *Einfache Kurvensingularitäten und torsionfreie Moduln.* Math. Ann. **270** (1985), 417–425.

[H] Harris, J.: *Theta–Characteristics on Algebraic Curves.* Trans. AMS **271** (1982), 611–638.

[He] Hesse, L.: *Ueber Determinanten und ihre Anwendungen in der Geometrie.* Gesammelte Werke, Verl. der Königlichen Akademie, München 1897.

[M] Mumford, D.: *Theta characteristics of an algebraic curve.* Ann. Sci. Éc. Norm. Supér., IV. Sér. **4** (1971), 181–192.

[P] Piontkowski, J.: *Theta–Characteristics on Singular Curves.* In preparation.

[S] Sorger, Ch.: *Thêta-caractéristiques des courbes tracées sur une surface lisse.* J. reine angew. Math. **435** (1993), 83–118.

[W] Wall, C. T. C.: *Nets of quadrics, and Theta-characteristics of singular curves.* Philos. Trans. R. Soc. London, Ser. A **289** (1978), 229–269.

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