Theta–Characteristics on Singular Curves

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May 1, 2007

Abstract

On a smooth curve a theta–characteristic is a line bundle L with square that is the canonical line bundle ω. The equivalent condition $\mathcal{H}om(L, \omega) \cong L$ generalizes well to singular curves, as applications show. More precisely, a theta–characteristic is a torsion–free sheaf $\mathcal{F}$ of rank 1 with $\mathcal{H}om(\mathcal{F}, \omega) \cong \mathcal{F}$. If the curve has non ADE–singularities then there are infinitely many theta–characteristics. Therefore, theta–characteristics are distinguished by their local type. The main purpose of this article is to compute the number of even and odd theta–characteristics (i.e. $\mathcal{F}$ with $h^0(C, \mathcal{F}) \equiv 0$ resp. $h^0(C, \mathcal{F}) \equiv 1$ modulo 2) in terms of the geometric genus of the curve and certain discrete invariants of a fixed local type.

Mathematics Subject Classification (2000). 14H60, 14H42, 14H40.

A theta–characteristic on a smooth projective curve C over $\mathbb{C}$ is a line bundle L whose square is the canonical bundle K. Let Θ be the set of all theta–characteristics. If g is the genus of the curve, then there are 2^{2g} theta–characteristics on C. Naturally, one would like to know the dimension of the linear systems $|L|$, $L \in \Theta$. However, these dimensions depend not only on the genus alone, but also on the complex structure of the curve. Nevertheless, using the his theory of theta–functions Riemann proved that the dimension modulo 2 depends only on the genus. If

$$\Theta^+ = \{L \in \Theta \mid h^0(L) \text{ is even.}\}$$
$$\Theta^- = \{L \in \Theta \mid h^0(L) \text{ is odd.}\}$$

are the even resp. odd theta–characteristics, then there are $2^{g-1}(2^g + 1)$ even and $2^{g-1}(2^g - 1)$ odd theta–characteristics.

Atiyah gave another analytic proof of this, and Mumford the first algebraic one [A, M]. Mumford's ideas were refined and extended by Harris to include singular curves into the theory — at least Gorenstein curves, for example plane curves [H]. On a singular curve a line bundle is defined to be a locally free sheaf of rank 1. Harris showed that the number of even and odd theta–characteristics can be computed in terms of the genus of the curve and certain discrete invariants of the singularaties. He also remarked that it would be desirable to be able to treat torsion–free sheaves of rank 1 alongside with the line bundles. This is what we want to do in this article.

Let C be a connected, reduced Gorenstein curve with structure sheaf $\mathcal{O}$ and (locally free) canonical sheaf ω. It makes no sense to call a torsion–free sheaf $\mathcal{F}$ of rank 1 a theta–characteristic if $\mathcal{F}\otimes\mathcal{F} = \omega$, because this condition implies that $\mathcal{F}$ is locally free. On a smooth curve $L^2 = K$ is equivalent to $L = \mathrm{Hom}(L, K)$, and this generalizes well:

Definition *A* theta–characteristic *on C is a torsion–free sheaf $\mathcal{F}$ of rank 1 which is isomorphic to $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \omega_C)$. The set of all theta–characteristics on C is denoted by Θ.*

We will see below that the curve C possesses infinitely many theta–characteristics as soon as it has a singularity worse than an ADE–singularity. Therefore, we will fix the local type of $\mathcal{F}$:

Definition *Let $\mathcal{F}$ be a torsion–free sheaf of rank 1, then the* local type *of $\mathcal{F}$ is the collection of modules $T = (\mathcal{F}_p)_{p\in \mathrm{Sing}\, C}$. The set of theta–characteristics of type T is denoted by Θ_T.*

Since the stalk of a torsion–free sheaf at a smooth point is free, the above definition captures all the local information. In particular, $\mathcal{F}$ is a line bundle iff its local type is $(\mathcal{O}_x)_{x\in \mathrm{Sing}\, C}$.

In Section 1 we will show that for any collection $T = (\mathcal{F}_p)_{p\in \mathrm{Sing}\, C}$ of self–dual $\mathcal{O}_p$–modules at the singular points of an integral curve C, there exists a theta–characteristic of type T. Further, we introduce the action of the line bundles of order 2 on Θ_T. Finally, the theta–characteristics on C are related to the theta–characteristics on a partial normalization of C. These results will be sufficient to deal with the curves with only ADE–singularities in Section 3. In Section 2 we solve the main problem which was posed above: Compute the number of even and odd theta–characteristics of a fixed local type T in terms of the geometric genus of the curve and certain discrete invariants of the type T.

Among the applications of this article are the following: In [S] Sorger asked which singular curves on a smooth surface have finitely many theta–characteristics. For an integral curve C Proposition 1.3 together with the results on the number of self–dual modules over a curve singularity [Pi2] imply that these are precisely the curves with only ADE–singularities. We can compute their number of theta–characteristics by using the tables given in Section 3.

Theta–characteristics in our general sense are used to find contact curves to plane curves or to obtain symmetric determinantal representations of plane curves. These ideas go back to Hesse for cubics and Dixon (1903) for smooth curves of arbitrary degree. Modern surveys are contained in [B2, M–B]. The generalized theta–characteristics correspond to systems of contact curves to the original curve. The notion of the local type provides an easy way to describe them. Previously, some authors classified these contact systems by examining the behavior of their base points in certain blow–ups. For the above correspondence it is essential to know the dimension of the linear system $H^0(\mathcal{F})$ of a theta–characteristic $\mathcal{F}$. Section 2 helps by computing the number of even and odd theta–characteristics of a fixed local type. For a curve of small degree or by using some another additional information the number of theta–characteristics with a linear system of a fixed dimension can be recovered.

Certain systems of contact curves yield nets of quadrics whose discriminant is the original curve. In [W1] Wall studied these nets, in particular on $\mathbb{C}^4$. In the third section he describes mostly conjecturally the relation between these nets and points of the compactified Jacobian of plane quartics. At least for integral curves these conjectures can be extended from quadrics with only A–singularities to all quartics and proved by first using Theorem [B2, 4.2] to relate the nets of quadrics to theta–characteristics and then using the results and tables of Section 3.

Finally, in Section 4 we give an application to the question in how many different ways a plane quartic can be written as the sum of three squares of conics.

1 Preliminary results

Our first aim is to ensure the existence of theta–characteristics with prescribed local type. We start with the following remark about torsion–free sheaves of rank 1. Here and in the following $\mathcal{K}$ will denote the function field sheaf on C.

Lemma 1.1 *Let C be a reduced curve.*

1. *For any $p \in \operatorname{Sing} C$ let $\mathcal{F}_p$ be a torsion–free $\mathcal{O}_p$–module of rank 1. Then there exists a torsion–free subsheaf $\mathcal{F}$ of $\mathcal{K}$ of rank 1 whose local type is $T = (\mathcal{F}_p)_{p\in \operatorname{Sing} C}$.*
2. *Let $\mathcal{F}$ be a torsion–free sheaf of rank 1. Then for any line bundle $\mathcal{L}$ the tensor product $\mathcal{F} \otimes \mathcal{L}$ is a torsion–free sheaf of rank 1 and the same local type as $\mathcal{F}$.*
3. *Two torsion–free sheaves $\mathcal{F}, \mathcal{G}$ of rank 1 and the same local type differ by the multiplication with some line bundle, i.e., $\mathcal{G} \cong \mathcal{F} \otimes \mathcal{L}$ for some line bundle $\mathcal{L}$.*

Proof. The sheaf $\mathcal{F}$ in the first statement must have the stalks $\mathcal{O}_p$ at the smooth points of C. Further, we may assume that $\mathcal{O}_p \subseteq \mathcal{F}_p \subseteq \mathcal{K}_p$, because any torsion–free module can be embedded in this way. These stalks can be glued together as

$$\mathcal{F}(U) = \left\{ f \in \mathcal{K}(U) \mid f|_{U\setminus \operatorname{Sing} C} \in \mathcal{O}(U \setminus \operatorname{Sing} C),\ \varinjlim f \in \mathcal{F}_p \right\},$$

see [Coo, 3.1.1 a]. The second statement is obvious. Third follows from the fact that isomorphisms between torsion–free sheaves are given locally by multiplying with elements of $\mathcal{K}^*$. These local multiplications define the line bundle $\mathcal{L}$, see [Coo, 3.1.3]. □

All line bundles of degree 0 on every component of C are collected in the generalized Jacobian JC. It possesses a natural structure of an algebraic group, in fact we have:

Lemma 1.2 *The generalized Jacobian is a divisible group.*

Proof. Compare [B1, 2.2]. Let $\pi : \tilde{C} \to C$ be the normalization of C and $\tilde{\mathcal{O}}$ the push forward of the structure sheaf of $\tilde{C}$. From the long exact sequence associated to

$$1 \to \mathcal{O}^* \to \tilde{\mathcal{O}}^* \to \prod_{p \in \operatorname{Sing} C} \tilde{\mathcal{O}}_p^*/\mathcal{O}_p^* \to 1$$

we get

$$0 \to \left(\prod_{p \in \operatorname{Sing} C} \tilde{\mathcal{O}}_p^*/\mathcal{O}_p^* \right) /(H^0(\tilde{\mathcal{O}}^*)/H^0(\mathcal{O}^*)) \to JC \xrightarrow{\pi^*} J\tilde{C} \to 0.$$

The first group is a product of multiplicative and additive groups, hence divisible. Since $J\tilde{C}$ is a product of complex tori, it is divisible as well. Thus JC is divisible. For future reference we note that the sequence splits as a sequence of Abelian groups. □

Now we can prove the existence of theta–characteristics of a prescribed local type.

Proposition 1.3 *Let C be an reduced curve.*

1. *Let $\mathcal{F}$ be a theta–characteristic on C. Then the local type of $\mathcal{F}$, $T = (\mathcal{F}_p)_{p \in \operatorname{Sing} C}$, is a collection of self–dual modules, i.e., $\operatorname{Hom}(\mathcal{F}_p, \omega_p) \cong \mathcal{F}_p$.*
2. *Any theta–characteristic $\mathcal{F}$ has degree $g_a - 1$, where g_a is the arithmetic genus.*
3. *Let C be irreducible, and for any $p \in \operatorname{Sing} C$ let $\mathcal{F}_p$ be a self–dual torsion–free $\mathcal{O}_p$–module of rank 1. Then there exists a theta–characteristic $\mathcal{F}$ of local type $T = (\mathcal{F}_p)_{p \in \operatorname{Sing} C}$.*

Proof. 1) This follows from the fact that the localization of the dualizing sheaf of C at a point is the dualizing module of $\mathcal{O}_p$ [Coo, 3.1.6].

2) The isomorphism $\mathcal{F} \cong \mathcal{H}om(\mathcal{F}, \omega)$ implies by [Coo, 3.1.6]

$$\deg \mathcal{F} = \deg \mathcal{H}om(\mathcal{F}, \omega) = (2g_a - 2) - \deg \mathcal{F}.$$

3) By Lemma 1.1.1 we find a torsion–free sheaf $\mathcal{G}$ of local type T. By assumption $\mathcal{H}om(\mathcal{G}, \omega)$ has the same local type as $\mathcal{G}$, hence there exists a line bundle $\mathcal{L}$ such that $\mathcal{H}om(\mathcal{G}, \omega) \cong \mathcal{G} \otimes \mathcal{L}$ by Lemma 1.1.3. By [Coo, 3.1.6/7] the degree of the line bundle $\mathcal{L}$ is

$$\deg \mathcal{L} = \deg \mathcal{H}om(\mathcal{G}, \omega) - \deg \mathcal{G} = -2 \deg \mathcal{G} + 2g_a - 2 = 2(g_a - \deg \mathcal{G} - 1).$$

Set $d = g_a - \deg \mathcal{G} - 1$ and let $p \in C$ be a smooth point. Then the line bundle $\mathcal{L}(-2dp)$ has degree 0 and by Lemma 1.2 there exists a line bundle $\mathcal{M}$ with $\mathcal{M}^2 \cong \mathcal{L}(-2dp)$. Setting $\mathcal{F} = \mathcal{G} \otimes \mathcal{M}(dp)$, we get [Ha, II, Ex. 5.1 / III, 6.7]

$$\begin{aligned} \mathcal{H}om(\mathcal{F}, \omega) &= \mathcal{H}om(\mathcal{G} \otimes \mathcal{M}(dp), \omega) \cong \mathcal{H}om(\mathcal{G}, \omega) \otimes \mathcal{M}^*(-dp) \\ &\cong \mathcal{G} \otimes \mathcal{L} \otimes \mathcal{M}^*(-dp) \cong \mathcal{G} \otimes \mathcal{M}(dp) = \mathcal{F}. \end{aligned}$$ □

By [Pi2] the number of self–dual modules of rank 1 over the local ring of a singularity is finite if and only if the singularity is an ADE–singularity or a partial resolution of it. Therefore, the proposition implies that there exist infinitely many theta–characteristics of any irreducible curve as soon as it has a singularity worse than an ADE–singularity.

The irreducibility assumption in the last statement cannot be dropped. As an example let $C = L_1 \cup L_2$ be the union of two distinct lines in $\mathbb{P}^2$. We claim that C does not posses a locally free theta–characteristic. Namely, assume that $\mathcal{F}$ is such. Then $\mathcal{H}om(\mathcal{F}, \omega) \cong \mathcal{F}$ is equivalent to $\mathcal{F}^2 \cong \omega$. Restricting to a line yields $(\mathcal{F} \mid_{L_1})^2 \cong \omega \mid_{L_1}$ and

$$2 \deg \mathcal{F} \mid_{L_1} = \deg \omega \mid_{L_1} = \deg \omega_{L_1} + L_1 \cdot L_2 = -2 + 1 = -1$$

by [C, Lemma 1.12], which is a contradiction. From now on we will always assume that there exists at least one theta–characteristic of the type in question.

Lemma 1.1 implies that JC acts on the torsion–free sheaves of rank 1. If $\mathcal{F}$ is a theta–characteristic, we cannot expect that $\mathcal{F} \otimes \mathcal{L}$ is also a theta–characteristic for every line bundle $\mathcal{L}$. However, it works for line bundles of order 2, we denote them by $J_2C \subseteq JC$.

Proposition 1.4 *The group J_2C acts on Θ_T.*

Proof. Let $\mathcal{F}$ be a theta–characteristic and $\mathcal{L} \in J_2C$. By Lemma 1.1 we only need to show that $\mathcal{F} \otimes \mathcal{L}$ is a theta-characteristic. From $\mathcal{L}^* \cong \mathcal{L}$ we get

$$\mathcal{H}om(\mathcal{F} \otimes \mathcal{L}, \omega) \cong \mathcal{L}^* \otimes \mathcal{H}om(\mathcal{F}, \omega) \cong \mathcal{L}^* \otimes \mathcal{F} \cong \mathcal{F} \otimes \mathcal{L}. \qquad \square$$

This action is transitive, but in general not faithful. To prove this and more, recall the following way to construct a partial normalization of a curve C:

Let $\mathcal{A}$ be an $\mathcal{O}$–algebra sheaf on C that is finitely generated as an $\mathcal{O}$–module. Then there exists a unique partial normalization $\check{\pi} : \check{C} \to C$ such that the push–down of the structure sheaf $\check{\mathcal{O}} := \mathcal{O}_{\check{C}}$ is $\mathcal{A}$. The functor $\check{\pi}_*$ induces an equivalence of categories from the category of quasi–coherent $\check{\mathcal{O}}$–modules on $\check{C}$ to the category of quasi–coherent $\mathcal{A}$–modules on C [Ha, II, Ex. 5.17]. For an $\mathcal{A}$–module sheaf $\mathcal{F}$ on C we denote by $\check{\mathcal{F}}$ or $\check{\pi}^! \mathcal{F}$ the sheaf on $\check{C}$ with $\check{\pi}_*(\check{\mathcal{F}}) = \mathcal{F}$. Further, there are natural isomorphisms $H^i(\check{C}, \check{\mathcal{F}}) \cong H^i(C, \mathcal{F})$ [Ha, III, Ex. 4.1]. Finally, we recall the following lemma of Beauville [B1, 2.1]:

Lemma 1.5 *Let $\mathcal{F}$ be a torsion–free sheaf of rank 1 on C and $\mathcal{A} = \mathcal{E}nd(\mathcal{F})$. Let $\mathcal{L}$ be a line bundle on C. Then $\mathcal{F} \otimes \mathcal{L} \cong \mathcal{F}$ if and only if $\check{\pi}^* \mathcal{L}$ is trivial.*

Proposition 1.6 *The group J_2C acts transitively on the theta–characteristics of a fixed local type T, Θ_T.*

Proof. For the transitivity let $\mathcal{F}, \mathcal{G}$ be two theta–characteristics of the same type. By Lemma 1.1.3 there exists a line bundle $\mathcal{L}$ with $\mathcal{G} \cong \mathcal{F} \otimes \mathcal{L}$. From the theta–characteristic property of $\mathcal{F} \otimes \mathcal{L}$ and $\mathcal{F}$ we get

$$\mathcal{F} \otimes \mathcal{L} \cong \mathcal{H}om(\mathcal{F} \otimes \mathcal{L}, \omega) \cong \mathcal{L}^* \otimes \mathcal{H}om(\mathcal{F}, \omega) \cong \mathcal{L}^* \otimes \mathcal{F}$$

or $\mathcal{F} \cong \mathcal{F} \otimes \mathcal{L}^2$. By Lemma 1.5 the line bundle $\check{\pi}^*\mathcal{L}^2 = (\check{\pi}^*\mathcal{L})^2$ is trivial, i.e., $\check{\pi}^*\mathcal{L} \in J_2\check{C}$. With the same arguments as in the proof of Lemma 1.2 one shows that JC is in a non–canonical way the direct product of $J\check{C}$ and another Abelian group G, $JC \cong J\check{C} \times G$, and hence $J_2C \cong J_2\check{C} \times G_2$. Therefore, we can find an element $\mathcal{M} \in J_2C$ such that $\check{\pi}^*\mathcal{M} = \check{\pi}^*\mathcal{L}$. Lemma 1.5 and $\check{\pi}^*(\mathcal{L} \otimes \mathcal{M}^*) = \check{\mathcal{O}}$ implies

$$\mathcal{G} \cong \mathcal{G} \otimes \mathcal{L} \otimes \mathcal{M}^* \cong \mathcal{F} \otimes \mathcal{M}^*,$$

showing that $\mathcal{G} \cong \mathcal{F} \otimes \mathcal{M}$ lies in the J_2C–orbit of $\mathcal{F}$. □

We will apply the above construction to the sheaf $\mathcal{A} = \mathcal{E}nd(\mathcal{F})$ for several $\mathcal{F}$ of the same local type. For this the following lemma is essential:

Lemma 1.7 *Let $\mathcal{F}$ be a torsion–free sheaf of rank 1 and local type T. The $\mathcal{O}$–algebra $\mathcal{A}_T = \mathcal{E}nd(\mathcal{F})$ depends only on the local type of $\mathcal{F}$.*

In particular, the partial normalization of C, $\check{\pi} : \check{C} \to C$, determined by $\mathcal{E}nd(\mathcal{F})$ depends only on the local type of $\mathcal{F}$.

Proof. By Lemma 1.1.3 any other sheaf $\mathcal{G}$ of the same local type as $\mathcal{F}$ is of the form $\mathcal{F} \otimes \mathcal{L}$ for a line bundle $\mathcal{L}$, thus

$$\mathcal{E}nd(\mathcal{G}) = \mathcal{H}om(\mathcal{F} \otimes \mathcal{L}, \mathcal{F} \otimes \mathcal{L}) \cong \mathcal{L}^* \otimes \mathcal{H}om(\mathcal{F}, \mathcal{F}) \otimes \mathcal{L} \cong \mathcal{E}nd(\mathcal{F}). \quad \square$$

Definition 1.8 *Let $\mathcal{A}$ be an $\mathcal{O}$–algebra on C like above. Let $T = (\mathcal{F}_p)_{p \in \operatorname{Sing} C}$ be a collection of $\mathcal{O}_p$–modules which posses a structure as $\mathcal{A}_p$–modules as well. Then the corresponding collection of $\check{\mathcal{O}}$–modules $\check{T} = (\check{\mathcal{F}}_q)_{q \in \check{\pi}^{-1}(\operatorname{Sing} C)}$ on $\check{C}$ is the collection chosen such that $\mathcal{F}_p = \prod_{q \in \check{\pi}^{-1}(p)} \check{\mathcal{F}}_q$.*

Here we slightly extended our notion of local type for notational convenience by allowing the q to run through some smooth points as well.

Proposition 1.9 *Let T be a local type on a reduced curve C, $\mathcal{A}$ its induced algebra sheaf and $\check{\pi} : \check{C} \to C$ the partial normalization constructed out of it.*

Then $\check{\pi}_*$ induces a bijection between the theta–characteristics of type T on C and the theta–characteristics of type $\check{T}$ on $\check{C}$, which preserves the dimension of the homology groups of the theta–characteristics.

Proof. It remains to show that a torsion–free sheaf $\mathcal{F}$ on C is a theta–characteristic iff $\check{\mathcal{F}}$ is a theta–characteristic on $\check{C}$. Recall [Ha, III, Ex. 7.2] that the dualizing sheaf $\check{\omega}$ on $\check{C}$ is $\check{\pi}^!\mathcal{H}om_{\mathcal{O}}(\check{\pi}_*\check{\mathcal{O}}, \omega) = \check{\pi}^!\mathcal{H}om_{\mathcal{O}}(\mathcal{A}, \omega)$. Let $\mathcal{F} = \check{\pi}_*\check{\mathcal{F}}$ be a theta–characteristic on C of local type T, in particular it is an $\mathcal{A}$–module. We want to show that $\check{\mathcal{F}}$ is a theta–characteristic on $\check{C}$. We have the isomorphisms

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \check{\pi}_*\check{\omega}) = \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}}(\mathcal{A}, \omega)) \cong \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \omega) \cong \mathcal{F};$$

here the middle isomorphism is given by locally evaluating at $1 \in \mathcal{A}$; its inverse is given by assigning to $\psi \in \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \omega)$ the map $\varphi(f)(a) := \psi(af)$.

Using the category equivalence induced by $\check{\pi}_*$, we find $\mathcal{H}om_{\check{\mathcal{O}}}(\check{\mathcal{F}}, \check{\omega}) \cong \check{\mathcal{F}}$, i.e., $\check{\mathcal{F}}$ is a theta–characteristic on $\check{C}$.

Now, if $\check{\mathcal{F}}$ is a theta–characteristic on $\check{C}$, the above argument can be reversed. $\check{\mathcal{F}} \cong \mathcal{H}om_{\check{\mathcal{O}}}(\check{\mathcal{F}}, \check{\omega})$ implies

$$\mathcal{F} = \check{\pi}_* \check{\mathcal{F}} \cong \mathcal{H}om_{\mathcal{A}}(\check{\pi}_* \check{\mathcal{F}}, \check{\pi}_* \check{\omega}) = \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}}(\mathcal{A}, \omega)) \cong \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \omega). \quad \square$$

On $\check{C}$ we have $\mathcal{E}nd(\check{\mathcal{F}}) = \check{\mathcal{O}}$ and in this situation the action of $J_2\check{C}$ on the theta–characteristics is as beautiful as possible.

Proposition 1.10 *Assume that for a fixed local type T on C the algebra sheaf $\mathcal{A}_T$ is trivial and Θ_T is non–empty. Then J_2C acts regularly on the theta–characteristics of type T.*

In particular, the number of theta–characteristics of type T equals the number of elements of J_2C.

Proof. The freeness of the action follows from Lemma 1.5, because here $\check{\pi}$ is the identity. Transitivity was proved above. $\square$

The last two propositions are already sufficient to study the even–/oddness of the theta–characteristics of curves with only ADE–singularities, see Section 3. The reader only interested only in those curves may skip the next section.

Unfortunately, this partial normalization construction has a drawback. The curve $\check{C}$ — even while being a partial normalization of C — may be worse than C in the sense that even if C is Gorenstein, $\check{C}$ need not be Gorenstein.

Example Let C be a curve with the Gorenstein singularity $x^4 - y^5$, i.e., the complete local ring at this point is $R = \mathbb{C}[[t^4, t^5]] \subseteq \tilde{R} = \mathbb{C}[[t]]$. The module $M = \langle 1, t\rangle + t^4\tilde{R}$ is self–dual, thus we can find a theta–characteristic $\mathcal{F}$ whose stalk at this singular point is M. However, the endomorphism ring of M, $A = \langle 1\rangle + t^4\tilde{R}$, is not Gorenstein. See [E, Ex. 21.11] for methods to compute the homomorphisms and dualizing modules in this case.

2 Even and odd theta–Characteristics

In this section we determine how many of the theta–characteristics of some fixed local type are even respectively odd. Harris solved this problem for the locally free theta–characteristics; we will improve his method such that other local types can be treated as well.

Fix a local type T for the theta–characteristics on the reduced curve C. We will assume throughout the whole section that Θ_T is non-empty, for example this is the case if C is irreducible (Proposition 1.3). We define

$$\Theta_T^+ = \{\mathcal{F} \in \Theta_T \mid h^0(\mathcal{F}) \text{ is even.}\}$$
$$\Theta_T^- = \{\mathcal{F} \in \Theta_T \mid h^0(\mathcal{F}) \text{ is odd.}\}$$

We will study the transitive action of J_2C on the theta–characteristic of this local type T. First, we search for an element L in J_2C such that the action of L on Θ_T interchanges the even and odd theta–characteristics, i.e., $L\Theta_T^+ = \Theta_T^-$. In particular, there are the same number of even and odd theta–characteristics in this case. However, we will not search in the whole group J_2C for such an L, but only in the subgroup Γ_2 consisting of the line bundles whose pullback to the normalization $\tilde{C}$ is trivial.

Before we start we fix some additional notation. The normalization map is again denoted by $\pi : \tilde{C} \to C$. We denote the structure sheaf of $\tilde{C}$ by $\tilde{\mathcal{O}}$ and its canonical sheaf by $\tilde{\omega}$. Further, we denote the push–down $\pi_*\tilde{\mathcal{O}}$ of $\tilde{\mathcal{O}}$ to C by $\tilde{\mathcal{O}}$ as well. There will be no confusion, because it will always be clear whether we are working on C or $\tilde{C}$, and the homology groups are canonically isomorphic (see remarks after Proposition 1.4). Now we have the following picture: From

$$0 \longrightarrow \mathcal{O}^* \longrightarrow \tilde{\mathcal{O}}^* \longrightarrow \prod_{p \in \operatorname{Sing} C} \tilde{\mathcal{O}}_p^*/\mathcal{O}_p^* \longrightarrow 0$$

we get the short exact sequence for the Jacobian

$$0 \longrightarrow \Gamma := \left(\prod_{p \in \operatorname{Sing} C} \tilde{\mathcal{O}}_p^*/\mathcal{O}_p^* \right) /(H^0(\tilde{\mathcal{O}}^*)/H^0(\mathcal{O}^*)) \longrightarrow JC \xrightarrow{\pi^*} J\tilde{C} \longrightarrow 0.$$

Because this sequence splits (see Lemma 1.2 and its proof), we obtain an exact sequence on the 2–torsion elements

$$0 \longrightarrow \Gamma_2 \longrightarrow J_2C \xrightarrow{\pi^*} J_2\tilde{C} \longrightarrow 0.$$

J_2C may not act faithfully on Θ_T, but by Lemma 1.5, the subgroup $\check{\Gamma}_2$ acting trivially on Θ_T can be computed as

$$\check{\Gamma}_2 = \{L \in J_2C \mid \check{\pi}^*L \text{ is trivial.}\} \subseteq \Gamma_2,$$

where $\check{\pi}$ is defined as before Lemma 1.5. We set $\bar{\Gamma}_2 = \Gamma_2/\check{\Gamma}_2$, then $J_2C/\check{\Gamma}_2$ and $\bar{\Gamma}_2$ act regularly on Θ_T. $\bar{\Gamma}_2 \cong (\mathbb{Z}/2\mathbb{Z})^k$ for some $k \in \mathbb{N}$. Since $J_2\tilde{C} \cong (\mathbb{Z}/2\mathbb{Z})^{2g}$, there are 2^{k+2g} theta–characteristics. We will show that the number of even or odd theta–characteristics is either 2^{k+2g-1}, $2^{k+g-1}(2^g+1)$ or $2^{k+g-1}(2^g-1)$.

We start the computation of Γ_2 at a point $p \in \operatorname{Sing} C$. The 2–torsion elements of $\Gamma_p := \tilde{\mathcal{O}}_p^*/\mathcal{O}_p^*$ are easily described [H, 2b]: If C has b_p branches in $p \in C$ then the canonical map $\{\pm 1\}^{b_p}/\{\pm(1,\dots,1)\} \to \Gamma_{p,2}$ is an isomorphism, in particular $\Gamma_{p,2} \cong (\mathbb{Z}/2\mathbb{Z})^{b_p-1}$. We have a natural map $\Gamma_{p,2} \to \Gamma_2$. For $q \in \pi^{-1}(p)$ we denote by e_q the line bundle in Γ_2 which is given by the section of $\tilde{\mathcal{O}}^*/\mathcal{O}^*$ that is -1 in a neighborhood of q and 1 in a neighborhood of the other points of $\pi^{-1}(\operatorname{Sing} C)$. Obviously, the e_q, $q \in \pi^{-1}(p)$, generate the $\mathbb{Z}/2\mathbb{Z}$–vector space $\Gamma_{p,2}$ subject to the relation $\sum_{q \in \pi^{-1}(p)} e_q = 0$, and Γ_2 is generated by the $\Gamma_{p,2}$ modulo the relations $\sum_{q \in C'} e_q = 0$ for every irreducible component C' of $\tilde{C}$.

We find the subgroup $\check{\Gamma}_2$ generated by the subgroups

$$\check{\Gamma}_{p,2} := \{e \in \Gamma_{p,2} \mid e \cdot \mathcal{F}_p = \mathcal{F}_p \text{ viewing } \mathcal{F}_p \subseteq \mathcal{K}_p\}.$$

The computation of the groups is purely local, thus it depends only on the local type of $\mathcal{F}$ and is easily executed.

Before stating the theorem we need some additional definitions. Recall that the *adjoint ideal* $\mathcal{I} \subseteq \mathcal{O}$ of the map $\pi : \tilde{C} \to C$ is the annihilator $\mathcal{A}nn_{\mathcal{O}}(\tilde{\mathcal{O}}/\mathcal{O}) = \mathcal{H}om_{\mathcal{O}}(\tilde{\mathcal{O}}, \mathcal{O})$. It is in fact an $\tilde{\mathcal{O}}$–ideal, and its pullback $\pi^*\mathcal{I}$ to $\tilde{C}$ defines the *adjoint divisor* D on $\tilde{C}$ by $\pi^*\mathcal{I} = \tilde{\mathcal{O}}(-D)$. If C is a Gorenstein curve — we assume that from now on, the δ*–invariant* is

$$\delta = \dim \tilde{\mathcal{O}}/\mathcal{O} = \dim \mathcal{O}/\mathcal{I} = \tfrac{1}{2} \dim \tilde{\mathcal{O}}/\mathcal{I},$$

in particular $\deg D = 2\delta$.

Now we extend these notions to torsion–free sheaves of rank 1 on C. We view the sheaf $\mathcal{F}$ again as being embedded into the sheaf of rational function $\mathcal{K} \supseteq \tilde{\mathcal{O}}$ on C. We define the adjoint ideal of $\mathcal{F}$ to be

$$\mathcal{I}_{\mathcal{F}} := \mathcal{A}nn_{\mathcal{O}}(\tilde{\mathcal{O}} \cdot \mathcal{F}/\mathcal{F}) = \mathcal{H}om_{\mathcal{O}}(\tilde{\mathcal{O}} \cdot \mathcal{F}, \mathcal{F}) \subseteq \mathcal{K}.$$

Clearly, $\mathcal{I}_{\mathcal{F}}$ is an $\tilde{\mathcal{O}}$–ideal, independent of the chosen embedding $\mathcal{F} \subseteq \mathcal{K}$, and $\mathcal{I}_{\mathcal{F}} \supseteq \mathcal{I}$ due to $\mathcal{I} \cdot \tilde{\mathcal{O}} \cdot \mathcal{F} \subseteq \mathcal{O} \cdot \mathcal{F} = \mathcal{F}$. We define the adjoint divisor $D_{\mathcal{F}}$ on $\tilde{C}$ to be the divisor with $\pi^*\mathcal{I}_{\mathcal{F}} = \tilde{\mathcal{O}}(-D_{\mathcal{F}})$. Finally, we set $\delta(\mathcal{F}) = \dim \tilde{\mathcal{O}} \cdot \mathcal{F}/\mathcal{F}$.

For $\mathcal{F} = \mathcal{O}$ these definitions reduce to the classical ones.

Lemma 2.1 *Let C be a reduced Gorenstein curve and $\mathcal{F}$ a theta–characteristic, then*

$$\delta(\mathcal{F}) = \dim \tilde{\mathcal{O}} \cdot \mathcal{F}/\mathcal{F} = \dim \mathcal{F}/\mathcal{I}_{\mathcal{F}} \cdot \mathcal{F} = \tfrac{1}{2} \dim \tilde{\mathcal{O}}/\mathcal{I}_{\mathcal{F}},$$

in particular $\deg D_{\mathcal{F}} = 2\delta(\mathcal{F})$.

Proof. As the statement is of local nature, we may verify the result for every point $p \in C$ separately. For notational convenience we suppress the subscript p. Since $\tilde{\mathcal{O}} \cdot \mathcal{F} \cong \tilde{\mathcal{O}}$ and the statement of the lemma does not depend on the embedding of $\mathcal{F}$ in $\mathcal{K}$, we may assume that $\tilde{\mathcal{O}} \cdot \mathcal{F} = \tilde{\mathcal{O}}$ and thus $\mathcal{I}_{\mathcal{F}} \cdot \mathcal{F} = \mathcal{I}_{\mathcal{F}}$ as well as $\mathcal{I} \cdot \mathcal{F} = \mathcal{I}$. Because C is Gorenstein at the point $p \in C$ we may use $\mathcal{O}$ as a dualizing module, thus $\dim \tilde{\mathcal{O}}/\mathcal{F} = \delta(\mathcal{F})$ implies that $\mathrm{Hom}(\mathcal{F}, \mathcal{O})/\mathrm{Hom}(\tilde{\mathcal{O}}, \mathcal{O}) = \mathrm{Hom}(\mathcal{F}, \mathcal{O})/\mathcal{I}$ has the same dimension $\delta(\mathcal{F})$. As $\mathcal{F}$ is self–dual, we get

$$\dim \tilde{\mathcal{O}} \cdot \mathrm{Hom}(\mathcal{F}, \mathcal{O})/\mathrm{Hom}(\mathcal{F}, \mathcal{O}) = \delta(\mathcal{F})$$

by the definition of $\delta(\mathcal{F})$ and further

$$\begin{aligned} \dim \tilde{\mathcal{O}} \cdot \mathrm{Hom}(\mathcal{F}, \mathcal{O})/\mathcal{I} &= \dim \tilde{\mathcal{O}} \cdot \mathrm{Hom}(\mathcal{F}, \mathcal{O})/\mathrm{Hom}(\mathcal{F}, \mathcal{O}) + \dim \mathrm{Hom}(\mathcal{F}, \mathcal{O})/\mathcal{I} \\ &= 2\delta(\mathcal{F}). \end{aligned}$$

We make the following

Claim. $\mathrm{Ann}_{\mathcal{O}} \tilde{\mathcal{O}} \cdot \mathrm{Hom}(\mathcal{F}, \mathcal{O})/\mathcal{I} = \mathcal{I}_{\mathcal{F}}$.

The lemma immediately follows from this claim: Because $\mathcal{F} \subset \mathcal{K}$ is of rank 1, $\mathrm{Hom}(\mathcal{F}, \mathcal{O}) \subset \mathcal{K}$ is of rank 1 as well and $\tilde{\mathcal{O}} \cdot \mathrm{Hom}(\mathcal{F}, \mathcal{O})$ isomorphic to $\tilde{\mathcal{O}}$; hence, the claim implies

$$\tilde{\mathcal{O}} \cdot \mathrm{Hom}(\mathcal{F}, \mathcal{O})/\mathcal{I} \cong \tilde{\mathcal{O}}/\mathcal{I}_{\mathcal{F}}.$$

We computed the dimension of the left hand side to be $2\delta(\mathcal{F})$ and thus $\dim \mathcal{F}/\mathcal{I}_{\mathcal{F}} = \dim \tilde{\mathcal{O}}/\mathcal{I}_{\mathcal{F}} - \dim \tilde{\mathcal{O}}/\mathcal{F} = \delta(\mathcal{F})$, thereby finishing the proof of the lemma.

To prove the claim, let r be an element of the annihilator ideal, i.e., $r\tilde{\mathcal{O}} \cdot \operatorname{Hom}(\mathcal{F}, \mathcal{O}) \subset \mathcal{I}$. Hence,

$$\begin{aligned} r\tilde{\mathcal{O}} &\subseteq \operatorname{Hom}(\tilde{\mathcal{O}} \cdot \operatorname{Hom}(\mathcal{F}, \mathcal{O}), \mathcal{I}) = \operatorname{Hom}(\tilde{\mathcal{O}} \cdot \operatorname{Hom}(\mathcal{F}, \mathcal{O}), \mathcal{O}) \\ &\subseteq \operatorname{Hom}(\operatorname{Hom}(\mathcal{F}, \mathcal{O}), \mathcal{O}) = \mathcal{F}; \end{aligned}$$

hereby, the first equality follows from the fact that any $\tilde{\mathcal{O}}$–submodule of $\mathcal{O}$ — in particular the image of $\tilde{\mathcal{O}} \cdot \operatorname{Hom}(\mathcal{F}, \mathcal{O})$ — is contained in $\mathcal{I}$ and the last equality follows from the obvious $\mathcal{F} \subseteq \operatorname{Hom}(\operatorname{Hom}(\mathcal{F}, \mathcal{O}), \mathcal{O})$ and $\dim \mathcal{F}/\mathcal{I} = \dim \operatorname{Hom}(\operatorname{Hom}(\mathcal{F}, \mathcal{O}), \mathcal{O})/\mathcal{I}$, which one obtains by applying the dualizing functor $\operatorname{Hom}(\_, \mathcal{O})$ twice to $\tilde{\mathcal{O}} \supseteq \mathcal{F} \supseteq \mathcal{I}$. The displayed formula says $r \in \operatorname{Hom}(\tilde{\mathcal{O}}, \mathcal{F}) = \mathcal{I}_{\mathcal{F}}$ which proves the inclusion of the annihilator ideal in the adjoint ideal.

For the reverse inclusion let $r \in \mathcal{I}_{\mathcal{F}} = \operatorname{Hom}(\tilde{\mathcal{O}}, \mathcal{F})$, i.e., $r\tilde{\mathcal{O}} \subseteq \mathcal{F}$. Recalling that we consider $\operatorname{Hom}(\mathcal{F}, \mathcal{O})$ as a subset of $\mathcal{K}$ acting by multiplication, we find

$$r\tilde{\mathcal{O}} \cdot \operatorname{Hom}(\mathcal{F}, \mathcal{O}) \subseteq \operatorname{Hom}(\mathcal{F}, \mathcal{O}) \cdot \mathcal{F} \subseteq \mathcal{O}.$$

Since $r\tilde{\mathcal{O}} \cdot \operatorname{Hom}(\mathcal{F}, \mathcal{O})$ is an $\tilde{\mathcal{O}}$–submodule of $\mathcal{O}$, we have in fact $r\tilde{\mathcal{O}} \cdot \operatorname{Hom}(\mathcal{F}, \mathcal{O}) \subseteq \mathcal{I}$ or equivalently $r \in \operatorname{Ann}_{\mathcal{O}}(\tilde{\mathcal{O}} \cdot \operatorname{Hom}(\mathcal{F}, \mathcal{O})/\mathcal{I})$. □

Obviously, the above definitions depend only on the stalks of $\mathcal{F}$ or even their completion. Therefore, for a theta–characteristic $\mathcal{F}$ of local type T the objects $\mathcal{I}_{\mathcal{F}}$, $\delta(\mathcal{F})$, $D_{\mathcal{F}}$ depend only on T and the notions of $\mathcal{I}_T$, $\delta(T) = \delta_T$, D_T are well–defined.

Theorem 2.2 *Let T be a local type for a theta–characteristic on a connected, reduced Gorenstein curve C and D_T its adjoint divisor on $\tilde{C}$. Further, let $q \in \tilde{C}$ be a point mapping to a singular point $p = \pi(q) \in C$ and $e_q \in J_2C$ be the line bundle on C constructed from it.*

If $\operatorname{mult}_q D_T$ is even then the action of e_q on the theta–characteristics Θ_T preserves the sets of even and odd theta–characteristics, otherwise it interchanges them.

In particular, if $\operatorname{mult}_q D_T$ is odd for some $q \in \pi^{-1}(\operatorname{Sing} C)$ then the number of even and odd theta–characteristics of local type T is 2^{k+2g-1}.

Proof. The proof follows Harris' proof for the locally free case [H, 2.12], but some steps become more involved. Let $\mathcal{F}$ be a theta–characteristic and $\mathcal{F}' = e_q \otimes \mathcal{F}$ its image under the action of e_q. We define $\tilde{\mathcal{F}} := \pi^*\mathcal{F}/\text{tor.} = \tilde{\mathcal{O}} \cdot \pi^*\mathcal{F}$ to be the pullback of $\mathcal{F}$ or $\mathcal{F}'$ modulo torsion and identify the sections of $\mathcal{F}$, $\mathcal{F}'$ with sections of $\pi_*\tilde{\mathcal{F}}$, i.e., $H^0(\mathcal{F}), H^0(\mathcal{F}') \subseteq H^0(\pi_*\tilde{\mathcal{F}}) = H^0(\tilde{\mathcal{F}})$. We continue our abuse of notation like writing $\tilde{\mathcal{O}}$ for $\tilde{\mathcal{O}}$ on $\tilde{C}$ itself as well as $\pi_*\tilde{\mathcal{O}}$ on C, we will write $\tilde{\mathcal{F}}$ for $\tilde{\mathcal{F}}$ on $\tilde{C}$ itself as well as $\pi_*\tilde{\mathcal{F}}$ on C. Mathematically, this is justified by the remarks after Proposition 1.4, because $\tilde{\mathcal{F}}$ is an $\tilde{\mathcal{O}}$–module.

We must show that

$$h^0(\tilde{\mathcal{F}}') - h^0(\tilde{\mathcal{F}}) \equiv \operatorname{mult}_q D_T \mod 2.$$

Since any section of $\tilde{\mathcal{F}}(-D_T)$ is a section of $\mathcal{F}$ as well as $\mathcal{F}'$, we compute modulo $\tilde{\mathcal{F}}(-D_T)$. Setting

$$\begin{aligned} V &= H^0(\tilde{\mathcal{F}}/\tilde{\mathcal{F}}(-D_T)) \\ \Lambda &= H^0(\mathcal{F}/\tilde{\mathcal{F}}(-D_T)) \subseteq V \\ \Lambda' &= H^0(\mathcal{F}'/\tilde{\mathcal{F}}(-D_T)) \subseteq V \\ \Sigma &= \operatorname{Im}(H^0(\tilde{\mathcal{F}}) \to V) \subseteq V \end{aligned}$$

we find

$$h^0(\mathcal{F}') - h^0(\mathcal{F}) = \dim(\Lambda' \cap \Sigma) - \dim(\Lambda \cap \Sigma).$$

Now, we want to define a symmetric bilinear form Q on V. We claim that on $\tilde{C}$

$$\tilde{\mathcal{F}} \cong \mathcal{H}om(\tilde{\mathcal{F}}, \tilde{\omega}(D_T)).$$

In the locally free case this is obvious, since then $\tilde{\mathcal{F}}^2 \cong (\pi^*\mathcal{F})^2 \cong \pi^*(\mathcal{F}^2) \cong \pi^*\omega \cong \tilde{\omega}(D)$. The general case of torsion–free sheaves is more involved.

From the exact sequence on C

$$0 \longrightarrow \mathcal{F} \longrightarrow \tilde{\mathcal{F}} \longrightarrow \mathcal{T} \longrightarrow 0,$$

where $\mathcal{T}$ is a torsion sheaf, we get by applying $\mathcal{H}om(\_, \omega)$

$$0 = \mathcal{H}om(\mathcal{T}, \omega) \longrightarrow \mathcal{H}om(\tilde{\mathcal{F}}, \omega) \longrightarrow \mathcal{H}om(\mathcal{F}, \omega).$$

Therefore,

$$0 \longrightarrow \tilde{\mathcal{O}} \cdot \mathcal{H}om(\tilde{\mathcal{F}}, \omega) \longrightarrow \tilde{\mathcal{O}} \cdot \mathcal{H}om(\mathcal{F}, \omega) \longrightarrow \mathcal{C} \longrightarrow 0,$$

where the cokernel $\mathcal{C}$ is supported on $\operatorname{Sing} C$, because $\tilde{\mathcal{F}}_x = \mathcal{F}_x$ for $x \in C \backslash \operatorname{Sing} C$. Thus we may compute the cokernel $\mathcal{C}$ locally at a point $x \in \operatorname{Sing} C$. As during the proof of Lemma 2.1 we suppress the index x and use the identification $\tilde{\mathcal{F}} = \tilde{\mathcal{O}}$ and $\omega = \mathcal{O}$. Because $\tilde{\mathcal{F}}$ and thus $\mathcal{H}om(\tilde{\mathcal{F}}, \omega)$ have an $\tilde{\mathcal{O}}$–module structure

$$\tilde{\mathcal{O}} \cdot \mathcal{H}om(\tilde{\mathcal{F}}, \omega) = \mathcal{H}om(\tilde{\mathcal{F}}, \omega) = \mathcal{H}om(\tilde{\mathcal{O}}, \mathcal{O}) = \mathcal{I}.$$

In the proof of Lemma 2.1 we showed that $\tilde{\mathcal{O}} \cdot \mathcal{H}om(\mathcal{F}, \mathcal{O})/\mathcal{I} \cong \tilde{\mathcal{O}}/\mathcal{I}_T$, i.e., the cokernel $\mathcal{C}$ is the structure sheaf $\mathcal{O}_{D_T}$ of D_T. Therefore, on the normalization $\tilde{C}$, where we are dealing only with line bundles on a smooth curve, this means

$$\pi^!(\tilde{\mathcal{O}} \cdot \mathcal{H}om(\mathcal{F}, \omega)) \cong \pi^!\mathcal{H}om(\pi_*\tilde{\mathcal{F}}, \omega) \otimes \tilde{\mathcal{O}}(D_T).$$

Finally, with $\mathcal{H}om(\mathcal{F}, \omega) \cong \mathcal{F}$ and the duality theorem [Ha, III, Ex. 7.2]

$$\begin{aligned} \tilde{\mathcal{F}} = \pi^!(\tilde{\mathcal{O}} \cdot \mathcal{F}) &\cong \pi^!\mathcal{H}om(\pi_*\tilde{\mathcal{F}}, \omega) \otimes \tilde{\mathcal{O}}(D_T) \cong \mathcal{H}om(\tilde{\mathcal{F}}, \tilde{\omega}) \otimes \tilde{\mathcal{O}}(D_T) \\ &\cong \mathcal{H}om(\tilde{\mathcal{F}}, \tilde{\omega}(D_T)). \end{aligned}$$

Now define the symmetric bilinear form Q on V by

$$Q(\sigma, \tau) = \sum_{p \in \pi^{-1}(\operatorname{Sing} C)} \operatorname{Res}_p(\sigma\tau).$$

The same argument as in [H] shows that Q is non–degenerate and $\Lambda, \Lambda', \Sigma$ are isotropic subspaces. In fact, they are all maximal isotropic subspaces of V; namely, by Lemma 2.1 the dimension of V is $2\delta_T$ and the dimensions of Λ and Λ' are δ_T; using $\tilde{\mathcal{F}} \cong \mathcal{H}om(\tilde{\mathcal{F}}, \tilde{\omega}(D_T))$ on $\tilde{C}$ and

$$\deg_{\tilde{C}} \tilde{\mathcal{F}} = \deg \mathcal{F} - (\delta - \delta(\mathcal{F})) = g_a(C) - 1 - \delta + \delta(\mathcal{F}) = g(\tilde{C}) - 1 + \delta_T$$

by [Coo, 3.2.4], the dimension of Σ can be computed with Riemann–Roch on $\tilde{C}$ as

$$\begin{aligned}\dim \Sigma &= h^0(\tilde{\mathcal{F}}) - h^0(\tilde{\mathcal{F}}(-D_T)) = h^0(\tilde{\mathcal{F}}) - h^0(\mathcal{H}om(\tilde{\mathcal{F}}, \tilde{\omega})) \\ &= \deg_{\tilde{C}} \tilde{\mathcal{F}} - g(\tilde{C}) + 1 = \delta_T.\end{aligned}$$

We follow Harris' proof further: by [H, 2.19]

$$h^0(\mathcal{F}') - h^0(\mathcal{F}) = \dim(\Lambda' \cap \Sigma) + \dim(\Lambda \cap \Sigma) = \dim(\Lambda' \cap \Lambda) + \delta_T.$$

$\Lambda' \cap \Lambda$ can be computed locally, it depends only on $\mathcal{F}_p$. As we are considering in the following all objects localized at the point p, we drop the supscript p for notational convenience. Denote by b the number of branches of C in p. We number the branches such that the first branch, Δ_1, is the one with $q \in \pi^{-1}(\Delta_1)$. Let Δ_0 be the sum of the remaining branches. Then a section $(\sigma_1, \ldots, \sigma_b) \in \pi_*\tilde{\mathcal{F}}$ lies in $\mathcal{F} \cap \mathcal{F}' = \Lambda' \cap \Lambda \bmod \pi_*\tilde{\mathcal{F}}(-D_T)$ if $(\sigma_1, \ldots, \sigma_b)$ and $(-\sigma_1, \sigma_2 \ldots, \sigma_b)$ are in $\mathcal{F}$ or equivalently if $(\sigma_1, 0, \ldots, 0)$ and $(0, \sigma_2 \ldots, \sigma_b)$ are in $\mathcal{F}$. Denoting by A_i the subspaces of sections of V that vanish on Δ_i we have $\Lambda \cap \Lambda' = A_0 \oplus A_1$.

We must compute the dimension of $A_0 \oplus A_1$. Let $\mathcal{J}_i$ be the kernel of the restriction map

$$\mathrm{res}_i : \mathcal{F} \longrightarrow \mathcal{F}|_{\Delta_i} = \mathcal{F} \otimes \mathcal{O}/I(\Delta_i),$$

where $I(\Delta_i)$ is the vanishing ideal of Δ_i. In particular, we have $\mathcal{F}|_{\Delta_i} = \mathcal{F}/\mathcal{J}_i$. The space A_i is $\mathcal{J}_i$ modulo $\mathcal{I}_T \cdot \mathcal{F} = \tilde{\mathcal{F}}(-D_T)$. Clearly, $\mathcal{J}_0 \cap \mathcal{J}_1 = 0$, i.e., the restriction map res_i is injective when restricted to $\mathcal{J}_{1-i}$, and we may identify $\mathcal{J}_{1-i}$ with its image in $\mathcal{F}|_{\Delta_i}$.

We define the intersection multiplicity of Δ_0 and Δ_1 with respect to $\mathcal{F}$ to be

$$m_{\mathcal{F}}(\Delta_0 \cdot \Delta_1) := \dim \mathcal{F}/\mathcal{J}_0 \oplus \mathcal{J}_1.$$

Since $\mathcal{I}_T \cdot \mathcal{F}$ is an $\tilde{\mathcal{O}}$–module, $\mathcal{I}_T \cdot \mathcal{F} \subseteq \mathcal{J}_0 \oplus \mathcal{J}_1$, thus

$$\begin{aligned}\dim \Lambda \cap \Lambda' &= \dim A_0 \oplus A_1 = \dim \mathcal{J}_0 \oplus \mathcal{J}_1/\mathcal{I}_T \cdot \mathcal{F} \\ &= \dim \mathcal{F}/\mathcal{I}_T \cdot \mathcal{F} - \dim \mathcal{F}/\mathcal{J}_0 \oplus \mathcal{J}_1 = \delta_T - m_{\mathcal{F}}(\Delta_0 \cdot \Delta_1).\end{aligned}$$

We will show the following generalization of Hironaka's lemma [JP, 5.4.11]

$$\delta(\mathcal{F}) = \delta(\mathcal{F}|_{\Delta_0}) + \delta(\mathcal{F}|_{\Delta_1}) - m_{\mathcal{F}}(\Delta_0 \cdot \Delta_1).$$

We denote by $\pi_i : \tilde{\Delta}_i \to \Delta_i$ the normalization of the curve germs Δ_i and by $\tilde{\mathcal{F}}_i$ the pullback of $\mathcal{F}|_{\Delta_i}$ via π_i modulo torsion. By definition

$$\delta(\mathcal{F}|_{\Delta_i}) = \dim(\pi_i)_*\tilde{\mathcal{F}}_i/\mathcal{F}|_{\Delta_i} = \dim(\pi_i)_*\tilde{\mathcal{F}}_i/\mathcal{J}_{1-i} - \dim \mathcal{F}|_{\Delta_i}/\mathcal{J}_{1-i}.$$

Using $\tilde{\mathcal{F}} = \pi_* \tilde{\mathcal{F}} = (\pi_0)_* \tilde{\mathcal{F}}_0 \oplus (\pi_1)_* \tilde{\mathcal{F}}_1$ and $\mathcal{F} \mid_{\Delta_i} / \mathcal{J}_{1-i} \cong (\mathcal{F}/\mathcal{J}_i)/\mathcal{J}_{1-i} \cong \mathcal{F}/(\mathcal{J}_0 \oplus \mathcal{J}_1)$, we obtain Hironaka's lemma

$$\begin{aligned}\delta(\mathcal{F}|_{\Delta_0}) + \delta(\mathcal{F}|_{\Delta_1}) &= \dim((\pi_0)_* \tilde{\mathcal{F}}_0 \oplus (\pi_1)_* \tilde{\mathcal{F}}_1)/(\mathcal{J}_1 \oplus \mathcal{J}_0) - 2 \dim \mathcal{F}/(\mathcal{J}_0 \oplus \mathcal{J}_1) \\ &= \dim \tilde{\mathcal{F}}/(\mathcal{J}_0 \oplus \mathcal{J}_1) - 2 \dim \mathcal{F}/(\mathcal{J}_0 \oplus \mathcal{J}_1) \\ &= \dim \tilde{\mathcal{F}}/\mathcal{F} - \dim \mathcal{F}/(\mathcal{J}_0 \oplus \mathcal{J}_1) = \delta_T - m_{\mathcal{F}}(\Delta_0 \cdot \Delta_1).\end{aligned}$$

With $\mathcal{F} = \mathcal{O}$ the above reduces to the standard results. The other way around, with the help of the above formulas we may substitute $\mathcal{O}$ by $\mathcal{F}$ and I by $\mathcal{J}$ in [H, 5.3–5.6] and obtain

$$\text{mult}_q(D_T) = 2\delta(\mathcal{F}|_{\Delta_1}) + m_{\mathcal{F}}(\Delta_0 \cdot \Delta_1).$$

Finally, putting everything together

$$\begin{aligned}h^0(\mathcal{F}') - h^0(\mathcal{F}) &= \delta_T + \dim \Lambda \cap \Lambda' = 2\delta_T - m_{\mathcal{F}}(\Delta_0 \cdot \Delta_1) \\ &= 2(\delta_T + \delta(\mathcal{F}|_{\Delta_1})) - \text{mult}_q(D_T) \equiv \text{mult}_q(D_T) \bmod 2. \quad \square\end{aligned}$$

For the determination of the number of even and odd theta–characteristics we may now assume that the action of Γ_2 on the theta–characteristics of local type T perserves the even– and oddness of the theta–characteristics.

Because of Theorem 2.2 we know that the divisor D_T is divisible by 2, set

$$E_T := \frac{1}{2} D_T \quad \text{and} \quad \varepsilon(\mathcal{F}) := \dim(\mathcal{F}/(\mathcal{F} \cap \mathcal{F} \cdot \tilde{\mathcal{O}}(-E_T))).$$

The multiplication takes place in $\mathcal{K}$, where the sheaves $\mathcal{F}$ and $\tilde{\mathcal{O}}(-E_T)$ are embedded. Obviously, the definition depends only on the stalks of $\mathcal{F}$ in the singular points of C, thus we will use $\varepsilon(T) = \varepsilon_T$ as well.

Theorem 2.3 *Let T a local type for a theta–characteristic on a connected, reduced Gorenstein curve, such that Θ_T is non–empty. Assume that the action of Γ_2 on the theta–characteristics of local type T perserves the even– and oddness of the theta–characteristics.*

Then the number of even theta–characteristics of local type T is

$$\begin{aligned}&2^{k+g-1}(2^g + 1) \quad \text{if } \varepsilon_T \text{ is even,} \\ &2^{k+g-1}(2^g - 1) \quad \text{otherwise}\end{aligned}$$

Proof. Let $\mathcal{F}$ be one of the theta–characteristics in question. We have shown during the proof of Theorem 2.2 that $\tilde{\mathcal{F}}^2 \cong \tilde{\omega}(D_T)$, i.e., the line bundle $\mathcal{M} = \tilde{\mathcal{F}}(-E_T)$ is a theta–characteristic on $\tilde{C}$. In this way we get a $\#\bar{\Gamma}_2 = 2^k$ to 1 map $\Theta_T(C) \to \Theta(\tilde{C})$. Due to our assumption on the action of Γ_2, all theta–characteristics in a fiber of this map are either even or odd. Since $\tilde{C}$ is smooth, $\tilde{C}$ has $2^{g-1}(2^g + 1)$ even and $2^{g-1}(2^g - 1)$ odd theta–characteristics, thus it is enough to show that

$$h^0(\mathcal{F}) = h^0(\mathcal{M}) + \varepsilon_T.$$

This can be done precisely as in [H, 2.22], one only needs to replace the notions for $\mathcal{O}$ by our more general notions for the torsion–free sheaf $\mathcal{F}$. $\square$

3 Curves with ADE–singularities

The results of the introductory section are sufficient to count the theta–characteristics on irreducible curves with only ADE–singularities.

First we study the local situation. Since all irreducible torsion–free modules over the one–dimensional ADE–singularities are known ([GK],[Coo, 2.4.2] or [Y, 9]), we can list the self–dual modules.

Theorem 3.1 *All torsion–free self–dual modules over the one–dimensional ADE–singularities together with their endomorphism ring are listed in the following table. Since the endomorphism ring of the self–dual module over a singularity is always isomorphic to the local ring of a partial resolution of the singularity, we list in the last column only the singularity type of this partial resolution. We use the symbol A_0 for one smooth branch and A_{-1} for two disjoint smooth branches.*

singularity, local ring	*self–dual modules*	*endo. ring*
$A_{2\delta-1}$ $R=\mathbb{C}[[(t,t),(t^\delta,0)]]$	$M_i=\left\langle 1,(t^{\delta-i},0)\right\rangle$ $i=0,\dots,\delta$	$A_{2\delta-1-2i}$
$A_{2\delta}$ $R=\mathbb{C}[[t^2,t^{2\delta+1}]]$	$M_i=\left\langle 1,t^{2\delta+1-2i}\right\rangle$ $i=0,\dots,\delta$	$A_{2\delta-2i}$
$D_{2\delta-2}$ $R=\mathbb{C}[[(t,t,0),$ $(t^{\delta-2},0,t)]]$	$M_0=R,\ N=\langle 1,(1,0,0)\rangle$ $N'=\langle 1,(0,1,0)\rangle$ $M_i=\left\langle 1,(0,0,1),(t^{\delta-i-1},0,0)\right\rangle$ $i=1,\dots,\delta-1$	$D_{2\delta-2},\ A_1\times A_0$ $A_1\times A_0$ $A_{2\delta-2i-3}\times A_0$
$D_{2\delta-1}$ $R=\mathbb{C}[[(t^2,0),$ $(t^{2\delta-3},t)]]$	$M_0=R,\ N=\langle 1,(t,0)\rangle$ $M_i=\left\langle 1,(1,0),(t^{2\delta-2i-1},0)\right\rangle$ $i=1,\dots,\delta-1$	$D_{2\delta-1},\ A_1$ $A_{2(\delta-i-1)}\times A_0$
E_6 $R=\mathbb{C}[[t^3,t^4]]$	$M_0=R,\ M_1=\left\langle 1,t^2\right\rangle$ $M_2=\mathbb{C}[[t]]$	$E_6,\ A_2$ A_0
E_7 $R=\mathbb{C}[[(t^2,t),(t^3,0)]]$	$M_0=R,\ M_1=\langle(1,0),(0,1)\rangle$ $M_2=\left\langle 1,(t,0),(t^2,0)\right\rangle$ $M_3=\mathbb{C}[[t]]^2$	$E_7,\ A_2\times A_0$ A_1 A_{-1}
E_8 $R=\mathbb{C}[[t^3,t^5]]$	$M_0=R,\ M_1=\left\langle 1,t^2,t^4\right\rangle$ $M_2=\mathbb{C}[[t]]$	$E_8,\ A_2$ A_0

In particular, $\mathrm{End}(M)\cong M$ *as R–modules for all the above modules.*

Because of $\mathrm{End}(M)\cong M$, M is a free $A:=\mathrm{End}(M)$–module, and Proposition 1.9 is very useful in this case. We formulate this case as a corollary.

Corollary 3.2 *Let C be a reduced curve with only ADE–singularities. Fix a local type T, and let be $\mathcal{A}$ its induced algebra sheaf and $\breve{\pi}:\breve{C}\to C$ the partial normalization constructed of it.*

Then $\breve{\pi}_*$ induces a bijection between the theta–characteristics of type T on C and the locally free theta–characteristics on $\breve{C}$, which preserves the dimension of the homology groups of the theta–characteristics.

The number and the parity type of the locally free theta–characteristics were determined by Harris [H], thereby we are able to determine the number and the parity type of the theta–characteristics of any local type for the ADE–case. We recall Harris' results and compute the missing invariants for the ADE–singularities.

Theorem 3.3 *Let C be an integral curve of geometric genus g with only ADE–singularities. Let b_p the number of branches of the curve C in a singular point $p \in \operatorname{Sing} C$. Set $k = \sum_p (b_p - 1)$.*

Then the number of locally free theta–characteristics of C is 2^{2g+k}. The number of even theta–characteristics of these, i.e., locally free theta–characteristics $\mathcal{F}$ with $h^0(\mathcal{F}) \equiv 0 \bmod 2$, is

2^{2g+k-1}	*if C has an A_{4l+1}, D_{4l+2}, or E_7 singularity*
$2^{g+k-1}(2^g+1)$	*if C has no A_{4l+1}, D_{4l+2}, or E_7 singularity and an even number of A_{8l+2}, A_{8l+3}, A_{8l+4}, D_{8l+3}, D_{8l+4}, D_{8l+5}, or E_6 singularities*
$2^{g+k-1}(2^g-1)$	*otherwise*

Proof. By [H] there are 2^{2g+k} locally free theta–characteristics and either 2^{2g+k-1}, $2^{g+k-1}(2^g+1)$, or $2^{g+k-1}(2^g-1)$ of them are even. The first case of 2^{2g+k-1} even locally free theta–characteristics can only occur if C has a multibranched singularity; more precisely, it occurs iff the adjoint divisor of the singularity on the normalization of C contains a point with odd multiplicity. For the multibranched ADE–singularities the multiplicities of the adjoint divisor are:

singularity	$A_{2\delta-1}$	$D_{2\delta-2}$	$D_{2\delta-1}$	E_7
multiplicities	δ, δ	$\delta-1, \delta-1, 2$	$2\delta-2, 2$	$5, 3$

Therefore, there are 2^{2g+k-1} even theta–characteristics iff the curve has an A_{4l+1}, D_{4l+2}, or E_7 singularity.

Now assume that the curve has none of these singularities then we have to decide whether there are $2^{g+k-1}(2^g+1)$ or $2^{g+k-1}(2^g-1)$ even locally free theta–characteristics. For this we have to compute the ε–invariants, which are for the ADE–singularities:

$A_{2\delta-1},\ \gcd(2,\delta)=2$	$A_{2\delta}$	$D_{2\delta-2},\ \gcd(2,\delta)=1$	$D_{2\delta-1}$	E_6	E_8
$\frac{\delta}{2}$	$\lceil \frac{\delta}{2} \rceil$	$\lfloor \frac{\delta}{2} \rfloor$	$\lfloor \frac{\delta}{2} \rfloor$	1	2

C has $2^{g+k-1}(2^g+1)$ even locally free theta–characteristics if and only if the sum of the ε–invariants is even. By the table this equivalent to the statement that the curve has an even number of A_{8l+3}, A_{8l+2}, A_{8l+4}, D_{8l+4}, D_{8l+3}, D_{8l+5}, or E_6 singularities. $\square$

With these results the computation of the number of all and the even theta–characteristics of any local type is easy: Take an irreducible curve C with only ADE–singularities and fix some local type T. Compute the geometric genus of C. Replace the singularities of C formally by the singularity types of the endomorphism rings of the self–dual modules in T using Theorem 3.1. Finally, use Theorem 3.3 to compute the number of all and the even locally free theta–characteristics of this new curve, which correspond to the theta–characteristics of type T on C by Corollary 3.2.

As an example we complete the example [H, 3b] of the unicursal quartics, i.e., the rational quartics with only nodes and cusps. In the following table let $i, j, k \in \{0, 1\}$ and $s := i + j + k$. The case of $i = j = k = 1$ is the original case of Harris.

$\operatorname{Sing} C$	local type	$\operatorname{Sing} \check{C}$	theta–char.	even
$3A_1$	(M_i, M_j, M_k)	$(3 - s)A_1$	2^{3-s}	$\lfloor 2^{2-s} \rfloor$
$A_1A_1A_2$	(M_i, M_j, M_k)	$(2 - i - j)A_1(1 - k)A_2$	2^{2-i-j}	$\lfloor 2^{1-i-j} \rfloor$
$A_1A_2A_2$	(M_i, M_j, M_k)	$(1 - i)A_1(2 - j - k)A_2$	2^{1-i}	$\lfloor 2^{-i} \rfloor$
$3A_2$	(M_i, M_j, M_k)	$(3 - s)A_2$	1	0

In the computation of all possible theta–characteristics of any type one has to consider all possible choices for the (M_i, M_j, M_k), for example for the quartics with three nodes we find $8 + 3 \cdot 4 + 3 \cdot 2 + 1 = 27$ theta–characteristics, of which $4 + 3 \cdot 2 + 3 \cdot 1 + 0 = 13$ are even.

The above results immediately extend to the reducible case if the existence of at least one theta–characteristic can be ensured — only k has to be computed as in Section 2. However, we have seen in Section 1 that not all local types are always possible on reducible curves.

4 Application: Quadratic representations

In the context of Hilbert's seventeenth problem Hilbert himself proved that a real homogeneous polynomial $F \in \mathbb{R}[x, y, z]$ of degree 4 which is non–negative on the reals is the sum of three squares of quadratic forms [Hi]. Recently, Rudin and Swan gave modern proofs of this theorem and Pfister, Powers and Reznick began constructive approaches to this theorem [R, Sw, P, PR]. One of the partially open questions is in how many essentially different ways F can be written as the sum of three squares. We will discuss the complex version of this question; to be precise we make the following definition:

Definition 4.1 *A quadratic representation of a plane quartic $V(F) \subset \mathbb{P}^2$ is an expression*

$$F = p^2 + q^2 + r^2$$

where p, q, r are homogeneous polynomials of degree 2.

Having found one quadratic representation we obtain another $F = (p')^2 + (q')^2 + (r')^2$ by setting $(p', q', r') = (p, q, r) \cdot A$ for any element A of the orthogonal group $O(3, \mathbb{C})$. Such representations are called equivalent.

Coble showed that on a smooth quartic the equivalence classes of quadratic representations are in one–to–one correspondence with the non–trivial 2–torsion points of the Jacobian of the quartic, in particular there are 63 representations [Cob]. Wall generalized this to irreducible singular quartics; however, he studied only basepoint–free quadratic representations of F, i.e., p, q, r have no common zero [W2]. Powers, Reznick, Scheiderer, and Sottile determined which of those are real [PRSS]. Here we want of drop the assumption of the base point freeness in the complex case.

Theorem 4.2 *Let $C = V(F) \subseteq \mathbb{P}^2$ be an irreducible plane quartic then there is a one–to–one correspondence between the set of equivalence classes of quadric representations of F and the set*

$$\Xi = \left\{ \mathcal{F} \mid \begin{array}{l} \mathcal{F} \textit{ globally generated torsion–free sheaf of rank 1 with} \\ \mathcal{F} \cong \mathcal{H}om(\mathcal{F}, \mathcal{O}_C(2)) \textit{ and } \mathcal{F} \ncong \mathcal{O}_C(1) \end{array} \right\}.$$

Proof. We will reformulate the question of the existence of a quadratic representation until we can apply a theorem of Beauville and Catanese. Let $Q(x, y, z) = x^2 + y^2 + z^2$ be the standard quadratic form on $\mathbb{C}^3$, thus we can rewrite a quadratic representation $F = p^2 + q^2 + r^2$ as $F = Q(p, q, r)$. We define another quadratic form of rank 3 by $Q'(x, y, z) = xy - z^2$. We might now ask for representations of F as $F = Q'(p, q, r)$, calling two representations $F = Q'(p, q, r)$ and $F = Q'(p', q', r')$ equivalent iff $(p', q', r') = (p, q, r) \cdot A$ for some element A of the orthogonal group with respect to the quadratic form Q'. However, since over $\mathbb{C}$ any two quadratic forms of the same rank are equivalent, we can derive one representation from the other:

$$Q(p, q, r) = Q'(p + iq, p - iq, ir) \quad \text{and} \quad Q'(p, q, r) = Q(\tfrac{p+q}{2}, \tfrac{p-q}{2}, ir).$$

Clearly, this bijection of representations preserves the notion of equivalence.

Now, we note that Q' is the determinant of the symmetric matrix $\left(\begin{smallmatrix} x & z \\ z & y \end{smallmatrix}\right)$, i.e., we are searching for representations of F as the determinant of a symmetric matrix whose entries are homogeneous polynomials of degree 2. Here, the notion of equivalence appears to be different. Two matrices $M = \left(\begin{smallmatrix} p & r \\ r & q \end{smallmatrix}\right)$ and $M' = \left(\begin{smallmatrix} p' & r' \\ r' & q' \end{smallmatrix}\right)$ are equivalent if there exists an $A \in G := \{B \in \mathrm{GL}(2, \mathbb{C}) \mid \det B = \pm 1\}$ such that $M = AM'A^t$. Nevertheless, G is a 2 to 1 covering of $O(3, \mathbb{C})$ and equivalent quadratic representations correspond to equivalent matrices. (See [FH, 7.16/17] and replace in Exercise 7.17 traceless matrices by symmetric matrices and gAg^{-1} by gAg^t.)

Let M be a matrix like above. By [B2, Theorem B] we obtain as the cokernel of M in the sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-2)^2 \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}^2 \longrightarrow \mathcal{F} \longrightarrow 0$$

a torsion–free sheaf of rank 1 on C with $\mathcal{F} \cong \mathcal{H}om(\mathcal{F}, \mathcal{O}_C(2))$. This short exact sequence is the minimal resolution of $\mathcal{F}$ as a sheaf on $\mathbb{P}^2$, and therefore unique up to isomorphism. In particular, $\mathcal{F} \ncong \mathcal{O}_C(1)$, because $\mathcal{O}_C(1)$ has the resolution $0 \to \mathcal{O}(-3) \to \mathcal{O}(1) \to \mathcal{O}_C(1) \to 0$.

Finally, assume we are given a torsion–free sheaf $\mathcal{F} \ncong \mathcal{O}_C(1)$ of rank 1 with $\mathcal{F} \cong \mathcal{H}om(\mathcal{F}, \mathcal{O}_C(2))$. Because the support of $\mathcal{F}$ is one–dimensional, $\mathcal{F}$ is arithmetically Cohen–Macaulay. Further, the bilinear form $\mathcal{F} \otimes \mathcal{F} \to \mathcal{O}_C(2)$ induced by $\mathcal{F} \cong \mathcal{H}om(\mathcal{F}, \mathcal{O}_C(2))$ is symmetric, because thinking of $\mathcal{F}$ as embedded in $\mathcal{K}$ any bilinear map $\mathcal{F} \otimes \mathcal{F} \to \mathcal{O}_C(2)$ is locally given by $(f_1, f_2) \mapsto \lambda f_1 f_2$ for some $\lambda \in \mathcal{K}$. Hence, by [B2, Theorem B] there is a symmetric minimal resolution of $\mathcal{F}$ of the form

$$0 \longrightarrow \bigoplus_{j=1}^{l} \mathcal{O}_{\mathbb{P}^2}(-d_j - 2) \xrightarrow{M} \bigoplus_{j=1}^{l} \mathcal{O}_{\mathbb{P}^2}(d_j) \longrightarrow \mathcal{F} \longrightarrow 0$$

with $M = M^t$ and $F = \det M$, thus $2l + \sum 2d_j = \deg F = 4$. Because $\mathcal{F}$ is globally generated, all d_i are non–negative. For $l = 1$ we get $d_1 = 1$ and $\mathcal{F}$ is the cokernel of $0 \to \mathcal{O}(-3) \stackrel{(F)}{\to} \mathcal{O}(1)$, i.e., $\mathcal{F} \cong \mathcal{O}_C(1)$, contradicting our assumption. Therefore, we must have $l = 2$ and $d_1 = d_2 = 0$, showing that M is a symmetric matrix whose entries are homogeneous of degree 2. $\square$

The set Ξ can now be examined with the help of the theta–characteristics: The canonical sheaf on a plane quartic is $\omega = \mathcal{O}_C(1)$. Let $\mathcal{L}$ be a line bundle with $\mathcal{L}^2 = \omega$ (Lemma 1.2). Then we have a bijection

$$\begin{array}{rccc} \Phi: & \Theta & \longrightarrow & \{\mathcal{F} \mid \mathcal{F} \text{ torsion–free sheaf of rank 1 with } \mathcal{F} \cong \mathcal{H}om(\mathcal{F}, \mathcal{O}_C(2))\} \\ & \mathcal{F} & \longmapsto & \mathcal{F} \otimes \mathcal{L}. \end{array}$$

By definition $\Phi(\mathcal{L}) = \mathcal{L}^2 = \mathcal{O}_C(1)$. Thus the set Ξ consists of the globally generated sheaves of $\Phi(\Theta \setminus \{\mathcal{L}\})$. Since irreducible plane quartics have only ADE–singularities, the results of Section 3 can be applied to compute Θ. That only the globally generated sheaves of $\Phi(\Theta \setminus \{\mathcal{L}\})$ lead to quadratic representations was pointed out to me by Claus Scheiderer. He is preparing an article in which he, among other things, describes the not globally generated sheaves of $\Phi(\Theta \setminus \{\mathcal{L}\})$ and thereby computes the cardinality of Ξ.

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