Affinely Smooth Developable Varieties of Low Gauss Rank

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Abstract

We study projective varieties whose image of Gauss map has dimension less or equal to four and which are smooth outside the hyperplane at infinity. We describe their geometric structure, and show in particular that they are uniruled by linear spaces which are larger than a priori expected.

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Let $X \subset \mathbb{P}^N$ be an irreducible projective variety of dimension n. Its Gauss map is the rational map

$$\gamma : X - - \rightarrow \mathbb{G}(n, N), \quad x \longmapsto \mathbb{T}_x X,$$

which assigns to every smooth point of X its projective tangent space as a point of the Grassmannian of n–planes in $\mathbb{P}^N$. The variety X is called developable if the dimension of the image of the Gauss map — the Gauss rank r of X — is less than n.

In this article we wish to study smooth affine varieties $X \subset \mathbb{C}^N$, which are developable. However, to describe their geometric structure it will be necessary to consider their behavior at infinity. Therefore, we view X as a projective variety in $\mathbb{P}^N$ which is smooth outside the hyperplane at infinity H_∞. We will call such a variety *affinely smooth*.

The fundamental result about developable varieties is that a general fiber of the Gauss map is a linear space of dimension $d = n - r$. X is singular along a hypersurface of a Gauss fiber F, the *focal hypersurface* of F. The closure of the union of all these focal hypersurfaces is the *focal variety* X_f of X.

The affine smoothness of X forces the focal hypersurfaces to be the intersection of the Gauss fiber and the hyperplane at infinity, in particular the focal variety lies in H_∞. For Gauss rank 1 Hartman and Nirenberg proved the following theorem which was reproven and extended in various geometric settings by several authors [HN, A, NP, O].

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Theorem. *An affinely smooth developable variety of Gauss rank 1 is a cone over a smooth curve whose vertex lies in the hyperplane at infinity.*

Akivis and Goldberg proved that an affinely smooth developable variety whose second fundamental form has a regular pencil of quadrics with distinct eigenvalues is always a cone [AG_3]. In contrast to this, Bourgain and Wu worked out an example of Gauss rank 2 which is not a cone [W]. Later, Akivis and Goldberg showed that this example is projectively equivalent to an earlier example of Sacksteder [AG_1, S]. Vitter as well as Dajczer and Gromoll proved that an affinely smooth developable variety of Gauss rank 2 is a union of $(n-1)$–planes if it is not a cone [DG, V]. This was refined in [P_1] to the following statement.

Theorem. *Let $X \subset \mathbb{P}^N$ be an affinely smooth developable variety of dimension n and Gauss rank 2 which is not a cone. Then there exists a unique curve C in the hyperplane at infinity such that X is the union of a one–dimensional family of $(n-1)$–planes that contain the $(n-2)$–th osculating planes of the curve C.*

Vitter also introduced the following concept: Let F be a general Gauss fiber and V its intersection with the hyperplane at infinity H_∞. Then the closure of the union of the linear Gauss fibers which intersect H_∞ in V is the *Gauss fiber cone with vertex V*. The $(n-1)$–planes in the above theorem are in fact the Gauss fiber cones. Wu and Zheng proved the existence of nontrivial Gauss fiber cones for $r = 3, 4$ [WZ].

Theorem. *Let X be an affinely smooth developable variety of dimension n and Gauss rank r less or equal to four. Then X has nontrivial Gauss fiber cones, i.e., they are of dimension greater than $d = n - r$.*

A priori these Gauss fiber cones are only cones with a $(d-1)$–dimensional vertex. Here we want to show that very often these Gauss fiber cones are linear spaces. Wu and Zheng gave also a criterion for this, but it applies in only a few cases [WZ, Theorem 2]. From our structure theorems we obtain in particular the following generalization of the theorem of Vitter and Dajczer–Gromoll.

Theorem. *Let $X \subset \mathbb{P}^N$ be an affinely smooth developable variety of dimension n and Gauss rank less or equal to four which is not a cone. Then X is a union of $(d+1)$–planes, where $d = n - r$.*

Unfortunately, the above mentioned method of Wu and Zheng for constructing a counter example cannot be modified to provide also a counter example to this theorem for $r \geq 5$, since it only produces quadrics which are necessarily uniruled by large linear subspaces. However, analyzing the 7–dimensional example $X \subset \mathbb{P}^8$ for Gauss rank $r = 5$, one sees that the appearing $(d+1)$–planes are not the union of Gauss fibers which indicates that they are artifacts of X being a quadric and a general affinely smooth developable variety X for $r \geq 5$ will not have them.

The main purpose of this article is to describe the structure of affinely smooth developable varieties of Gauss rank 3. We need to recall three definitions:

The *dual variety* X^* of a developable variety with Gauss fiber dimension d is *degenerate* if its dimension is less than the expected one, $N - 1 - d$.

At a general point of $x \in X$ there exists a linear subspace $\mathcal{A}$ of nilpotent matrices of the endomorphisms of the tangent space $\mathbb{T}_x X$ modulo the linear

Gauss fiber F_x through x. We call $\mathcal{A} \subset \mathrm{End}(\mathbb{T}_x X/F_x)$ the *fiber movement system at infinity*. Its invariants, $l = \max\{\mathrm{rank}\, A \mid A \in \mathcal{A}\}$, the rank of a general matrix, and $b = \dim \sum_{A\in\mathcal{A}} \mathrm{Im}\, A$, the dimension of the span of all images of $A \in \mathcal{A}$, were already used to show that the focal variety X_f has dimension $d + l - 1 \leq n - 2$ and Gauss rank b [P$_1$, Theorem 3].

A variety $X \subset \mathbb{P}^N$ of dimension n and Gauss rank r is a *twisted* $(n-1)$–*plane of type* $(k_1, \ldots, k_r) \in \mathbb{N}^r$ with $\sum k_\varrho = n - r$ if it can be constructed in the following way: There exist r curves $C_\varrho \subset \mathbb{P}^N$ and a correspondence between them, i.e., a curve $C \subseteq C_1 \times \ldots \times C_r$ which projects surjectively onto each factor, such that X is the union of the one–dimensional family of $(n-1)$–planes that are the span of the k_ϱ–th osculating spaces to the curves C_ϱ at corresponding points. Hereby, we use that the zeroth osculating space is the point itself and the first the tangent line.

Any variety that is the union of a one–dimensional family of codimension one planes is a twisted plane of some type. Furthermore, the focal variety of a twisted plane of type $(k_1, \ldots, k_r)$ is a twisted plane of type $(k_1 - 1, \ldots, k_r - 1)$ over the same curves, where the possibly appearing negative numbers and the corresponding directing curves have to be left out.

With this definition the affinely smooth developable varieties of Gauss rank 2 which are not cones are twisted $(n-1)$–planes of type $(0, n-2)$ where the last directing curve lies in H_∞.

Finally, we can state our structure theorem for Gauss rank 3.

Theorem. *Let $X \subset \mathbb{P}^N$ be an affinely smooth developable variety of Gauss rank 3 which is not a cone. With the fiber movement system $\mathcal{A}$ belonging to a general point of X, we define the following invariants of X:*

$$a = \dim \mathcal{A}$$
$$l = \max\{\mathrm{rank}\, A \mid A \in \mathcal{A}\} = \textit{rank of general matrix of } \mathcal{A}.$$

According to the values of these invariants, we have the following geometric descriptions of X:

$\underline{l = 1,\ a = 1}$*: The focal variety X_f of X is the $(n-4)$–th osculating scroll of a unique curve $C \subset H_\infty$. X is the union of the one–dimensional family of Gauss fiber cones that are $(n-1)$–dimensional cones whose vertices are the $(n-3)$–th osculating spaces to the curve C.*

If X has a degenerated dual variety, then X is a twisted $(n-1)$–plane of type $(0, 0, n-3)$ where the last directing curve lies in H_∞.

$\underline{l = 1,\ a = 2}$*: X is a twisted $(n-1)$–plane of type $(0, k_2, k_3)$ with $k_2, k_3 \geq 1$ where the last two directing curves lie in H_∞. Its Gauss fiber cones are the $(n-1)$–planes.*

$\underline{l = 2}$*: The focal variety of X has dimension $n-2$ and Gauss rank 2. Further, it has an asymptotic $(n-3)$–plane in each tangent space. The variety X itself is the union of the two–dimensional family $\mathcal{G}$ of the $(n-2)$-dimensional linear Gauss fiber cones, each of which contains an asymptotic plane of X_f.*

X can also be seen as the union of a one–dimensional family of Gauss

rank 1 varieties. To be precise, let Y be an integral manifold of the asymptotic distribution on X_f and $\mathcal{G}'$ be the one-dimensional subfamily of $\mathcal{G}$ which contains the asymptotic $(n-3)$–planes of X_f along Y. Define the variety $Z \subseteq X$ to be the union of the $(n-2)$–planes of $\mathcal{G}'$. Then Z has dimension $n-1$ and Gauss rank 1, and its Gauss fibers are the family $\mathcal{G}'$.

If $a = 1$, then $\dim X = 4$, *otherwise* $\dim X \geq 5$.

A direct computation shows that the descriptions in the above Theorem can also be read as ways how to construct a variety of the corresponding type if the occurring objects are chosen general enough. However, while the focal variety of the constructed variety will lie in H_∞, additional singularities — even outside H_∞ — may occur. Further, if in the $l = 2$ case the asymptotic submanifolds of X_f, which is supposed to become the focal variety of X, are linear, additional technical conditions must be imposed on the family $\mathcal{G}$.

An analogous structure theorem for Gauss rank 4 shows that nine cases have to be distinguished. An extended version of this article containing this theorem and its proof can be obtained from the author.

1 The Setup

The structure theorems will be proven with the help of Cartan's moving frame method, for an introduction see the books [AG$_2$, L]. We will use the notations of [P$_1$], which we will recall briefly.

Let $X \subset \mathbb{P}^N$ be an irreducible variety which is smooth outside the hyperplane at infinity $H_\infty \subset \mathbb{P}^N$. Denote by n the dimension of X and by d the dimension of a general Gauss fiber. We adapt the frame such that

$\{e_0\}$	is a general point of X,
$\{e_0, \ldots, e_d\}$	is the linear Gauss fiber F of X through $\{e_0\}$,
$\{e_0, \ldots, e_n\}$	is the tangent space $\mathbb{T}_{e_0}X$ of X in $\{e_0\}$,
$\{e_1, \ldots, e_N\}$	is the hyperplane at infinity H_∞,
$\{e_1, \ldots, e_d\}$	is the Gauss fiber cone vertex.

Here we use the curly brackets to indicate the linear span of the enclosed elements. Using the index ranges $1 \leq \delta, \varepsilon \leq d$, $d+1 \leq i, j \leq n$, and $n+1 \leq \mu, \nu \leq N$, the infinitesimal movement of the frame is given by

$$\begin{aligned}
de_0 &= \omega^0 e_0 + \omega^\delta e_\delta + \omega^i e_i \\
de_\delta &= \omega_\delta^\varepsilon e_\varepsilon + \omega_\delta^i e_i \\
de_i &= \omega_i^\delta e_\delta + \omega_i^j e_j + \omega_i^\mu e_\mu \\
de_\mu &= \omega_\mu^\delta e_\delta + \omega_\mu^i e_i + \omega_\mu^\nu e_\nu .
\end{aligned}$$

Note that the Gauss fiber cone vertex $\{e_1, \ldots, e_d\}$ is fixed if $de_\delta = 0$ modulo $\{e_1, \ldots, e_d\}$, i.e., if $\omega_\delta^i = 0$ for all δ, i. This distribution is integrable, and an integral manifold is a Gauss fiber cone.

By differentiating $\omega^\mu = \omega^\mu_\delta = 0$ and using Cartan's lemma, one finds functions $a^i_{\delta j}$, q^μ_{ij} such that

$$\omega^i_\delta = a^i_{\delta j}\omega^j \qquad \text{and} \qquad \omega^\mu_i = q^\mu_{ij}\omega^j.$$

Let $A_\delta = (a^i_{\delta j})^i_j$, $\mathcal{A} = \{A_\delta\}$, $Q^\mu = (q^\mu_{ij})_{ij}$, and $\mathcal{Q} = \{Q^\mu\}$. These invariantly defined linear subspaces, $\mathcal{A}$ and $\mathcal{Q}$, of the endomorphisms of $\mathbb{T}_{e_0}X/F$ resp. of bilinear forms on $\mathbb{T}_{e_0}X/F$ are called the *fiber movement system (at infinity)* resp. the *(nondegenerated part of) the second fundamental form* of X in $\{e_0\}$. Due to our assumption that X is smooth outside H_∞, the matrices $A \in \mathcal{A}$ are nilpotent. Furthermore, the matrices Q and QA for $A \in \mathcal{A}$, $Q \in \mathcal{Q}$ are symmetric. This holds for any developable variety and follows from the symmetry of the second fundamental form along the Gauss fiber. Such linear systems $\mathcal{A}$, $\mathcal{Q}$ were studied by Wu and Zheng [WZ, Prop. 2 and 3] and their results were refined in [P_1, Prop. 2]. We will use the following:

Let l be the rank of a general matrix of $\mathcal{A}$. Then for Gauss rank $r = 3$ there exists a basis of $\mathbb{C}^3$ such that $\mathcal{A}$ is contained in the following linear systems of matrices

$l = 1$	$l = 2$
$\begin{pmatrix} 0 & 0 & s \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & s & * \\ 0 & 0 & s \\ 0 & 0 & 0 \end{pmatrix}$

The linear system $\mathcal{A}$ always contains the matrix with $s = 1$ and the other entry set to zero. In particular, the linear system $\mathcal{A}$ has a nontrivial common kernel [WZ] . The systems $\mathcal{Q}$ which belong to the above systems $\mathcal{A}$ have also been computed and will be recalled when needed.

Before we treat the different cases separately, we will show that if $l \geq 2$, $\dim \mathcal{A} = 1$, and X is not a cone, then $\dim X = r + 1$. We adapt the frame such that $\operatorname{rank} A_1 = l$ and $A_\varepsilon = 0$ for $2 \leq \varepsilon \leq d$. Then we differentiate $\omega^i_\varepsilon = 0$ to obtain

$$0 = d\omega^i_\varepsilon = -\omega^i_1 \wedge \omega^1_\varepsilon.$$

Since there are $l \geq 2$ linear independent 1–forms ω^i_1, this implies $\omega^1_\varepsilon = 0$ and $de_\varepsilon = 0 \mod \{e_\varepsilon\}$. Therefore, $\{e_\varepsilon\}$ is a fixed linear space and X — as the union of the linear spaces $\{e_1, e_\varepsilon\}$ — is a cone over it. Thus if X is not a cone, we must have $d = 1$ and $\dim X = r + 1$.

2 The Proof for Gauss Rank 3

Now we treat the different cases — according to the invariants of the linear system $\mathcal{A}$ at a general point — separately.

Case $l = 1,\ a = 1$. We adapt the frame such that

$$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ Q^{n+1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & q^{n+1}_1 & q^{n+1}_2 \\ 1 & q^{n+1}_2 & q^{n+1}_3 \end{pmatrix},\ Q^\mu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q^\mu_1 & q^\mu_2 \\ 0 & q^\mu_2 & q^\mu_3 \end{pmatrix},$$

and $A_\varepsilon = 0$, where $2 \leq \varepsilon \leq d$, $n + 2 \leq \mu \leq N$. In particular, we have $\omega^{n-2}_\varepsilon = \omega^{n-1}_1 = \omega^n_1 = 0$ and $\omega^{n-2}_1 = \omega^n$. We differentiate these equalities to obtain some useful relations:

$$
\begin{aligned}
0 = d\omega_\varepsilon^{n-2} &= -\omega_1^{n-2} \wedge \omega_\varepsilon^1 = -\omega^n \wedge \omega_\varepsilon^1 && \Rightarrow \omega_\varepsilon^1 = f_1 \omega^n \\
0 = d\omega_1^{n-1} &= -\omega_{n-2}^{n-1} \wedge \omega_1^{n-2} = \omega^n \wedge \omega_{n-2}^{n-1} && \Rightarrow \omega_{n-2}^{n-1} = f_2 \omega^n \\
0 = d\omega_1^{n} &= -\omega_{n-2}^{n} \wedge \omega_1^{n-2} = \omega^n \wedge \omega_{n-2}^{n} && \Rightarrow \omega_{n-2}^{n} = f_3 \omega^n \\
0 = d(\omega_1^{n-2} - \omega^n) &= -\omega_1^{n-2} \wedge \omega_1^1 - \omega_{n-2}^{n-2} \wedge \omega_1^{n-2} + \omega^n \wedge \omega^0 + \omega_i^n \wedge \omega^i \\
&= \omega^{n-1} \wedge (-\omega_{n-1}^n) + \omega^n \wedge (\ldots) && \Rightarrow \omega_{n-1}^n = f_4 \omega^{n-1} + f_5 \omega^n
\end{aligned}
$$

for some suitable functions $f_1, \ldots, f_5$.

Now we can examine the focal variety X_f of X. Its dimension is $n-3$ since from

$$de_1 = \omega_1^\varepsilon e_\varepsilon + \omega_1^{n-2} e_{n-2} \mod \{e_1\}$$

and the fact that X_f contains the linear space $\{e_1, \ldots, e_d\}$, we see that the tangent space of X_f at the general point e_1 is $\{e_1, \ldots, e_{n-2}\}$. The second fundamental form of X_f is

$$
\begin{aligned}
\mathrm{I\!I}_{X_f, e_1} = d^2 e_1 &= \omega_1^{n-2} (\omega_{n-2}^{n-1} e_{n-1} + \omega_{n-2}^n e_n + \omega_{n-2}^{n+1} e_{n+1}) \\
&= (\omega_1^{n-2})^2 (f_2 e_{n-1} + f_3 e_n + e_{n+1}) \mod \{e_1, \ldots, e_{n-2}\}.
\end{aligned}
$$

Thus X_f has Gauss rank 1. Since X is not a cone, X_f is not a cone. Therefore X_f has to be a $(d-1)$–th osculating scroll of a unique curve $C \subset H_\infty$.

We turn to the one–dimensional family of Gauss fiber cones of X given by the distribution $\omega_\delta^i = 0$ for all i, δ, i.e. $\omega^n = 0$. Each of which is a priori a cone with a $(d-1)$–dimensional vertex, but we will show that it is a cone with a d–dimensional vertex. Since

$$de_0 = \omega^1 e_1 + \omega^\varepsilon e_\varepsilon + \omega^{n-2} e_{n-2} + \omega^{n-1} e_{n-1} \mod \{e_0, \omega^n\},$$

the tangent space of the Gauss fiber cone G at e_0 is $\{e_0, \ldots, e_{n-1}\}$. The second fundamental form of G — using the index range $n-2 \le k \le n-1$ — is

$$
\begin{aligned}
\mathrm{I\!I}_{G, e_0} &= \omega^k \omega_k^n e_n + \omega^k \omega_k^{n+1} e_{n+1} + \omega^{n-1} \omega_{n-1}^\mu e_\mu \\
&= (\omega^{n-1})^2 (f_4 e_n + q_1^{n+1} e_{n+1} + q_1^\mu e_\mu) \mod \{e_1, \ldots, e_{n-2}, \omega^n\}.
\end{aligned}
$$

Thus G has only Gauss rank 1, and its Gauss fibers are $\{e_0, \ldots, e_{n-2}\}$. The linear space $\{e_1, \ldots, e_{n-2}\}$, which is the tangent space to X_f at any of the smooth points of $\{e_1, \ldots, e_d\}$, is fixed on G because

$$de_1 = de_\varepsilon = de_{n-2} = 0 \mod \{e_1, \ldots, e_{n-2}, \omega^n\}.$$

Therefore, the Gauss fiber cone is the union of a one–dimensional family of $(d+1)$–planes containing the d–th osculating space to the curve C; hence, it is a cone with the d–th osculating space of C as vertex.

We treat the special case where X has a degenerate dual variety. This is equivalent to the fact that the linear system $\mathcal{Q}$ of the second fundamental form contains only matrices of rank less than 3 [L, 7.3], i.e., $q_1^{n+1} = q_1^\mu = 0$, and due to $\operatorname{Sing} \mathcal{Q} = 0$ we may assume $q_2^{n+1} = 0$ and $q_2^\mu = 1$. We claim that in this case the Gauss fiber cones are $(n-1)$–planes. This will be implied if the second fundamental form of each Gauss fiber cone vanishes. By our above computations it only remains to show that $f_4 = 0$. We get this by differentiating $\omega^{n+1}_{n-1} = 0$:

$$
\begin{aligned}
0 = d\omega^{n+1}_{n-1} &= -\omega^{n+1}_{n-2} \wedge \omega^{n-2}_{n-1} - \omega^{n+1}_{n} \wedge \omega^{n}_{n-1} - \omega^{n+1}_{\mu} \wedge \omega^{\mu}_{n-1} \\
&= -f_4\omega^{n-2} \wedge \omega^{n-1} + \omega^n \wedge (\ldots).
\end{aligned}
$$

Summarizing the above computations, we see that X is the union of the one–dimensional family of $(n-1)$–planes, the linear Gauss fiber cones, containing the d–th osculating space of the curve $C \subset H_\infty$.

Case $l = 1,\ a = 2$. Here we have $\operatorname{Im} \mathcal{A} = \ker \mathcal{A}$, and the statement follows from [P_1, Corollary 11] in view of [WZ, Theorem 2] or [P_1, Theorem 6].

Case $l = 2$. We adapt the frame such that

$$
A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ A_\varepsilon = \begin{pmatrix} 0 & 0 & t_\varepsilon \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ Q^{n+1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},
$$

$$
\text{and } Q^\mu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & q^\mu_1 \\ 0 & q^\mu_1 & q^\mu_2 \end{pmatrix},
$$

where $2 \le \varepsilon \le d$, $n+2 \le \mu \le N$ and $t_\varepsilon = 0$ if $a = 1$.

The Gauss fiber cones of X are the integral manifolds of the distribution $\omega^i_\delta = 0$ for all δ, i, i.e., of the distribution $\omega^{n-1} = \omega^n = 0$. We claim that a Gauss fiber cone G is a linear space. It is enough to show that the second fundamental form of G vanishes. On G we have

$$
\begin{aligned}
de_0 &= \omega^1 e_1 + \omega^\varepsilon e_\varepsilon + \omega^{n-2} e_{n-2} \quad \operatorname{mod} \{e_0, \omega^{n-1}, \omega^n\} \\
\mathrm{II}_{G,e_0} = d^2 e_0 &= (\omega^1\omega^{n-1}_1 + \omega^{n-2}\omega^{n-1}_{n-2}) e_{n-1} + \omega^{n-2}\omega^n_{n-2} e_n + \omega^{n-2}\omega^{n+1}_{n-2} e_{n+1} \\
&\qquad\qquad \operatorname{mod} \{e_0, \ldots, e_{n-2}, \omega^{n-1}, \omega^n\}.
\end{aligned}
$$

We know that $\omega^{n-1}_1 = \omega^{n+1}_{n-2} = \omega^n$ vanish on G. To compute ω^{n-1}_{n-2} and ω^n_{n-2}, we differentiate $\omega^n_1 = 0$, $\omega^{n-1}_1 = \omega^{n+1}_{n-2}$, and $\omega^{n-1}_1 = \omega^n$. With the index range $n-2 \le k \le n-1$ we have

$$
\begin{aligned}
0 &= d\omega^n_1 = -\omega^n_k \wedge \omega^k_1 = \omega^{n-1} \wedge \omega^n_{n-2} + \omega^n \wedge \omega^n_{n-1} \\
0 &= d(\omega^{n-1}_1 - \omega^{n+1}_{n-2}) = -\omega^{n-1}_1 \wedge \omega^1_1 - \omega^{n-1}_k \wedge \omega^k_1 + \omega^{n+1}_i \wedge \omega^i_{n-2} + \omega^{n+1}_{n+1} \wedge \omega^{n+1}_{n-2} \\
&\quad = \omega^{n-2} \wedge \omega^n_{n-2} + \omega^{n-1} \wedge (2\omega^{n-1}_{n-2}) + \omega^n \wedge (\ldots) \\
0 &= d(\omega^{n-1}_1 - \omega^n) = -\omega^{n-1}_1 \wedge \omega^1_1 - \omega^{n-1}_k \wedge \omega^k_1 + \omega^n \wedge \omega^0 + \omega^n_i \wedge \omega^i \\
&\quad = \omega^{n-2} \wedge (-\omega^n_{n-2}) + \omega^{n-1} \wedge (\omega^{n-1}_{n-2} - \omega^n_{n-1}) + \omega^n \wedge (\ldots).
\end{aligned}
$$

From the first equation we get by Cartan's Lemma

$$
\omega^n_{n-2} = f_1\omega^{n-1} + f_2\omega^n \quad \text{and} \quad \omega^n_{n-1} = f_2\omega^{n-1} + f_3\omega^n.
$$

From the second we obtain

$$
2\omega^{n-1}_{n-2} = f_1\omega^{n-2} + f_4\omega^{n-1} + f_5\omega^n.
$$

Plugging this into the third equation, we find $f_1 = 0$; hence,

$$\omega^n_{n-2} = f_2\omega^n, \quad \omega^n_{n-1} = f_2\omega^{n-1} + f_3\omega^n, \quad \omega^{n-1}_{n-2} = \frac{f_4}{2}\omega^{n-1} + \frac{f_5}{2}\omega^n.$$

All these terms vanish on the Gauss fiber cone G and therefore also the second fundamental form of G, i.e., G is a linear space.

Now we turn to the irreducible focal variety X_f of X. The point e_1 is a general point of X_f, and the tangent space $\mathbb{T}_{e_1}X_f$ is the image of

$$de_1 = \omega^\varepsilon_1 e_\varepsilon + \omega^{n-2}_1 e_{n-2} + \omega^{n-1}_1 e_{n-1} \mod \{e_1\}.$$

Since X_f is the union of the linear spaces $\{e_1, \ldots, e_d\}$, the tangent space $\mathbb{T}_{e_1}X_f$ must contain this linear space $\{e_1, \ldots, e_d\}$. Because ω^{n-2}_1 and ω^{n-1}_1 are linear independent, the tangent space $\mathbb{T}_{e_1}X_f$ is $\{e_1, \ldots, e_{n-1}\}$, and hence the dimension of X_f is $n-2$.

We can compute the second fundamental form of X_f easily as

$$\begin{aligned}
\mathrm{I\!I}_{X_f,e_1} &= \omega^{n-2}_1(\omega^n_{n-2}e_n + \omega^{n+1}_{n-2}e_{n+1}) + \omega^{n-1}_1(\omega^n_{n-1}e_n + \omega^{n+1}_{n-1}e_{n+1} + \omega^\mu_{n-1}e_\mu) \\
&= (2f_2\omega^{n-2}_1\omega^{n-1}_1 + f_3(\omega^{n-1}_1)^2)e_n + 2\omega^{n-2}_1\omega^{n-1}_1 e_{n+1} + q^\mu_1(\omega^{n-1}_1)^2 e_\mu \\
&\qquad \mod \{e_1, \ldots, e_{n-1}\}.
\end{aligned}$$

Thus X_f is of Gauss rank 2 and has $\{e_1, \ldots, e_d\}$ as Gauss fiber. Further, it has the asymptotic space $\{e_1, \ldots, e_{n-2}\} = \{\omega^{n-1}_1\}^\perp$. This asymptotic space is the intersection of the linear Gauss fiber cone $G = \{e_0, \ldots, e_{n-2}\}$ with the hyperplane at infinity. We can consider this asymptotic distribution $\omega^{n-1}_1 = 0 \Leftrightarrow \omega^n = 0$ on X_f as well as on X. By the Theorem of Frobenius it is completely integrable on both varieties since

$$\begin{aligned}
d\omega^{n-1}_1 &= -\omega^{n-1}_1 \wedge \omega^1_1 - \omega^{n-1}_{n-2} \wedge \omega^{n-2}_1 - \omega^{n-1}_{n-1} \wedge \omega^{n-1}_1 \\
&= -(\tfrac{f_4}{2}\omega^{n-2}_1 + \tfrac{f_5}{2}\omega^{n-1}_1) \wedge \omega^{n-2}_1 = 0 \mod \{\omega^{n-1}_1\}.
\end{aligned}$$

Now let Y and $Z \supset Y$ be integral manifolds of this distribution on X_f resp. X. Then Z is the union of the Gauss fiber cones G that contain the tangent spaces of Y or equivalently the asymptotic planes of X_f along Y. It remains to show that Z has Gauss rank 1 and has the Gauss fiber cones G as Gauss fibers. We compute the second fundamental form of Z. On Z we have

$$\begin{aligned}
de_0 &= \omega^1 e_1 + \omega^\varepsilon e_\varepsilon + \omega^{n-2}e_{n-2} + \omega^{n-1}e_{n-1} \mod \{e_0, \omega^n\} \\
\mathrm{I\!I}_{Z,e_0} &= d^2e_0 = (\omega^{n-2}\omega^n_{n-2} + \omega^{n-1}\omega^n_{n-1})e_n + (\omega^{n-2}\omega^{n+1}_{n-2} + \omega^{n-1}\omega^{n+1}_{n-1})e_{n+1} \\
&\quad + \omega^{n-1}\omega^\mu_{n-1}e_\mu = (\omega^{n-1})^2(f_2e_n + e_{n+1}) \mod \{e_0, \ldots, e_{n-1}, \omega^n\}.
\end{aligned}$$

Clearly, the singular locus of $\mathrm{I\!I}_{Z,e_0}$ is the linear space $\{e_0, \ldots, e_{n-2}\}$, the Gauss fiber cone of X.

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