Affinely Smooth Developable Varieties of Low Gauss Rank
Extended Version

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Abstract
We study projective varieties whose image of Gauss map has dimension less or equal to four and which are smooth outside the hyperplane at infinity. We describe their geometric structure, and show in particular that they are uniruled by linear spaces which are larger than a priori expected.


Key words. developable varieties, tangentially degenerate varieties, degenerate Gauss mapping.

Let $X \subset \mathbb{P}^N$ be an irreducible projective variety of dimension $n$. Its Gauss map is the rational map

$$\gamma : X \dashrightarrow \mathbb{G}(n, N), \quad x \mapsto T_x X,$$

which assigns to every smooth point of $X$ its projective tangent space as a point of the Grassmannian of $n$–planes in $\mathbb{P}^N$. The variety $X$ is called developable if the dimension of the image of the Gauss map — the Gauss rank $r$ of $X$ — is less than $n$.

In this article we wish to study smooth affine varieties $X \subset \mathbb{C}^N$, which are developable. However, to describe their geometric structure it will be necessary to consider their behavior at infinity. Therefore, we view $X$ as a projective variety in $\mathbb{P}^N$ which is smooth outside the hyperplane at infinity $H_\infty$. We will call such a variety affinely smooth.

The fundamental result about developable varieties is that a general fiber of the Gauss map is a linear space of dimension $d = n - r$. $X$ is singular along a hypersurface of a Gauss fiber $F$, the focal hypersurface of $F$. The closure of the union of all these focal hypersurfaces is the focal variety $X_f$ of $X$.

The affine smoothness of $X$ forces the focal hypersurfaces to be the intersection of the Gauss fiber and the hyperplane at infinity, in particular the focal

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variety lies in $H_\infty$. For Gauss rank 1 Hartman and Nirenberg proved the following theorem which was reproven and extended in various geometric settings by several authors [HN, A, NP, O].

**Theorem.** An affinely smooth developable variety of Gauss rank 1 is a cone over a smooth curve whose vertex lies in the hyperplane at infinity.

Akivis and Goldberg proved that an affinely smooth developable variety whose second fundamental form has a regular pencil of quadrics with distinct eigenvalues is always a cone [AG3]. In contrast to this, Bourgain and Wu worked out an example of Gauss rank 2 which is not a cone [W]. Later, Akivis and Goldberg showed that this example is projectively equivalent to an earlier example of Sacksteder [AG1, S]. Vitter as well as Dajczer and Gromoll proved that an affinely smooth developable variety of Gauss rank 2 is a union of $(n-1)$–planes if it is not a cone [DG, V]. This was refined in [P1] to the following statement.

**Theorem.** Let $X \subset \mathbb{P}^N$ be an affinely smooth developable variety of dimension $n$ and Gauss rank 2 which is not a cone. Then there exists a unique curve $C$ in the hyperplane at infinity such that $X$ is the union of a one–dimensional family of $(n-1)$–planes that contain the $(n-2)$–th osculating planes of the curve $C$.

Vitter also introduced the following concept: Let $F$ be a general Gauss fiber and $V$ its intersection with the hyperplane at infinity $H_\infty$. Then the closure of the union of the linear Gauss fibers which intersect $H_\infty$ in $V$ is the Gauss fiber cone with vertex $V$. The $(n-1)$–planes in the above theorem are in fact the Gauss fiber cones. Wu and Zheng proved the existence of nontrivial Gauss fiber cones for $r = 3, 4$ [WZ].

**Theorem.** Let $X$ be an affinely smooth developable variety of dimension $n$ and Gauss rank $r$ less or equal to four. Then $X$ has nontrivial Gauss fiber cones, i.e., they are of dimension greater than $d = n – r$.

A priori these Gauss fiber cones are only cones with a $(d-1)$–dimensional vertex. Here we want to show that very often these Gauss fiber cones are linear spaces. Wu and Zheng gave also a criterion for this, but it applies in only a few cases [WZ, Theorem 2]. From our structure theorems we obtain in particular the following generalization of the theorem of Vitter and Dajczer–Gromoll.

**Theorem.** Let $X \subset \mathbb{P}^N$ be an affinely smooth developable variety of dimension $n$ and Gauss rank less or equal to four which is not a cone. Then $X$ is a union of $(d + 1)$–planes, where $d = n – r$.

Unfortunately, the above mentioned method of Wu and Zheng for constructing a counter example cannot be modified to provide also a counter example to this theorem for $r \geq 5$, since it only produces quadrics which are necessarily uniruled by large linear subspaces. However, analyzing the 7–dimensional example $X \subset \mathbb{P}^8$ for Gauss rank $r = 5$, one sees that the appearing $(d + 1)$–planes are not the union of Gauss fibers which indicates that they are artifacts of $X$ being a quadric and a general affinely smooth developable variety $X$ for $r \geq 5$ will not have them.

The main purpose of this article is to describe the structure of affinely smooth developable varieties of Gauss rank 3 and 4. We need to recall three definitions:
The dual variety $X^*$ of a developable variety with Gauss fiber dimension $d$ is degenerate if its dimension is less than the expected one, $N - 1 - d$.

At a general point of $x \in X$ there exists a linear subspace $A$ of nilpotent matrices of the endomorphisms of the tangent space $T_x X$ modulo the linear Gauss fiber $F_x$ through $x$. We call $A \subset \text{End}(T_x X/F_x)$ the fiber movement system at infinity. Its invariants, $l = \max\{\text{rank } A \mid A \in A\}$, the rank of a general matrix, and $b = \dim \sum_{A \in A} \text{Im } A$, the dimension of the span of all images of $A \in A$, were already used to show that the focal variety $X_f$ has dimension $d + l - 1 \leq n - 2$ and Gauss rank $b$ [P1, Theorem 3].

A variety $X \subset \mathbb{P}^N$ of dimension $n$ and Gauss rank $r$ is a twisted $(n-1)$–plane of type $(k_1, \ldots, k_r) \in \mathbb{N}^r$ with $\sum k_d = n - r$ if it can be constructed in the following way: There exist $r$ curves $C_\varphi \subset \mathbb{P}^N$ and a correspondence between them, i.e., a curve $C \subseteq C_1 \times \ldots \times C_r$ which projects surjectively onto each factor, such that $X$ is the union of the one–dimensional family of $(n-1)$–planes that are the span of the $k_\varphi$–th osculating spaces to the curves $C_\varphi$ at corresponding points. Hereby, we use that the zeroth osculating space is the point itself and the first the tangent line.

Any variety that is the union of a one–dimensional family of codimension one planes is a twisted plane of some type. Furthermore, the focal variety of a twisted plane of type $(k_1, \ldots, k_r)$ is a twisted plane of type $(k_1 - 1, \ldots, k_r - 1)$ over the same curves, where the possibly appearing negative numbers and the corresponding directing curves have to be left out.

With this definition the affinely smooth developable varieties of Gauss rank 2 which are not cones are twisted $(n-1)$–planes of type $(0, n-2)$ where the last curve lies in $H_\infty$.

Finally, we can state our structure theorem for Gauss rank 3. An analogous one for Gauss rank 4 can be found in Section 3.

**Theorem.** Let $X \subset \mathbb{P}^N$ be an affinely smooth developable variety of Gauss rank 3 which is not a cone. With the fiber movement system $A$ belonging to a general point of $X$, we define the following invariants of $X$:

\[
a = \dim A \\
l = \max\{\text{rank } A \mid A \in A\} = \text{rank of general matrix of } A.
\]

According to the values of these invariants, we have the following geometric descriptions of $X$:

- **$l = 1, a = 1$**: The focal variety $X_f$ of $X$ is the $(n-4)$–th osculating scroll of a unique curve $C \subset H_\infty$. $X$ is the union of the one–dimensional family of Gauss fiber cones that are $(n-1)$–dimensional cones whose vertices are the $(n-3)$–th osculating spaces to the curve $C$.

  If $X$ has a degenerated dual variety, then $X$ is a twisted $(n-1)$–plane of type $(0, 0, n-3)$ where the last directing curve lies in $H_\infty$.

- **$l = 1, a = 2$**: $X$ is a twisted $(n-1)$–plane of type $(k_2, k_3)$ with $k_2, k_3 \geq 1$ where the last two directing curves lie in $H_\infty$. Its Gauss fiber cones are the $(n-1)$–planes.
The focal variety of $X$ has dimension $n-2$ and Gauss rank 2. Further, it has an asymptotic $(n-3)$–plane in each tangent space. The variety $X$ itself is the union of the two–dimensional family $\mathcal{G}$ of the $(n-2)$-dimensional linear Gauss fiber cones, each of which contains an asymptotic plane of $X_f$.

$X$ can also be seen as the union of a one-dimensional family of Gauss rank 1 varieties. To be precise, let $Y$ be an integral manifold of the asymptotic distribution on $X_f$ and $\mathcal{G}'$ be the one-dimensional subfamily of $\mathcal{G}$ which contains the asymptotic $(n-3)$–planes of $X_f$ along $Y$. Define the variety $Z \subseteq X$ to be the union of the $(n-2)$–planes of $\mathcal{G}'$. Then $Z$ has dimension $n-1$ and Gauss rank 1, and its Gauss fibers are the family $\mathcal{G}'$.

If $a=1$, then $\dim X = 4$, otherwise $\dim X \geq 5$.

A direct computation shows that the descriptions in the above Theorem can also be read as ways how to construct a variety of the corresponding type if the occurring objects are chosen general enough. However, while the focal variety of the constructed variety will lie in $H_\infty$, additional singularities — even outside $H_\infty$ — may occur. Further, if in the $l=2$ case the asymptotic submanifolds of $X_f$, which is supposed to become the focal variety of $X$, are linear, additional technical conditions must be imposed on the family $\mathcal{G}$.

### 1 The Setup

The structure theorems will be proven with the help of Cartan’s moving frame method, for an introduction see the books [AG$_2$, L]. We will use the notations of [P$_1$], which we will recall briefly.

Let $X \subset \mathbb{P}^N$ be an irreducible variety which is smooth outside the hyperplane at infinity $H_\infty \subset \mathbb{P}^N$. Denote by $n$ the dimension of $X$ and by $d$ the dimension of a general Gauss fiber. We adapt the frame such that

\[
\{e_0\} \quad \text{is a general point of } X,
\]

\[
\{e_0, \ldots, e_d\} \quad \text{is the linear Gauss fiber } F \text{ of } X \text{ through } \{e_0\},
\]

\[
\{e_0, \ldots, e_n\} \quad \text{is the tangent space } T_{e_0}X \text{ of } X \text{ in } \{e_0\},
\]

\[
\{e_1, \ldots, e_N\} \quad \text{is the hyperplane at infinity } H_\infty,
\]

\[
\{e_1, \ldots, e_d\} \quad \text{is the Gauss fiber cone vertex.}
\]

Here we use the curly brackets to indicate the linear span of the enclosed elements. Using the index ranges $1 \leq \delta, \varepsilon \leq d$, $d+1 \leq i, j \leq n$, and $n+1 \leq \mu, \nu \leq N$, the infinitesimal movement of the frame is given by

\[
de_0 = \omega^0 e_0 + \omega^\delta e_\delta + \omega^i e_i
\]

\[
de_\delta = \omega_\delta^\varepsilon e_\varepsilon + \omega_\delta^i e_i
\]

\[
de_i = \omega_i^\delta e_\delta + \omega_i^j e_j + \omega_i^\mu e_\mu
\]

\[
de_\mu = \omega_\mu^\delta e_\delta + \omega_\mu^i e_i + \omega_\mu^\nu e_\nu.
\]

Note that the Gauss fiber cone vertex $\{e_1, \ldots, e_d\}$ is fixed if $de_\delta = 0$ modulo $\{e_1, \ldots, e_d\}$, i.e., if $\omega^\delta_i = 0$ for all $\delta, i$. This distribution is integrable, and an integral manifold is a Gauss fiber cone.
By differentiating $\omega^\mu = \omega^\mu_\delta = 0$ and using Cartan’s lemma, one finds functions $a^i_{\delta j}$, $q^\mu_{ij}$ such that
\[ \omega^\mu_i = a^i_{\delta j} \omega^\mu_j \quad \text{and} \quad \omega^\mu_i = q^\mu_{ij} \omega^\mu_j. \]

Let $A_\delta = (a^i_{\delta j})^i_j$, $A = \{A_\delta\}$, $Q^\mu = (q^\mu_{ij})^i_j$, and $Q = \{Q^\mu\}$. These invariantly defined linear subspaces, $A$ and $Q$, of the endomorphisms of $T_eX/F$ resp. of bilinear forms on $T_eX/F$ are called the fiber movement system (at infinity) resp. the (nondegenerated part of) the second fundamental form of $X$ in $\{e_0\}$. Due to our assumption that $X$ is smooth outside $H_\infty$, the matrices $A \in A$ are nilpotent. Furthermore, the matrices $Q$ and $QA$ for $A \in A$, $Q \in Q$ are symmetric. This holds for any developable variety and follows from the symmetry of the second fundamental form along the Gauss fiber. Such linear systems $A, Q$ were studied by Wu and Zheng [WZ, Proposition 2 and 3]. Their results were refined to the following classification in [P1, Proposition 2].

**Proposition.** Let $A$ be a nontrivial linear system of endomorphisms of $\mathbb{C}^r$ and $Q$ a linear system of symmetric bilinear forms of $\mathbb{C}^r$ with

1. every $A \in A$ is nilpotent,
2. the bilinear form $Q(\cdot, A(\cdot))$ is symmetric for every $A \in A$ and $Q \in Q$,
3. $\text{Sing } Q = \{v \in \mathbb{C}^r | Q(v, \mathbb{C}^r) = 0 \ \forall Q \in Q\} = 0$.

Let $l$ be the rank of a general matrix of $A$. Then there exists a basis of $\mathbb{C}^r$ such that $A$ is contained in the following linear systems of matrices

<table>
<thead>
<tr>
<th>$r \setminus l$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
</table>
| 3               | \[
\begin{pmatrix}
0 & 0 & s \\
0 & 0 & * \\
0 & 0 & 0
\end{pmatrix}
\] | \[
\begin{pmatrix}
0 & s & * \\
0 & 0 & s \\
0 & 0 & 0
\end{pmatrix}
\] | \[
\begin{pmatrix}
0 & s & * \\
0 & 0 & s \\
0 & 0 & 0
\end{pmatrix}
\] |
| 4               | \[
\begin{pmatrix}
0 & 0 & 0 & s \\
0 & 0 & 0 & * \\
0 & 0 & 0 & * \\
0 & 0 & 0 & 0
\end{pmatrix}
\] | \[
\begin{pmatrix}
0 & 0 & s & * \\
0 & 0 & 0 & s \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\] | \[
\begin{pmatrix}
0 & s & t & * \\
0 & 0 & s & t \\
0 & 0 & 0 & s \\
0 & 0 & 0 & 0
\end{pmatrix}
\] |

$^i$ If the system $Q$ contains a matrix of full rank, then $t = u$, otherwise $t = 0$.

The linear system $A$ always contains the matrix with $s = 1$ and all other entries set to zero.

In particular, the linear system $A$ has a nontrivial common kernel.

The systems $Q$ which belong to the above systems $A$ have also been computed in [WZ] or [P1] and will be recalled when needed.

Before we treat the different cases separately, we will show that if $l \geq 2$, $\dim A = 1$, and $X$ is not a cone, then $\dim X = r + 1$. We adapt the frame such
that rank $A_1 = l$ and $A_\varepsilon = 0$ for $2 \leq \varepsilon \leq d$. Then we differentiate $\omega_\varepsilon^{l} = 0$ to obtain

$$0 = d\omega_\varepsilon^{l} = -\omega_1^{l} \wedge \omega_\varepsilon^{l}.$$ 

Since there are $l \geq 2$ linear independent 1-forms $\omega_1^{l}$, this implies $\omega_1^{l} = 0$ and $d\varepsilon = 0 \mod \{e_\varepsilon\}$. Therefore, $\{e_\varepsilon\}$ is a fixed linear space and $X$ — as the union of the linear spaces $\{e_1, e_\varepsilon\}$ — is a cone over it. Thus if $X$ is not a cone, we must have $d = 1$ and dim $X = r + 1$.

### 2 The Proof for Gauss Rank 3

Now we treat the different cases — according to the invariants of the linear system $A$ at a general point — separately.

**Case l = 1, a = 1.** We adapt the frame such that

$$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q'^{n+1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & q_1^{n+1} & q_2^{n+1} \\ 1 & q_2^{n+1} & q_3^{n+1} \end{pmatrix}, \quad Q'^{l} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q_1^{l} & q_2^{l} \\ 0 & q_2^{l} & q_3^{l} \end{pmatrix},$$

and $A_\varepsilon = 0$, where $2 \leq \varepsilon \leq d, n + 2 \leq \mu \leq N$. In particular, we have $\omega_\varepsilon^{n-2} = \omega_1^{n-1} = \omega_1^{n} = 0$ and $\omega_1^{n-2} = \omega_1^{n}$. We differentiate these equalities to obtain some useful relations:

$$0 = d\omega_\varepsilon^{n-2} = -\omega_1^{n-2} \wedge \omega_\varepsilon^{1} = -\omega_1^{n} \wedge \omega_\varepsilon^{1} \quad \Rightarrow \omega_1^{1} = f_1 \omega_1^{n}$$

$$0 = d\omega_1^{n-1} = -\omega_1^{n-1} \wedge \omega_1^{1} = -\omega_1^{n} \wedge \omega_1^{n-2} \quad \Rightarrow \omega_1^{n-2} = f_2 \omega_1^{n}$$

$$0 = d\omega_1^{n} = -\omega_1^{n} \wedge \omega_1^{1} = -\omega_1^{n} \wedge \omega_1^{n} \quad \Rightarrow \omega_1^{n} = f_3 \omega_1^{n}$$

$$0 = d(\omega_\varepsilon^{n-2} - \omega_1^{n}) = -\omega_1^{n-2} \wedge \omega_1^{1} - \omega_1^{n-2} \wedge \omega_1^{n-2} \wedge \omega_1^{n} \wedge \omega_1^{n} \wedge \omega_1^{n} \wedge \omega_1^{n} \wedge \omega_1^{n} \wedge \omega_1^{n} \wedge \omega_1^{n} \wedge \omega_1^{n}$$

$$\Rightarrow \omega_1^{n} = f_4 \omega_1^{n-1} + f_5 \omega_1^{n}$$

for some suitable functions $f_1, \ldots, f_5$.

Now we can examine the focal variety $X_f$ of $X$. Its dimension is $n - 3$ since from

$$de_1 = \omega_\varepsilon^{e_\varepsilon} + \omega_1^{n-2} e_{n-2} \quad \text{mod} \{e_1\}$$

and the fact that $X_f$ contains the linear space $\{e_1, \ldots, e_d\}$, we see that the tangent space of $X_f$ at the general point $e_1$ is $\{e_1, \ldots, e_{n-2}\}$. The second fundamental form of $X_f$ is

$$\Pi_{X_f, e_1} = d^2 e_1 = \omega_1^{n-2}(\omega_1^{n-1} e_{n-1} + \omega_1^{n-2} e_n + \omega_1^{n+1} e_{n+1})$$

$$= (\omega_1^{n-2})^2(f_2 e_{n-1} + f_3 e_n + e_{n+1}) \quad \text{mod} \{e_1, \ldots, e_{n-2}\}.$$

Thus $X_f$ has Gauss rank 1. Since $X$ is not a cone, $X_f$ is not a cone. Therefore $X_f$ has to be a $(d - 1)$-th osculating scroll of a unique curve $C \subset H_\infty$.

We turn to the one-dimensional family of Gauss fiber cones of $X$ given by the distribution $\omega_1^{i} = 0$ for all $i, \delta$, i.e. $\omega_1^{i} = 0$. Each of which is a priori a cone with a $(d - 1)$-dimensional vertex, but we will show that it is a cone with a $d$-dimensional vertex. Since

$$de_0 = \omega_\varepsilon^{e_\varepsilon} + \omega_1^{n-2} e_{n-2} + \omega_1^{n-1} e_{n-1} \quad \text{mod} \{e_0, \omega_1^{n}\},$$


the tangent space of the Gauss fiber cone \( G \) at \( e_0 \) is \( \{e_0, \ldots, e_{n-1}\} \). The second fundamental form of \( G \) — using the index range \( n - 2 < k \leq n - 1 \) — is
\[
\mathbb{II}_{G,e_0} = \omega^k \omega_k^0 e_n + \omega^k \omega_k^{n+1} e_{n+1} + \omega^{n-1} \omega_{n-1}^\mu e_\mu
\]
\[
= (\omega^{n-1})^2 (f_4 e_1 + q_1^{n+1} e_{n+1} + q_1^\mu e_\mu) \pmod{\{e_1, \ldots, e_{n-2}, \omega^n\}}.
\]
Thus \( G \) has only Gauss rank 1, and its Gauss fibers are \( \{e_0, \ldots, e_{n-2}\} \). The linear space \( \{e_1, \ldots, e_{n-2}\} \), which is the tangent space to \( X_f \) at any of the smooth points of \( \{e_1, \ldots, e_d\} \), is fixed on \( G \) because
\[
de_1 = de_\varepsilon = de_{n-2} = 0 \pmod{\{e_1, \ldots, e_{n-2}, \omega^n\}}.
\]
Therefore, the Gauss fiber cone is the union of a one-dimensional family of \((d + 1)\)-planes containing the \( d \)-th osculating space to the curve \( C \); hence, it is a cone with the \( d \)-th osculating space of \( C \) as vertex.

We treat the special case where \( X \) has a degenerate dual variety. This is equivalent to the fact that the linear system \( \mathcal{Q} \) of the second fundamental form contains only matrices of rank less than 3 [L, 7.3], i.e., \( q_1^{n+1} = q_1^\mu = 0 \), and due to \( \text{Sing} \mathcal{Q} = 0 \) we may assume \( q_2^{n+1} = 0 \) and \( q_2^\mu = 1 \). We claim that in this case the Gauss fiber cones are \((n - 1)\)-planes. This will be implied if the second fundamental form of each Gauss fiber cone vanishes. By our above computations it only remains to show that \( f_4 = 0 \). We get this by differentiating \( \omega_1^{n+1} = 0 \):
\[
0 = d\omega_1^{n+1} = -\omega_1^{n+1} \wedge \omega_1^{n-2} = -\omega_1^{n+1} \wedge \omega_1^{n-1} - \omega_1^{n+1} \wedge \omega_1^\mu
\]
\[
= -f_4 \omega_1^{n-2} \wedge \omega_1^{n-1} + \omega_1^{n} \wedge (\ldots).
\]
Summarizing the above computations, we see that \( X \) is the union of the one-dimensional family of \((n - 1)\)-planes, the linear Gauss fiber cones, containing the \( d \)-th osculating space of the curve \( C \subset H_{\infty} \).

Case \( \ell = 1, a = 2 \). Here we have \( \text{Im} A = \ker A \), and the statement follows from [P1, Corollary 11] in view of [WZ, Theorem 2] or [P1, Theorem 6].

Case \( \ell = 2 \). We adapt the frame such that
\[
A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_\varepsilon = \begin{pmatrix} 0 & 0 & t_\varepsilon \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q^{n+1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},
\]
and \( Q^\mu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & q_1^\mu \\ 0 & q_1^\mu & q_2^\mu \end{pmatrix} \),
where \( 2 \leq \varepsilon \leq d, n + 2 \leq \mu \leq N \) and \( t_\varepsilon = 0 \) if \( a = 1 \).

The Gauss fiber cones of \( X \) are the integral manifolds of the distribution \( \omega_1^\delta = 0 \) for all \( \delta, i \), i.e., of the distribution \( \omega_1^{n-1} = \omega_1^n = 0 \). We claim that the Gauss fiber cone \( G \) is a linear space. It is enough to show that the second fundamental form of \( G \) vanishes. On \( G \) we have
\[
de e_0 = \omega^1 e_1 + \omega^\varepsilon e_\varepsilon + \omega^{n-2} e_{n-2} \pmod{\{e_0, \omega^{n-1}, \omega^n\}}
\]
\[
\mathbb{II}_{G,e_0} = d^2 e_0 = (\omega^1 \omega_1^{n-1} + \omega^{n-2} \omega_1^{n-1}) e_{n-1} + \omega^{n-2} \omega_1^{n-2} e_n + \omega^{n-2} \omega_1^{n+1} e_{n+1} \pmod{\{e_0, \ldots, e_{n-2}, \omega^{n-1}, \omega^n\}}.
\]
We know that \( \omega_{n-1}^1 = \omega_{n-1}^{n+1} = \omega^n \) vanish on \( G \). To compute \( \omega_{n-2}^1 \) and \( \omega_{n-2}^n \), we differentiate \( \omega^n = 0 \), \( \omega_{n-1}^1 = \omega_{n-1}^{n+1} \), and \( \omega_{1}^{n-1} = \omega^n \). With the index range \( n-2 \leq k \leq n-1 \) we have

\[
\begin{align*}
0 &= d\omega_1^n = -\omega_1^n \wedge \omega_1^k = \omega^{n-1} \wedge \omega_{n-2}^n + \omega^n \wedge \omega_{n-1}^1 \\
0 &= d(\omega_{n-1}^1 - \omega_{n-1}^{n+1}) = -\omega_{n-1}^1 \wedge \omega_1^k - \omega_{k+1}^{n+1} \wedge \omega_{n-2}^n + \omega_{n+1}^1 \wedge \omega_{n-2}^n + \omega^n \wedge \omega_{n-1}^1 \\
&= \omega^{n-2} \wedge \omega_{n-2}^n + \omega^{n-1} \wedge (2\omega_{n-2}^n) + \omega^n \wedge (\ldots) \\
0 &= d(\omega_{n-1}^1 - \omega^n) = -\omega_{n-1}^1 \wedge \omega_1^k - \omega_{k+1}^1 \wedge \omega^n + \omega^n \wedge \omega_{n-1}^1 \\
&= \omega^{n-2} \wedge \omega_{n-2}^n + \omega^{n-1} \wedge (\omega_{n-2}^n - \omega_{n-1}^1) + \omega^n \wedge (\ldots).
\end{align*}
\]

From the first equation we get by Cartan’s Lemma

\[
\omega_{n-2}^n = f_1\omega^{n-1} + f_2\omega^n \quad \text{and} \quad \omega_{n-1}^n = f_3\omega^{n-1} + f_4\omega^n.
\]

From the second we obtain

\[
2\omega_{n-2}^n = f_1\omega^{n-2} + f_4\omega^{n-1} + f_5\omega^n.
\]

Plugging this into the third equation, we find \( f_1 = 0 \); hence,

\[
\begin{align*}
\omega_{n-2}^n &= f_2\omega^n, \\
\omega_{n-1}^n &= f_3\omega^{n-1} + f_4\omega^n, \\
\omega_{n-2}^{n-1} &= f_5\omega^{n-1} + f_6\omega^n.
\end{align*}
\]

All these terms vanish on the Gauss fiber cone \( G \) and therefore also the second fundamental form of \( G \), i.e., \( G \) is a linear space.

Now we turn to the irreducible focal variety \( X_f \) of \( X \). The point \( e_1 \) is a general point of \( X_f \), and the tangent space \( T_{e_1}X_f \) is the image of

\[
d\epsilon_1 = \omega_1^1e_2 + \omega_{1}^{n-2}e_{n-2} + \omega_{n-1}^1e_{n-1} \mod \{e_1\}.
\]

Since \( X_f \) is the union of the linear spaces \( \{e_1, \ldots, e_d\} \), the tangent space \( T_{e_1}X_f \) must contain this linear space \( \{e_1, \ldots, e_d\} \). Because \( \omega_{1}^{n-2} \) and \( \omega_{1}^{n-1} \) are linear independent, the tangent space \( T_{e_1}X_f \) is \( \{e_1, \ldots, e_{n-1}\} \), and hence the dimension of \( X_f \) is \( n-2 \).

We can compute the second fundamental form of \( X_f \) easily as

\[
\Pi_{X_f,e_1} = \omega_{n-2}^1(\omega_{n-1}^2e_n + \omega_{n-2}^1e_{n-1}) + \omega_{n-1}(\omega_{n-1}^1e_n + \omega_{n-1}^1e_{n-1} + \omega_{n-1}^n e_{1})
\]

\[
= (2f_2\omega_{n-2}^1 - \omega_{n-1}^1)^2e_n + 2\omega_{n-2}^1\omega_{n-1}^1e_{n+1} + \frac{1}{2} \omega_{n-1}^m \omega_{n-1}^r e_{1} \mod \{e_1, \ldots, e_{n-1}\}.
\]

Thus \( X_f \) is of Gauss rank 2 and has \( \{e_1, \ldots, e_d\} \) as Gauss fiber. Further, it has the asymptotic space \( \{e_1, \ldots, e_{n-2}\} = \{\omega_{n-1}^{-1}\} \). This asymptotic space is the intersection of the linear Gauss fiber cone \( G = \{e_0, \ldots, e_{n-2}\} \) with the hyperplane at infinity. We can consider this asymptotic distribution \( \omega_{1}^{n-1} = 0 \Leftrightarrow \omega^n = 0 \) on \( X_f \) as well as on \( X \). By the Theorem of Frobenius it is completely integrable on both varieties since

\[
d\omega_{1}^{n-1} = -\omega_{1}^{n-1} \wedge \omega_{1}^{1} - \omega_{n-2}^1 \wedge \omega_{n-1}^2 + \omega_{n-1}^1 \wedge \omega_{n-1}^1 \wedge \omega_{n-1}^n \wedge \omega_{n-1}^1 = - (\frac{d}{2} \omega_{n-2}^1 + \frac{d}{2} \omega_{n-1}^1) \wedge \omega_{n-2}^1 = 0 \mod \{\omega_{1}^{n-1}\}.
\]
Now let $Y$ and $Z \supset Y$ be integral manifolds of this distribution on $X_f$ resp. $X$. Then $Z$ is the union of the Gauss fiber cones $G$ that contain the tangent spaces of $Y$ or equivalently the asymptotic planes of $X_f$ along $Y$. It remains to show that $Z$ has Gauss rank 1 and has the Gauss fiber cones $G$ as Gauss fibers.

We compute the second fundamental form of $Z$. On $Z$ we have

$$d e_0 = \omega^1 e_1 + \omega^2 e_2 + \omega^n e_n - 2 e_{n-2} + \omega^{n-1} e_{n-1} \mod \{e_0, \omega^n\}$$

$$H_{Z,e_0} = d^2 e_0 = (\omega^{n-2} \omega_{n-2} + \omega^{n-1} \omega_{n-1}) e_n + (\omega^{n-2} \omega_{n-2} + \omega^{n-1} \omega_{n-1}) e_{n+1}$$

$$+ \omega^{n-1} \omega_n e_{n} = (\omega^{n-1})^2 (f_2 e_n + e_{n+1}) \mod \{e_0, \ldots, e_{n-1}, \omega^n\}.$$ 

Clearly, the singular locus of $H_{Z,e_0}$ is the linear space $\{e_0, \ldots, e_{n-2}\}$, the Gauss fiber cone of $X$.

3 The Case of Gauss Rank 4

Here we prove the structure theorem for Gauss rank 4. Unfortunately, the descriptions will not always be detailed enough to yield methods for the constructions of varieties of the corresponding type.

**Theorem.** Let $X \subset \mathbb{P}^N$ be an affinely smooth developable variety of Gauss rank 4 and Gauss fiber dimension $d = n - 4$ which is not a cone. With the fiber movement system $A$ belonging to a general point of $X$, we define the following invariants of $X$:

- $a = \dim A$
- $b = \dim \sum_{A \in A} \text{Im } A$
- $l = \max\{\text{rank } A \mid A \in A\} = \text{rank of general matrix of } A$.

According to the values of these invariants, we have the following geometric descriptions of $X$:

1. $(l = 1, \ a = 1)$ The focal variety $X_f$ of $X$ is the $(d-1)$-th osculating scroll of a unique curve $C \subset H_\infty$. $X$ is the union of the one-dimensional family of Gauss fiber cones that are $(n-1)$-dimensional cones whose vertices are the $d$-th osculating spaces to the curve $C$.

2. $(l = 1, \ a = 2)$ $X_f$ is a twisted $(d-1)$-plane of type $(k_1, k_2)$ of two curves at infinity. $X$ is the union of the one-dimensional family of Gauss fiber cones, which are $(n-1)$-dimensional cones over the $(d+1)$-planes of the twisted $(d+1)$-plane of type $(k_1 + 1, k_2 + 1)$ of the same curves as the one above.

3. $(l = 1, \ a = 3)$ $X$ is a twisted $(n-1)$-plane of type $(0, k_2, k_3, k_4)$ with $k_2, k_3, k_4 \geq 1$ where the last three curves lie in $H_\infty$. Its Gauss fiber cones are the $(n-1)$-planes.

4. $(l = 2, \ A^2 = 0)$ The focal variety $X_f$ is an $(n-3)$-dimensional variety of Gauss rank 2. $X$ is the union of the two-dimensional family of the linear Gauss fiber cones, which contain the tangent spaces to $X_f$. 

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5. \((l = 2, a = 1, \mathcal{A}^2 \neq 0, X^* \text{ nondegenerate})\) \(X\) has dimension 5 and is the union of the two-dimensional family of the three-dimensional Gauss fiber cones, which are uniruled by \(2\)-planes. The \(2\)-planes of a Gauss fiber cone with vertex \(V \subset H_\infty\) intersect the tangent spaces \(\mathbb{T}_vX\), \(v \in V\), of the focal variety \(X_f\) in codimension one.

6. \((l = 2, a = b = 2, \mathcal{A}^2 \neq 0, X^* \text{ nondegenerate})\) \(X\) is the union of its two-dimensional family of Gauss fiber cones, each of which is an \((n-2)\)-dimensional cone over a \(d\)-plane in \(H_\infty\). Such a \(d\)-plane is asymptotic in the tangent spaces of \(X_f\) along the Gauss fiber cone vertex.

7. \((l = 2, b = 3, \mathcal{A}^2 \neq 0, X^* \text{ nondegenerate})\) \(X_f\) is a twisted \(d\)-plane of type \((k_1, k_2, k_3)\). \(X\) is the union of the three-dimensional family of linear Gauss fiber cones, which contain a \(d\)-plane of this twisted \(d\)-plane.

8. \((l = 2, \mathcal{A}^2 \neq 0, X^* \text{ degenerate})\) \(X\) is the union of its two-dimensional family of linear Gauss fiber cones. \(X_f\) has dimension \(n-2\) and Gauss rank \(b\). The linear Gauss fiber cone for a general vertex \(V \subset X_f\) intersects the tangent spaces \(\mathbb{T}_vX_f\), \(v \in V\), in a fixed linear space of dimension \(n-3\). \(X\) can also be seen as the union of a one-dimensional family of Gauss rank 1 varieties, whose Gauss fibers are the linear Gauss fiber cones of \(X\).

9. \((l = 3)\) \(X\) is the union of a two-dimensional family of Gauss rank 1 varieties whose Gauss fibers are the linear Gauss fiber cones of \(X\). The intersection of the linear Gauss fiber cone with \(H_\infty\) is an asymptotic plane in the tangent spaces of \(X_f\).

For \(l \geq 2\) the condition \(a = 1\) is equivalent to \(\dim X = 5\).

To prove this theorem, we have again to treat the different cases separately.

Case \(l = 1, a = 1\). The proof is analogous to the one for Gauss rank 3.

Case \(l = 1, a = 2\). We adapt the frame such that

\[
A_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_d = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q^\mu = \begin{pmatrix} 0 & 0 & 0 & q_4^\mu \\ 0 & 0 & 0 & q_2^\mu \\ 0 & 0 & q_3^\mu & q_4^\mu \\ q_1^\mu & q_2^\mu & q_4^\mu & q_5^\mu \end{pmatrix},
\]

and \(A_\varepsilon\) is 0 with the index ranges \(2 \leq \varepsilon \leq d - 1\), \(n + 1 \leq \mu \leq N\), in particular \(\omega_\varepsilon = 0\) and \(\omega_1 = \omega_2 = \omega_3 = \omega_4 = 0\). We start by examining a general Gauss fiber cone \(G\) of \(X\), given by the integrable distribution \(\omega^n = 0\). Using the index range \(n - 3 \leq k \leq n - 1\), its tangent space at a general point \(e_0\) is the image of

\[
d e_0 = \omega^1 e_1 + \omega^2 e_2 + \omega^3 e_3 + \omega^4 e_4 \mod \{e_0, \omega^n\},
\]

i.e., it is \(\{e_0, \ldots, e_{n-1}\}\). Its second fundamental form can be computed as

\[
\Pi_{G,e_0} = d^2 e_0 = \omega^k \omega^l_{\mu} e_n + \omega^k \omega^l_{\mu} e_\mu \mod \{e_0, \ldots, e_{n-1}, \omega^n\}.
\]

To determine the unknown forms \(\omega^k_n\), we differentiate \(\omega^n = \omega_d = 0\),

\[
0 = d \omega_1^\mu = -\omega_{n-3}^\mu \land \omega_{n-3}^\mu = \omega_n^\mu \land \omega_{n-3}^\mu = f_1 \omega_n^\mu
\]

\[
0 = d \omega_2^\mu = -\omega_{n-2}^\mu \land \omega_{n-2}^\mu = \omega_n^\mu \land \omega_{n-2}^\mu = f_2 \omega_n^\mu,
\]

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and further \( \omega_1^{n-3} = \omega_n \),
\[
0 = d(\omega_1^{n-3} - \omega_n) = -\omega_1^{n-3} \land \omega_1^3 - \omega_1^{n-3} - \omega_1^{n-3} + \omega_n \land \omega^0 + \omega_1^0 \land \omega^1 = -\omega_1^{n-1} \land \omega_n - \omega_n \land (\ldots) \Rightarrow \omega_n = f_3 \omega_1^{n-1} + f_4 \omega_n.
\]

Therefore,
\[
\mathbb{V}_{G, e_0} = (\omega_1^{n-1})^2 (f_3 e_n + q_4 e_n) \mod \{e_0, \ldots, e_{n-1}, \omega_n\}.
\]

We see that \( G \) is either of Gauss rank 1 with Gauss fibers \( \{e_0, \ldots, e_{n-3}\} \) or even linear. We claim that in any case \( G \) is a cone over \( \{e_1, \ldots, e_{n-2}\} \). This is equivalent to the fact that \( \{e_1, \ldots, e_{n-2}\} \) is a fixed linear space on \( G \), which follows from
\[
\begin{align*}
&d e_1 = d e_2 = d e_3 = 0 \mod \{e_1, \ldots, e_{n-2}, \omega_n\} \\
&d e_{n-3} = \omega_1^{n-1} e_{n-1} + \omega_n e_n + \omega_1^0 e_\mu = 0 \mod \{e_1, \ldots, e_{n-2}, \omega_n\} \\
&d e_{n-2} = \omega_1^{n-1} e_{n-2} + \omega_n e_n + \omega_1^0 e_\mu = 0 \mod \{e_1, \ldots, e_{n-2}, \omega_n\}.
\end{align*}
\]

Here we used \( \omega_1^{n-3} = \omega_n = 0 \mod \{\omega_n\} \), which can be derived analogously to \( \omega_1^{n-3} = \omega_{n-2} = 0 \mod \{\omega_n\} \).

From [P, 1, Theorem 9] we know that \( X_f \) is the union of the one-dimensional family of \((d - 1)\)-planes \( \{e_1, \ldots, e_d\} \) and has Gauss rank 2. By the classification of Gauss rank 2 varieties [P2] it is therefore a twisted \((d - 1)\)-plane of type \((k_1, k_2)\) of two curves \( C_1, C_2 \in \mathcal{H}_\infty \). The movement of the \((d - 1)\)-planes \( \{e_1, \ldots, e_d\} \) is characterized by the span of \( \{e_1, \ldots, e_d\} \) and the image of \( d e_1, \ldots, d e_d \). (This is the associated curve \( \Phi(1) \) of \( \Phi = \{e_1, \ldots, e_d\} \) in the notation of [P, 1, Section 4].)

We have
\[
\begin{align*}
&d e_1 = \omega_n e_{n-3} \mod \{e_1, \ldots, e_d\} \\
&d e_2 = 0 \mod \{e_1, \ldots, e_d\} \\
&d e_3 = \omega_n e_{n-2} \mod \{e_1, \ldots, e_d\},
\end{align*}
\]

hence the common image is the \((d + 1)\)-plane \( \{e_1, \ldots, e_{n-2}\} \). This \((d + 1)\)-plane is on the one hand by the above computation a \((d + 1)\)-plane of the twisted \((d + 1)\)-plane of type \((k_1 + 1, k_2 + 1)\) of the curves \( C_1, C_2 \in \mathcal{H}_\infty \) and on the other hand a vertex of the Gauss fiber cone \( G \).

Case \( l = 1, a = 3 \). Here we have \( \text{Im} \mathcal{A} = \ker \mathcal{A} \), and the statement follows from [P, 1, Corollary 11] in view of [WZ, Theorem 2] or [P, 1, Theorem 6].

Case \( l = 2, \mathcal{A}^2 = 0 \). Here we are again in the situation that \( \text{Im} \mathcal{A} = \ker \mathcal{A} \); hence, the Gauss fiber cones are linear. By [P, 1, Theorem 9] \( X_f \) has Gauss rank 2, and one easily checks that every Gauss fiber cone contains a tangent space of \( X_f \).

Case \( l = 2, \mathcal{A}^2 \neq 0 \), \( X^* \) nondegenerate. It was shown in [P, 1, Appendix] that the linear system \( \mathcal{Q} \) belonging to this linear system \( \mathcal{A} \) has the form
\[
\begin{pmatrix}
0 & 0 & 0 & q_1 \\
0 & q_2 & 0 & q_3 \\
0 & 0 & q_2 & q_4 \\
q_1 & q_3 & q_4 & q_5
\end{pmatrix}.
\]
By the assumption that $X^*$ is nondegenerate, $Q$ contains a matrix of full rank \cite[7.3]{L}, i.e., one with $q_1 q_2 \neq 0$. By scaling this matrix we may assume $q_1 = 1$. Then the transformation

$$T = \begin{pmatrix}
1 & -\frac{1}{2} q_3 - \frac{1}{2} q_4 + \frac{1}{2} q_5 \\
0 & 1 & 0 & -\frac{1}{2} q_3 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

takes $Q$ to a matrix with $q_3 = q_4 = q_5 = 0$ and maps $A$ to itself. Hence, we can adapt the frame such that

$$A_1 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad A_\varepsilon = \begin{pmatrix}
0 & 0 & s_\varepsilon & t_\varepsilon \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & s_\varepsilon \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad Q^{n+1} = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & q_2^{n+1} & 0 \\
1 & 0 & 0 & 0
\end{pmatrix},$$

and

$$Q^\mu = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & q_2^\mu \\
0 & 0 & q_2^\mu & q_3^\mu \\
0 & q_3^\mu & q_4^\mu & q_5^\mu
\end{pmatrix}$$

with $q_2^{n+1} \neq 0$ and the index ranges $2 \leq \varepsilon \leq d$ and $n + 2 \leq \mu \leq N$. Again, to prove our geometric statements, we need to express several 1–forms in terms of the semi–basic forms. We use the index range $n - 3 \leq k \leq n - 2$ and start by differentiating $\omega_1^{n-1} = \omega_1^n = 0$,

$$0 = d\omega_1^n = -\omega^n_k \wedge \omega_1^n = \omega_1^{n-2} \wedge \omega_{n-3}^n + \omega^n \wedge \omega_{n-2}^n$$

and obtain

$$\omega_{n-3}^n = f_1 \omega_1^{n-2} + f_2 \omega^n$$

$$\omega_{n-2}^n = f_2 \omega_1^{n-2} + f_3 \omega^n$$

$$\omega_{n-3}^n = f_4 \omega_1^{n-2} + f_5 \omega^n$$

$$\omega_{n-2}^n = s_\varepsilon \omega_1^n + f_5 \omega^{n-2} + f_6 \omega^n.$$  

Next from $\omega_{1}^{n-2} = \omega_{1}^{n+1}$ we get

$$0 = d(\omega_{1}^{n-2} - \omega_{1}^{n+1}) = -\omega_{1}^{n-2} \wedge \omega_1^n - \omega_1^n \wedge \omega_{1}^{n-2} + \omega_1^{n+1} \wedge \omega_1^n + \omega_1^{n+1} \wedge \omega_{1}^{n-3} + \omega_1^{n+1} \wedge \omega_{1}^{n-3}$$

$$= \omega_{1}^{n-2} \wedge (2\omega_{1}^{n-2} - f_1 \omega_1^{n-3} - f_4 q_2^n + \omega_1^n) + \omega_1^n \wedge (\ldots)$$

and further

$$\omega_{1}^{n-2} = \frac{f_1}{2} \omega_1^{n-3} + \frac{f_4 q_2^n}{2} \omega_1^{n-1} + f_6 \omega^n.$$  

Finally, by differentiating $\omega_{1}^{n-2} = \omega^n$,

$$0 = d(\omega_{1}^{n-2} - \omega_1^n) = -\omega_{1}^{n-2} \wedge \omega_1^n - \omega_1^n \wedge \omega_{1}^{n-2} + \omega_1^n \wedge \omega_1^n + \omega_1^n \wedge \omega_1^n$$

$$= -\frac{1}{2} f_1 \omega_1^{n-3} \wedge \omega_1^{n-2} + \omega_1^{n-2} \wedge (\omega_1^{n-1} - \frac{1}{2} f_4 q_2^n \omega_1^{n-2} + \omega_1^n \wedge (\ldots),$$

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we find \( f_1 = 0 \) and
\[
\omega_{n-1}^n = -\frac{f_4 q_2^n}{2} \omega_{n-2}^n + f_9 \omega_{n-1}^n + f_{10} \omega^2.
\]

Now we consider the distribution \( \omega^2 = \omega_{n-1} = \omega^0 = 0 \). For \( b = 3 \) this is the integrable distribution whose integral manifolds are the Gauss fiber cones. For \( b = 2 \) this distribution is only a subdistribution of that distribution, which is given this time by \( \omega^2 = \omega^0 = 0 \). However, this distribution \( \omega^2 = \omega_{n-1} = \omega^0 = 0 \) is also integrable for \( b = 2 \) by the Theorem of Frobenius, since we already know that \( \omega_{n-2} = \omega = 0 \) is also integrable for \( \omega_{n-2} = \omega = 0 \).

It remains to show that \( \omega^2 = \omega_{n-1} = \omega^0 = 0 \).

\[
d\omega_{n-1}^n = -\omega_{n-1}^n \wedge \omega^0 - \omega_{n-1}^i \wedge \omega^i = 0 \mod \{\omega_{n-2}^2, \omega_{n-1}^n, \omega^n\}
\]

by \( \omega_{n-3}^n = 0 \mod \{\omega_{n-2}^2, \omega_{n-1}^n\} \). Let \( L \) be an integrable manifold of this distribution. We want to prove that \( L \) is linear. Its tangent space at a general point \( e_0 \in \{e_0, \ldots, e_{n-3}\} \) as the image of
\[
d_{e_0} = \omega^1 e_1 + \omega^2 e_2 + \omega_{n-1} e_{n-3} \mod \{e_0, \omega_{n-2}, \omega_{n-1}, \omega^n\}
\]

Its second fundamental form,
\[
\Pi_{L,e_0} = d^2_{e_0} = (\omega_1 \omega_{n-2}^1 + \omega_{n-3} \omega_{n-2}^1) e_{n-2} + (\omega^1 \omega^2 e_1 + \omega^2 \omega_{n-2}^2) e_{n-1}
+ \omega^3 \omega^2 e_0 + \omega^4 \omega^3 e_{n-1} = 0 \mod \{e_0, \omega_{n-2}, \omega_{n-1}, \omega^n\}
\]

vanishes by \( \omega_{n-3}^n = \omega_{n-3}^n = \omega_{n-3}^n = 0 \mod \{\omega_{n-2}, \omega_{n-1}, \omega^n\} \), hence, \( L \) is linear.

Unfortunately, we cannot prove much more about the structure of \( X \) in case of \( a = 1 \). We can only note that the linear spaces \( L = \{e_0, \ldots, e_{n-3}\} \) of the Gauss fiber cone \( G \) intersect the tangent space to \( X_f \) at the point \( e_1 \), which is \( \{e_1, \ldots, e_{n-2}\} \), in codimension one. It seems possible that this intersection moves, while \( L \) moves in \( G \). However, if it does not, which will be a special case, then \( X \) has an analogous geometric description as the one which will be given for the case \( a = b = 2 \).

For this case \( a = b = 2 \), where \( s_\epsilon = 0 \) and some \( t_\epsilon \neq 0 \), we will now show that the Gauss fiber cone \( G \) is a cone over the \( d \)-plane \( \{e_1, \ldots, e_{n-3}\} \), i.e., all linear spaces \( L \) inside \( G \) contain this \( d \)-plane. We show that this \( d \)-plane is fixed on the Gauss fiber cone, which was given by the distribution \( \omega^2 = \omega^n = 0 \). Because of
\[
\begin{align*}
d_{e_1} & = \omega_{n-2} e_{n-2} = 0 \mod \{e_1, \ldots, e_{n-3}, \omega^2, \omega^n\} \\
d_{e_\epsilon} & = 0 \mod \{e_1, \ldots, e_{n-3}, \omega^2, \omega^n\} \\
d_{e_{n-3}} & = \omega_{n-3} e_{n-2} + \omega_{n-3} e_{n-1} + \omega_{n-3} e_n + \omega_{n-3} e_{n+1} \\
& = \frac{1}{2} f_{q_2^n} + \omega_{n-2} e_{n-2} \mod \{e_1, \ldots, e_{n-3}, \omega^2, \omega^n\},
\end{align*}
\]

it remains to show that \( f_4 = 0 \). This follows from differentiating \( \omega_{n-3} = 0 \):
\[
0 = d\omega_{n-3} = -\omega_{n-3} \wedge \omega_{n-3} = -f_4 \epsilon q_2^n \omega_{n-2} \wedge \omega^n.
\]
We turn to the focal variety \( X_f \). Its tangent space at the general point \( e_1 \) is \( \{ e_1, \ldots, e_{n-2} \} \) as the image of
\[
d e_1 = \omega^1_1 e_1 + \omega^{n-3}_1 e_{n-3} + \omega^{n-2}_1 e_{n-2} \mod \{ e_1 \}.
\]
Since its second fundamental form is
\[
\ddot{I}_{X_f, e_1} = d^2 e_1 = (\omega^{n-3}_1 \omega^{n-1}_1 + \omega^{n-2}_1 \omega^{n-2}_1) e_{n-1} + (\omega^{n-3}_1 \omega^{n-3}_1 + \omega^{n-2}_1 \omega^{n-2}_1) e_n
\]
\[
+ (\omega^{n-3}_1 \omega^{n+1}_1 + \omega^{n-2}_1 \omega^{n+1}_1) e_{n+1} + \omega^{n-2}_1 \omega^{\mu}_1 e_{\mu}
\]
\[
= \omega^{n-3}_1 \omega^{n-2}_1 (2f_3 e_{n-1} + 2f_2 e_n + 2e_{n+1})
\]
\[
+ (\omega^{n-2}_1 \omega^2_e)^2 (f_6 e_{n-1} + f_3 e_n + q_2^4 e_{\mu}) \mod \{ e_1, \ldots, e_{n-2} \},
\]
the vertex of the Gauss fiber cone, \( \{ e_1, \ldots, e_{n-3} \} \), is an asymptotic space of \( X_f \) at the general point \( e_1 \) — and hence all points — of the a priori Gauss fiber cone vertex \( \{ e_1, \ldots, e_d \} \).

At last, we treat the case of \( b = 3 \), i.e., some \( s_e \neq 0 \). Here we want to show that in addition to \( f_1 = 0 \) one has \( f_4 = f_7 = 0 \) and \( f_3 = f_2 \). We get this by differentiating \( \omega^n_e = \omega^{n-2}_e = 0 \):
\[
0 = d \omega^n_e = -\omega^{n-3}_e \wedge \omega^{n-3}_e - \omega^{n-1}_e \wedge \omega^{n-1}_e
\]
\[
= \frac{1}{2} q_2 f_2^3 \omega^{n+1}_e \wedge \omega^n_e + (f_2 - f_3) s_e \omega^{n-1}_e \wedge \omega^n_e
\]
\[
0 = d \omega^{n-2}_e = -\omega^{n-2}_e \wedge \omega^{n-2}_e - \omega^{n-3}_e \wedge \omega^{n-3}_e - \omega^{n-2}_e \wedge \omega^{n-1}_e
\]
\[
= -s_e f_7 \omega^{n-2}_e \wedge \omega^n_e + \omega^n_e \wedge (\ldots).
\]

The key observation for this case is that the distribution \( \omega^n = 0 \) is integrable by the Theorem of Frobenius since
\[
d \omega^n = -\omega^n \wedge \omega^n = 0 \mod \{ \omega^n \}
\]
using what we just proved. With the index range \( n-3 \leq k \leq n-1 \) we have on an integral manifold \( Y \)
\[
d e_0 = \omega^1_1 e_1 + \omega^2_1 e_2 + \omega^k_1 e_k \mod \{ e_0, \omega^n \},
\]
\[
\ddot{I}_{Y, e_0} = d^2 e_0 = \omega^k_1 \omega^k_1 e_n + \omega^k_1 \omega^{n+1}_k e_{n+1} + \omega^k_1 \omega^\mu_k e_{\mu}
\]
\[
= (\omega^{n-2}_1)^2 (f_2 e_{n+1} + e_{n+1}) + (\omega^{n-1}_1)^2 (f_2 e_n + q_2^4 e_{n+1} + q_2^4 e_{\mu}) \mod \{ e_0, e_1, \ldots, e_{n-1}, \omega^n \}.
\]

Therefore, \( Y \) has Gauss rank 2, its Gauss fibers are the linear Gauss fiber cones \( L \) of \( X \), and possesses a pair of conjugate \( (n-2) \)-planes. We claim that \( Y \) is in fact a cone over \( \{ e_1, \ldots, e_{n-3} \} \subset L \). We must show that this space is fixed on \( Y \). This follows from
\[
d e_1 = \omega^{n-2}_1 e_{n-2} = 0 \mod \{ e_1, \ldots, e_{n-3}, \omega^n \}
\]
\[
d e_x = 0 \mod \{ e_1, \ldots, e_{n-3}, \omega^n \}
\]
\[
d e_{n-3} = \omega^{k+1}_1 e_{k+1} + \omega^{n+1}_n e_{n+1} = 0 \mod \{ e_1, \ldots, e_{n-3}, \omega^n \}.
\]

(This distribution, \( \omega^n = 0 \), is also integrable in the case of \( a = b = 2 \). There one can also show that \( Y \) is a variety of Gauss rank 2 whose Gauss fibers are
the linear Gauss fiber cones of $X$ and has a pair of conjugate $(n - 2)$–planes in
general tangent space. However, it seems that $Y$ does not have to be a cone
with an $d$–dimensional vertex.

One can also consider the distribution $\omega^n = 0$ on the focal variety $X_f$. Its
integral manifold $Z$ has the tangent space $\{e_1, \ldots, e_{n-3}\}$ and must be linear
by the above consideration. Hence, $X_f$ is uniruled by codimension one planes.
Since it has Gauss rank 3 by [P1, Theorem 9], it is a twisted $d$–plane of type
$(k_1, k_2, k_3)$ of three curves in $H_\infty$ [P1, Section 4].

Case $l = 2$, $A^2 \neq 0$, $X^*$ degenerate. Since $X^*$ is degenerate, the linear system
$Q$ cannot contain a matrix of full rank [L, 7.3]. From [P1, Appendix] we know
that elements of $Q$ are of the form

$$Q = \begin{pmatrix} 0 & 0 & 0 & q_1 \\ 0 & q_1 & 0 & q_3 \\ 0 & 0 & 0 & q_4 \\ q_1 & q_3 & q_4 & q_5 \end{pmatrix}.$$ 

Due to $\text{Sing } Q = 0$, the linear system $Q$ must contain the matrices

$$Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & q_3 \\ 0 & 0 & 0 & 0 \\ 1 & q_3 & 0 & q_5 \end{pmatrix} \quad \text{and} \quad Q' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_3' \\ 0 & 0 & 0 & 1 \\ 0 & q_3' & 1 & q_5' \end{pmatrix}.$$ 

A transformation of the basis of $\mathbb{C}^4$ with

$$T = \begin{pmatrix} 1 & -\frac{1}{2}q_3 & 0 & \frac{3}{2}q_3 - \frac{1}{2}q_5 \\ 0 & 1 & 0 & -\frac{1}{2}q_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

takes the entries $q_3, q_5$ to zero while leaving the elements of $A$ fixed. Therefore
we may adapt the frame such that

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_{\varepsilon} = \begin{pmatrix} 0 & 0 & 0 & s_{\varepsilon} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{\varepsilon} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q^{n+1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and

$$Q^\mu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_1^\mu \\ 0 & 0 & 0 & q_2^\mu \\ 0 & q_1^\mu & q_2^\mu & q_3^\mu \end{pmatrix}$$

with the index ranges $2 \leq \varepsilon \leq d$ and $n + 2 \leq \mu \leq N$, in addition we will use the
index range $n - 3 \leq k \leq n - 2$. To prove our statements about the structure of
$X$ we will need the following relations

$$\omega^n_{n-3}, \omega^n_{n-1} \quad = 0 \mod \{\omega^n\}$$
$$\omega^{n-2}_{n-3}, \omega^{n-2}_{n-1}, \omega^{n-1}_{n-3}, \omega^n_{n-2} = 0 \mod \{\omega^{n-2}, \omega^n\},$$

(\ast)
which we derive now. Differentiating $\omega^{n-1}_1 = \omega^n_1 = 0$,

\[
0 = d\omega^{n-1}_1 = -\omega^{n-1}_c \wedge \omega^1_1 - \omega^{n-1}_k \wedge \omega^1_k = \omega^{n-2} \wedge \omega^{n-1}_1 + \omega^n \wedge (\ldots)
\]

\[
0 = d\omega^n_k = -\omega^n_0 \wedge \omega^n_k = \omega^{n-2} \wedge \omega^n_{n-3} + \omega^n \wedge \omega^n_{n-2},
\]

we find $\omega^{n-3}_n = f_1 \omega^{n-2} + f_2 \omega^n$ and $\omega^{n-1}_{n-3}, \omega^n_{n-2} = 0 \mod \{\omega^{n-2}, \omega^n\}$. Now we compare the differentials of $\omega^{n-2}_1 = \omega^n$ and $\omega^{n-2}_1 = \omega^{n+1}_{n-3}$,

\[
0 = d(\omega^{n-2}_1 - \omega^n) = -\omega^{n-2}_1 \wedge \omega^n_1 - \omega^{n-2}_k \wedge \omega^n_k + \omega^n \wedge \omega^0 + \omega^n \wedge \omega^i
\]

\[
= \omega^{n-2} \wedge (\omega^{n-2}_{n-3} + f_1 \omega^{n-3}) + \omega^{n-1} \wedge (-\omega^{n-1}_{n-1}) + \omega^n \wedge (\ldots)
\]

\[
0 = d(\omega^{n-2}_1 - \omega^{n+1}_{n-3}) = -\omega^{n-2}_1 \wedge \omega^n_1 - \omega^{n-2}_k \wedge \omega^n_k + \omega^{n+1}_{n-3} \wedge \omega^n_1 + \omega^{n+1}_{n-3} \wedge \omega^n_{n-3}
\]

\[
= \omega^{n-2} \wedge (2\omega^{n-2}_{n-3} - f_1 \omega^{n-3}) + \omega^n \wedge (\ldots).
\]

From $\omega^{n-2}_1 + f_1 \omega^{n-3} = 0 \mod \{\omega^{n-2}, \omega^{n-1}, \omega^n\}$ and $2\omega^{n-2}_{n-3} - f_1 \omega^{n-3} = 0 \mod \{\omega^{n-2}, \omega^n\}$ we conclude $f_1 = 0$ and $\omega^{n-2}_{n-3} = 0 \mod \{\omega^{n-2}, \omega^n\}$. In addition we now realize that $\omega^{n-1}_{n-1} = 0 \mod \{\omega^{n-1}, \omega^n\}$. Finally, differentiating $\omega^{n+1}_{n-1} = 0$,

\[
0 = d\omega^{n+1}_{n-1} = -\omega^{n+1}_{n-3} \wedge \omega^n_{n-1} - \omega^{n+1}_{n-2} \wedge \omega^n_{n-2} - \omega^{n+1}_{n-1} \wedge \omega^n_{n-1} + \omega^n_{n-1} \wedge \omega^n_{n-1}
\]

\[
= -\omega^{n-2} \wedge \omega^n_{n-1} - \omega^{n-2} \wedge \omega^n_{n-1} - \omega^n \wedge (\ldots)
\]

yields $\omega^{n}_{n-1} = 0 \mod \{\omega^{n-3}, \omega^{n-2}, \omega^n\}$, hence $\omega^n_{n-1} = 0 \mod \{\omega^n\}$. We also see now that $\omega^{n-2}_1 = 0 \mod \{\omega^{n-2}, \omega^n\}$.

A Gauss fiber cone $G$ of $X$ is given by the integrable distribution $\omega^{n-2} = \omega^n = 0$. Its tangent space at $e_0$ is the image of

\[
de_0 = \omega^1 e_1 + \omega^3 e_3 + \omega^{n-3} e_{n-3} + \omega^{n-1} e_{n-1} \mod \{e_0, \omega^{n-2}, \omega^n\},
\]

i.e., it is $\{e_0, \ldots, e_{n-3}, e_{n-1}\}$. Its second fundamental form

\[
\mathbb{H}_{e_0} = d^2 e_0 = (\omega^{n-3} \omega^{n-2} + \omega^{n-1} \omega^n) e_{n-2} + (\omega^{n-3} \omega^{n-2} + \omega^{n-1} \omega^n) e_n
\]

\[
+ \omega^{n-3} \omega^{n-2} e_{n+1} + \omega^{n-1} \omega^n e_{n-1} = 0 \mod \{e_0, \ldots, e_{n-3}, e_{n-1}, \omega^{n-2}, \omega^n\}
\]

vanishes by $(\ast)$. Therefore $G$ is the linear space $\{e_0, \ldots, e_{n-3}, e_{n-1}\}$. From [P1, Theorem 9] we know that $X_f$ has Gauss rank $b$, and its tangent space at the general point $e_1$ is $\{e_1, \ldots, e_{n-2}\}$. The linear Gauss fiber cone intersects this tangent space $T_{e_1} X_f$ in the linear space $\{e_1, \ldots, e_{n-3}\}$, which is a fixed $d$-plane along the Gauss fiber cone vertex $V = \{e_1, \ldots, e_d\} \subset X_f$ since

\[
de_1 = de_3 = de_{n-3} = 0 \mod \{e_1, \ldots, e_{n-3}, \omega^{n-2}, \omega^n\}.
\]

Now we note that the distribution $\omega^n = 0$ is integrable by the theorem of Frobenius, since

\[
d\omega^n = -\omega^n \wedge \omega^0 - \omega^n \wedge \omega^i = 0 \mod \{\omega^n\}
\]

by $(\ast)$. An integral manifold $Y$ has in a general point $e_0$ the tangent space $\{e_0, \ldots, e_{n-1}\}$ because — using the index range $n - 3 \leq k \leq n - 1$ — one has

\[
de_{e_0} = \omega^1 e_1 + \omega^3 e_3 + \omega^k e_k \mod \{e_0, \omega^n\}.
\]

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Next we consider $\omega$. Using the relation

$$\mathbb{I}_{Y,0} = d^2 e_0 = \omega^k (\omega_k e_n + \omega^{n+1}_k e_{n+1} + \omega^\mu e_\mu)$$

$$= (\omega^{n-2})^2 (f e_n + e_{n+1}) \mod \{e_0, \ldots, e_{n-1}, \omega^n\}.$$ 

Thus it has Gauss rank 1, and its Gauss fibers, $\{e_0, \ldots, e_{n-3}, e_{n-1}\}$, are the linear Gauss fiber cones of $X$.

**Case $l = 3$.** We start with adapting the frame such that

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & t_2 & u_2 \\ 0 & 0 & s_2 & t_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q^{n+1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and

$$Q^\mu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & q^\mu_1 & q^\mu_1 \\ 0 & q^\mu_2 & q^\mu_2 & q^\mu_2 \\ 0 & q^\mu_2 & q^\mu_2 & q^\mu_2 \end{pmatrix}$$

with the index ranges $2 \leq \varepsilon \leq d$ and $n + 2 \leq \mu \leq N$. In addition, we will use the index range $n - 3 \leq k \leq n - 1$. Our first task is to express $\omega_i$ for $i > j$ in terms of the forms $\{\omega^{n-3}, \ldots, \omega^n\}$ for later use. Differentiating $\omega_1^n = 0$ yields

$$0 = d\omega_n^n = -\omega_k^k \land \omega_k^1 = \omega^{n-2} \land \omega_{n-2}^n + \omega_{n-2}^n + \omega^n \land \omega_{n-1}^n,$$

thus

$$\omega_{n-3}^n = f_1 \omega^{n-2} + f_2 \omega^{n-1} + f_3 \omega^n$$

$$\omega_{n-2}^n = f_4 \omega^{n-1} + f_5 \omega^n$$

$$\omega_{n-1}^n = f_6 \omega^{n-1} + f_7 \omega^n.$$ 

Using the relation $\omega_2^n = \omega^n$, we obtain modulo $omega^n$

$$0 = d(\omega_1^n - \omega^n) = -\omega_1^n \land \omega_1^2 - \omega_1^1 \land \omega_1^1 + \omega^n \land \omega^0 + \omega^n \land \omega^1$$

$$= \omega_{n-3}^n \land (-\omega_{n-3}^n) + \omega_{n-2}^n \land (\omega_{n-1}^n - \omega_{n-2}^n) + \omega_{n-1}^n \land (\omega_{n-1}^n - \omega_{n-1}^n)$$

and therefore

$$\omega_{n-3}^n = -f_1 \omega_{n-3}^n + g_1 \omega_{n-2}^n + g_2 \omega_{n-1}^n + g_3 \omega^n$$

$$\omega_{n-2}^n = g_4 \omega_{n-2}^n + (g_2 - f_3) \omega_{n-1}^n + g_4 \omega_{n-1}^n + g_5 \omega^n.$$ 

Next we consider $\omega_{n-3}^{n+1} = \omega^n$ and its differential modulo $\omega^n$

$$0 = d(\omega^n - \omega_{n-3}^{n+1}) = -\omega^n \land \omega^0 - \omega_1^n \land \omega^1 + \omega_{n-1}^n \land \omega_{n-2}^n + \omega_{n-2}^n \land \omega_{n-3}^n$$

$$= \omega_{n-3}^n \land (2 \omega_{n-3}^n) + \omega_{n-2}^n \land (\omega_{n-1}^n + \omega_{n-2}^n) + \omega_{n-1}^n \land (\omega_{n-2}^n + \omega_{n-1}^n)$$

Plugging in the terms from above, we find $f_1 = 0$ by looking at the coefficient of $\omega_{n-3}^n \land \omega_{n-2}^n$ and further

$$\omega_{n-3}^n = 2f_2 \omega_{n-3}^n + (g_2 + f_3 - f_1) \omega_{n-2}^n + h_1 \omega_{n-1}^n + h_2 \omega^n.$$


Finally, we show that \( g_1 = f_2 = 0 \) in addition to \( f_1 = 0 \). The differentials of \( \omega_n^{n-2} = \omega^{n-1} \) and \( \omega_n^{n-3} = f_2 \omega^{n-1} + f_3 \omega^n \) modulo \( \{ \omega_n^{n-1}, \omega^{n} \} \) are

\[
0 = d(\omega_n^{n-2} - \omega^{n-1}) = -\omega_n^{n-2} \wedge \omega_n^{n-3} + \omega_n^{n-3} \wedge \omega_n^{n-3} + \omega_n^{n-2} \wedge \omega^{n-2} \\
= (3f_2 + g_1)\omega_n^{n-2} \wedge \omega_n^{n-3} \mod \{ \omega_n^{n-1}, \omega^{n} \}
\]

\[
0 = d(\omega_n^{n-3} - f_2 \omega^{n-1} - f_3 \omega^n) = -\omega_n^{n-2} \wedge \omega_n^{n-3} + f_2 \omega_n^{n-1} \wedge \omega^{n-3} + f_2 \omega_n^{n-1} \wedge \omega_n^{n-3} \mod \{ \omega_n^{n-1}, \omega^n \}.
\]

From \( 0 = 3f_2 + g_1 = f_2 (g_1 - f_2) \), we conclude \( g_1 = f_2 = 0 \).

Now we can examine \( X \). We start by showing that a general Gauss fiber cone \( G \), which is given by the integrable distribution \( \omega_n^{n-2} = \omega^{n-1} = \omega = 0 \), is linear. The tangent space of \( G \) at \( e_0 \) is the image of

\[
d e_0 = \omega \cdot e_1 + \omega_n \cdot e_1 + \omega_n^{n-3} e_{n-3} \mod \{ e_0, \omega_n^{n-2}, \omega^{n-1}, \omega_n \},
\]
i.e., it is \( \{ e_0, \ldots, e_{n-3} \} \). The second fundamental form of \( G \),

\[
\mathbb{II}_{G,e_0} = \omega_n^{n-3} e_{k+1} + \omega_n^{n-3} e_{n+1} \mod \{ e_0, \ldots, e_{n-3}, \omega_n^{n-2}, \omega^{n-1}, \omega_n \},
\]
vanishes, because \( \omega_n^{n-3} e_{n+1} = 0 \mod \{ \omega_n^{n-2}, \omega^{n-1}, \omega_n \} \) by our computations above.

Thus \( G \) is the linear space \( \{ e_0, \ldots, e_{n-3} \} \).

Next we note that the distribution \( \omega_n^{n-1} = \omega = 0 \) is integrable by the Theorem of Frobenius, since

\[
d \omega_n^{n-2} = -\omega_n^{n-2} \wedge \omega_n^{n-2} \wedge \omega_n^{n-2} \wedge \omega_n^{n-2} = 0 \mod \{ \omega_n^{n-1}, \omega_n \},
\]

\[
d \mathbb{II}_{n-1} = -\mathbb{II}_1 \wedge \mathbb{II}^1 \wedge \mathbb{II}_n \wedge \mathbb{II}_n = 0 \mod \{ \omega_n^{n-1}, \omega_n \},
\]

where we used again that \( \omega_n^{n-2} \wedge \omega_n^{n-1} = 0 \mod \{ \omega_n^{n-2}, \omega^{n-1} \} \).

Let \( Y \) be an integral manifold of this distribution. Its tangent space at the point \( e_0 \) is \( \{ e_0, \ldots, e_{n-2} \} \) as the image of

\[
d e_0 = \omega \cdot e_1 + \omega_n \cdot e_1 + \omega_n^{n-3} e_{n-3} + \omega_n^{n-2} e_{n-2} \mod \{ e_0, \omega_n^{n-1}, \omega_n \}.
\]

Its second fundamental form is

\[
\mathbb{II}_{Y,e_0} = d^2 e_0 = (\omega_n^{n-3} e_{n-1} + \omega_n^{n-2} e_{n-2}) e_{n-1} + (\omega_n^{n-3} e_{n-3} + \omega_n^{n-2} e_{n-2}) e_n \\
+ \omega_n^{n-3} e_{n+1} + \omega_n^{n-2} e_{n+1} + \omega_n^{n-2} e_{n+1} \mod \{ e_0, \ldots, e_{n-3}, \omega_n^{n-1}, \omega_n \}.
\]

Therefore, \( Y \) has Gauss rank 1 and has \( \{ e_0, \ldots, e_{n-3} \} \) — the linear Gauss fiber cones of \( X \) — as Gauss fibers.

Finally, we want to make a remark about how the linear Gauss fiber cones relate to the focal variety \( X_f \). The second fundamental form of \( X_f \) can be easily computed to be

\[
\mathbb{II}_{X_f,e_0} = (2f_1 \omega_n^{n-3} \omega_n^{n-1} + f_4 (\omega_n^{n-2})^2 + 2f_5 \omega_n^{n-2} \omega_n^{n-1} + f_{6} (\omega_1^{n-2})^2) e_n \\
+ (2\omega_n^{n-3} \omega_n^{n-1} + (\omega_n^{n-2})^2) e_{n+1} + (2f_{6} \omega_n^{n-2} \omega_n^{n-1} + g_2 (\omega_1^{n-2})^2) e_{n+1} \mod \{ e_1, \ldots, e_{n-1} \}.
\]

We see that the intersection of the linear Gauss fiber cones with the hyperplane \( H_\omega \), the linear space \( \{ e_1, \ldots, e_{n-3} \} \), is an asymptotic space of \( \mathbb{II}_{X_f,e_0} \), which is in fact fixed along the Gauss fiber cone vertex \( \{ e_1, \ldots, e_d \} \).
4 Constructions for Gauss Rank 3

In this final section we will show that the descriptions in the structure theorem for Gauss rank 3 can be read as ways how to construct developable varieties with a prescribed focal variety. Since the twisted planes are well understood, only the following remains to be proved for the \( l = 1 \) case.

**Proposition 1** Let \( C \) be a curve and \( \mathcal{G} \) a two-dimensional family of \((n-2)\)-planes such that every \((n-3)\)-th osculating plane of \( C \) is contained in a one-dimensional subfamily of \( \mathcal{G} \). Let \( X \) be the union of the planes of \( \mathcal{G} \). If \( \mathcal{G} \) is chosen general enough, then \( X \) is a developable variety whose focal variety is the \((n-4)\)-th osculating scroll of \( C \).

**Proof.** We will use the language of \([FP]\) for the local computations, which is very convenient for constructions. Let \( \varphi(s) \) be a local lifting of a parameterization of \( C \). Then the \((n-3)\)-th osculating planes of \( C \) are \( \{\varphi, \ldots, \varphi^{(n-3)}\} \).

We choose the family \( \mathcal{G} \) by choosing a parameterized surface \( \psi(s,t) \) and setting \( \mathcal{G}(s,t) := \{\varphi, \ldots, \varphi^{(n-3)}, \psi\} \).

Then \( X \) is locally the image of
\[
\Phi : (C^2, 0) \times C^{n-2} \times C \to \tilde{X} \quad (s, t, \lambda, \mu) \mapsto \sum_{i=0}^{n-3} \lambda_i \varphi^{(i)}(s) + \mu \psi(s, t).
\]

Its tangent space is
\[
T_{\varphi(s,t,\lambda,\mu)} \tilde{X} = \left\{ \varphi, \ldots, \varphi^{(n-3)}, \psi, \lambda_{n-3} \varphi^{(n-2)} + \mu \frac{\partial \psi}{\partial s}, \mu \frac{\partial \psi}{\partial t} \right\};
\]
hence, it depends precisely on \( s, t \), and the ratio \((\lambda_{n-3} : \mu)\) if \( \psi \) was chosen general enough. Therefore, \( X \) has Gauss rank 3. An adapted parameterization of \( X \) is given by \( \Phi(s, t, u, \lambda, \lambda_{n-3} u) \), whose images for fixed \((s, t, u)\) are the Gauss fibers of \( X \). For general \((s, t, u)\) the dimension of \( T_{\Phi(s,t,u,\lambda,\lambda_{n-3} u)} \tilde{X} \) will drop only for \( \lambda_{n-3} = 0 \); hence, the focal variety of \( X \) is the image of \( \Phi(s, t, u, \lambda_0, \ldots, \lambda_{n-4}, 0, 0) \), i.e., the \((n-4)\)-th osculating scroll of \( C \).

As already mentioned in the introduction the \( l = 2 \) case gets very technically if the prescribed focal variety is ruled by codimension one planes. Therefore, we restrict ourselves to show the following:

**Proposition 2** Let \( Y \) be a variety of dimension \( n-2 \) and Gauss rank 2 which has an asymptotic \((n-3)\)-plane in each tangent space, but such that the integral submanifolds of this distribution are not linear. Choose a 2-dimensional family \( \mathcal{G} \) containing the asymptotic \((n-2)\)-planes of \( Y \) which has also the additional property described in the Theorem. Let \( X \) be the union of these planes. If the family \( \mathcal{G} \) was chosen sufficiently general, \( X \) will be developable of Gauss rank 3 and \( l = 2 \), and its focal variety will lie inside \( H_\infty \).
Proof. Let
\[ \Phi : \mathbb{C}^2 \times \mathbb{C}^{n-3} \longrightarrow \hat{\mathbb{Y}} \]
\[ (s, t, \lambda) \longmapsto \sum_{i=0}^{n-4} \lambda_i \varrho_i(s, t) \]
be an adapted parameterization of \( \hat{\mathbb{Y}} \), i.e., the Gauss fibers of \( \hat{\mathbb{Y}} \) are spanned by the vectors \( \{ \varrho_i \} \), and the asymptotic submanifolds are the images of \( \Phi \) for fixed \( t \). The asymptotic spaces and the tangent spaces, which depend only on \( s \) and \( t \), are — assuming \( \varrho_0 \) is general —
\[ A(s, t) := \left\{ \varrho_i, \frac{\partial \varrho_i}{\partial s} \right\} = \left\{ \varrho_i, \frac{\partial \varrho_i}{\partial s} \right\} \]
\[ T(s, t) \hat{\mathbb{Y}} := \left\{ \varrho_i, \frac{\partial \varrho_i}{\partial s}, \frac{\partial \varrho_i}{\partial t} \right\} = \left\{ \varrho_i, \frac{\partial \varrho_i}{\partial s}, \frac{\partial \varrho_i}{\partial t} \right\}. \]
Since \( \frac{\partial}{\partial s} \) is an asymptotic direction, we have \( \frac{\partial^2 \varrho_0}{(\partial s)^2} \notin T(s, t) \hat{\mathbb{Y}} \). We claim that \( \frac{\partial^2 \varrho_0}{(\partial s)^2} \notin A(s, t) \) for some \( i \), w.l.o.g. \( i = 0 \). Otherwise, arbitrary high derivatives of \( \varrho_i \) with respect to \( s \) lie in \( A(s, t) \) by induction; hence arbitrary high derivatives of \( \Phi \) with respect to \( s \) and \( \lambda \) lie in \( A(s, t) \), and we conclude that the image of \( \Phi \) with fixed \( t \) is the linear space \( A(s, t) \), contradicting our assumptions. Therefore,
\[ \frac{\partial^2 \varrho_0}{(\partial s)^2} \neq 0 \mod A(s, t) \quad \text{and} \quad T(s, t) \hat{\mathbb{Y}} = A(s, t) + \left\{ \frac{\partial^2 \varrho_0}{(\partial s)^2} \right\}, \]
and we find functions \( \zeta_i(s, t) \) with
\[ \frac{\partial \varrho_i}{\partial t} = \zeta_i \frac{\partial^2 \varrho_0}{(\partial s)^2}. \]
Next we choose the family \( \mathcal{G} \) by choosing a parameterized surface \( \varphi(s, t) \) not contained in \( H_\infty \) and setting
\[ \mathcal{G}(s, t) := A(s, t) + \{ \varphi(s, t) \} = \left\{ \varrho_i, \frac{\partial \varrho_i}{\partial s}, \varphi \right\} = \left\{ \varrho_i, \frac{\partial \varrho_i}{\partial s}, \varphi \right\}. \]
Now the family \( s \mapsto \mathcal{G}(s, t) \) is supposed to swept out a Gauss rank 1 variety, hence
\[ \left( \left( \frac{\partial \varrho_i}{\partial s}, \frac{\partial^2 \varrho_0}{(\partial s)^2} \frac{\partial \varphi}{\partial s} \right) + \mathcal{G} \right)/\mathcal{G} = \left( \left( \frac{\partial^2 \varrho_0}{(\partial s)^2} \frac{\partial \varphi}{\partial s} \right) + \mathcal{G} \right)/\mathcal{G} \]
is one-dimensional [FP, 2.3.5], i.e., using \( \frac{\partial^2 \varrho_0}{(\partial s)^2} \notin \mathcal{G} \) there exists a function \( \xi(s, t) \) with
\[ \frac{\partial \varphi}{\partial s} = \xi \frac{\partial^2 \varrho_0}{(\partial s)^2} \mod \mathcal{G}. \]
The variety \( X \) is locally the image of
\[ \Psi : \mathbb{C}^2 \times \mathbb{C}^{n-1} \longrightarrow \tilde{X} \]
\[ (s, t, \lambda, \mu, \nu) \longmapsto \sum_{i=0}^{n-4} \lambda_i \varrho_i(s, t) + \mu \frac{\partial \varrho_0}{\partial t} + \nu \varphi. \]
Its tangent space is
\[ T_{\Psi(s, t, \lambda, \mu, \nu)} \tilde{X} = \left\{ \varrho_i, \frac{\partial \varrho_i}{\partial s}, \varphi, \mu \frac{\partial^2 \varrho_0}{(\partial s)^2} + \nu \frac{\partial \varphi}{\partial s}, \sum \lambda_i \frac{\partial \varrho_i}{\partial t} + \mu \frac{\partial \varrho_0}{\partial t} + \nu \frac{\partial \varphi}{\partial t} \right\}, \]
\[ = \left\{ \varrho_i, \frac{\partial \varrho_i}{\partial s}, \varphi, (\mu + \nu \xi) \frac{\partial^2 \varrho_0}{(\partial s)^2}, (\sum \lambda_i \xi_i) \frac{\partial^2 \varrho_0}{(\partial s)^2} + \mu \frac{\partial \varrho_0}{\partial t} + \nu \frac{\partial \varphi}{\partial t} \right\}. \]
Note that for a sufficient general choice of $\varphi$ the $\frac{\partial \varphi}{\partial t}$ will not be contained in the span of the other occurring vectors. Therefore, the tangent space of $X$ is precisely constant along the image of $\Psi$ with fixed $s$, $t$ and fixed ratio of $\mu$ and $\nu$. This shows that $X$ has Gauss rank 3. Further, an adapted parameterization of $X$ is given by $\Psi(s,t,\lambda,\mu,\mu u)$ whose image for fixed $(s,t,u)$ are the Gauss fibers of $X$. For general $(s,t,u)$ the dimension of $T_{\Psi(s,t,\lambda,\mu,\mu u)}\hat{X}$ will drop only at $\mu = 0$, i.e., the focal variety of $X$ is given by the image of $\Psi(s,t,\lambda,0,0)$, which is $Y \subset H_\infty$.

References


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