

Linear Symmetric Determinantal Hypersurfaces

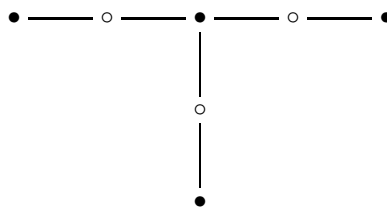
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The question which equations of hypersurfaces in the complex projective space can be expressed as the determinant of a matrix whose entries are linear forms is classical. In 1844 Hesse proved that a smooth plane cubic has three essentially different linear symmetric representations [He]. Dixon showed in 1904 that for smooth plane curves linear symmetric determinantal representations correspond to ineffective theta-characteristics, i.e., ineffective divisor classes whose double is the canonical divisor [Di]. Barth proved the corresponding statement for singular plane curves [B]. The general case for any hypersurface was treated by Catanese [C], Meyer–Brandis [M–B], and Beauville [Be].

Any plane curve has a linear symmetric determinantal representation [Be, 4.4], but every linear symmetric determinantal surface is singular. Already Salmon knew that such a surface of degree n possesses in general $\binom{n+1}{3}$ nodes [S, p. 495] and Cayley examined the position of these [Ca]. Catanese studied these surfaces with only nodes in a more general context [C]. Here we are dealing mainly with the question which combinations of singularities can occur on a linear symmetric determinantal cubic or quartic surface. For the cubics we find all their linear symmetric representations and obtain in particular the following theorem.

Theorem. *There are four types of linear symmetric determinantal cubic surfaces with isolated singularities. The combinations of their singularities are given by the subgraphs of \tilde{E}_6*



which are obtained by removing some of the white dots. In addition, all nonnormal cubics with the exception of the union of a smooth quadric with a transversal plane are linear symmetric determinantal cubics.

The combination of isolated singularities which occur on a linear symmetric determinantal quartic can be described similarly — only with more Dynkin diagrams as starting points for the splitting process.

The author’s original motivation for this study was the desire to understand linear maps from a vector space V into the space of symmetric matrices, which

occur for example in the examination of focal varieties, see for example [FP, 2.2.4]. Such a map can be understood as a symmetric matrix M whose entries are linear forms on V and $\det M$ describes the locus of V which is mapped to symmetric matrices of reduced rank. For $\dim V = 2$ such maps are classified up to the choice of coordinates classically [G, 12.6]. The case of $n = 2$ and arbitrary dimension of V is easy and the case of $n = 3$ is treated in course of proving the above Theorem. For $n = 4$ and $\dim V = 3$ the classification can be obtained with the methods used here if the linear symmetric determinantal quartic is a normal rational surface. However, if the quartic has only rational singularities, the below methods are not constructive because Torelli type theorems are used. This corresponds to the fact that while every possible combination of rational double points on a quartic is known by the work of Urabe and Yang [U1, U2, Y], equations for most of these surfaces are unknown.

The author is indebted to Y.-G. Yang who sent his program for the enumeration of the combinations of rational singularities on quartics to the author. Further, the author thanks J. Nagel and D. van Straten for several discussions.

1 General definitions and statements

Definition 1.1 *Let $M \in \text{Sym}(n, V^*)$ be a symmetric $n \times n$ -matrix whose entries are linear forms on a vector space V over \mathbb{C} . If $F := \det M$ is not zero, then it determines a linear symmetric (determinantal) hypersurface of degree n in $\mathbb{P}(V)$. Two matrix representations M and M' of F are equivalent if there is a $T \in \text{GL}(n, \mathbb{C})$ with $M' = T^t M T$. A matrix representation M will be called nondegenerate if the induced map $V \rightarrow \text{Sym}(n, \mathbb{C})$, $v \mapsto M(v)$, is injective.*

We note that the hypersurface F of a degenerate matrix representation M will be a cone over the kernel of the induced map.

Often M will be obtained by choosing some matrices A_0, \dots, A_N and setting $M := \sum_{i=0}^N x_i A_i$, where the x_i are a basis of $(\mathbb{C}^{N+1})^*$. The representation M will be *nondegenerate* if the matrices A_0, \dots, A_N are linearly independent. Choosing different generators A'_0, \dots, A'_N of the space $\text{span}\{A_0, \dots, A_N\}$ corresponds to a projective transformation of \mathbb{P}^N . Thus the hypersurface $F = \det M$ is determined up to projective equivalence by the choice of the linear space $\mathcal{A} := \text{span}\{A_i\} \subseteq \text{Sym}(n, \mathbb{C})$. In fact, we may view F as the intersection of $\mathbb{P}(\mathcal{A}) \subseteq \mathbb{P}(\text{Sym}(n, \mathbb{C}))$ with the general determinantal hypersurface $V(\det)$ or a cone over such a construction, in case we started with a degenerate representation.

One might expect that the linear symmetric hypersurfaces form a Zariski-closed subset of all hypersurfaces of degree n . However, this may be false because the map

$$\mathbb{P}(\text{Sym}(n, V^*)) \dashrightarrow \mathbb{P}(\text{polynomials of degree } n), [M] \mapsto [\det M],$$

is only a rational map and not regular for $n \geq 2$; hence, the set of linear symmetric hypersurfaces is only constructible.

As it is well known, the locus of corank 1 matrices is precisely singular along the locus of corank ≥ 2 matrices. Therefore, singularities of F appear if either

$\mathbb{P}(\mathcal{A})$ intersects $V(\det)$ at a corank ≥ 2 matrix or tangentially at a corank 1 matrix. We use this in the following definition.

Definition 1.2 *A singular point $x \in F$ is called an essential singularity if the corank of $M(x)$ is greater or equal to 2, otherwise an accidental singularity.*

The accidental singularities are difficult to control. Luckily, we can prove that for small sizes of the matrix M only certain types of singularities can occur.

Proposition 1.3 *Let F be a linear symmetric determinantal hypersurface of degree n in \mathbb{P}^N . Then the isolated accidental singularities of F are of corank less or equal to $n - N - 1$. (Here corank denotes the corank of the Hesse matrix of F at the singular point.)*

In particular, a linear symmetric cubic in \mathbb{P}^3 has no isolated accidental singularities, a quartic only nodes, and a quintic only A_k -singularities.

Before we start with the proof of the proposition, we show the following lemma, which enables us to identify some of the nonisolated singularities of a linear symmetric hypersurface. This statement was already known to Salmon [S, p. 495].

Lemma 1.4 *Let $M = (m_{ij})$ be a linear symmetric $n \times n$ -matrix with $m_{11} = 0$. Then the hypersurface $F = \det M$ is singular along $V(m_{12}, \dots, m_{1n})$.*

Proof. We expand the determinant F by the Leibniz formula. Then each summand of

$$\frac{\partial F}{\partial x_j} = \sum_{\sigma \in S(n)} \sum_{i=1}^n \operatorname{sgn} \sigma \cdot m_{1\sigma(1)} \cdots \frac{\partial m_{i\sigma(i)}}{\partial x_j} \cdots m_{n\sigma(n)},$$

contains $\frac{\partial m_{11}}{\partial x_j} = 0$, $m_{1\sigma(1)}$, or $m_{\sigma^{-1}(1)1} = m_{1\sigma^{-1}(1)}$; hence, it vanishes on $V(m_{12}, \dots, m_{1n})$. \square

Proof of Proposition 1.3. Assume that we are examining the point $p = (1 : 0 : \dots : 0)$. Because p is an accidental singularity, corank $A_0 = 1$ and we can choose coordinates on the \mathbb{C}^r such that

$$A_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We set $x_0 = 1$ and write

$$M = \begin{pmatrix} f_{11} & f_{12} & f_{13} & \cdots & f_{1n} \\ f_{12} & 1 + f_{22} & f_{23} & \cdots & f_{2n} \\ f_{13} & f_{23} & \ddots & & f_{3n} \\ \vdots & \vdots & & \ddots & \vdots \\ f_{1n} & f_{2n} & f_{3n} & \cdots & 1 + f_{nn} \end{pmatrix} \quad \text{with } f_{ij} \in \mathbb{C}[x_1, \dots, x_N].$$

Obviously, the linear part of $F = \det M$ is f_{11} , which has to vanish for p to be singular. Looking at

$$F = \det M = \sum_{\sigma \in S(n)} \operatorname{sgn} \sigma \cdot m_{1\sigma(1)} \cdot \dots \cdot m_{n\sigma(n)},$$

we see that the quadratic part of F is due to the summands where $n - 2$ of the $m_{i\sigma(i)}$ are of order 0, i.e., where σ is a transposition of 1 with $i \in \{2, \dots, n\}$; hence the quadratic part of F is

$$-\sum_{i=2}^n (f_{1i})^2.$$

The Hessian of F in p is the associated symmetric $N \times N$ -matrix S of this quadric. Our task is to show that the rank of S is at least $2N - n + 1$. By Lemma 1.4 there are N linearly independent forms among f_{12}, \dots, f_{1n} because the point p is an isolated singularity. Let us assume that f_{12}, \dots, f_{1N+1} are linearly independent, then the associated symmetric $N \times N$ -matrix \tilde{S} of the quadric $-\sum_{i=2}^{N+1} f_{1i}^2$ has rank N . The symmetric matrices S_{N+2}, \dots, S_n associated to $f_{1N+2}^2, \dots, f_{1n}^2$ have rank 1 or 0. From $\tilde{S} = S + \sum_{i=N+2}^n S_i$ we find

$$N = \operatorname{rank} \tilde{S} \leq \operatorname{rank} S + \sum_{i=N+2}^n \operatorname{rank} S_i \leq \operatorname{rank} S + n - N - 1,$$

thus $\operatorname{rank} S \geq 2N - n + 1$. □

Remark. We will see soon that an essential singularity of a linear symmetric hypersurface can never be an A_{2k} -singularity, but an accidental singularity may as well be one. For example the quintic given as the determinant of the matrix

$$\begin{pmatrix} 0 & x & y & z & \sqrt{-1}z \\ x & w+z & 0 & 0 & 0 \\ y & 0 & w+y & 0 & 0 \\ z & 0 & 0 & w+x & \sqrt{-1}z \\ \sqrt{-1}z & 0 & 0 & \sqrt{-1}z & w \end{pmatrix}$$

has an A_2 -singularity at $(1 : 0 : 0 : 0)$. It seems likely that as the size of the matrix increases all types of singularities will occur as accidental singularities.

We turn to the examination of the essential singularities. First, we will count them. The following statement was already known to Salmon [S, p. 495].

Proposition 1.5 *The general linear symmetric determinantal hypersurface F of degree n has only essential singularities and its singular locus has codimension 2 and degree $\binom{n+1}{3}$.*

In particular, a general linear symmetric surface $F \subset \mathbb{P}^3$ has $\binom{n+1}{3}$ essential A_1 -singularities.

Proof. For the first statement we view F as $V(\det) \cap \mathbb{P}(\mathcal{A}) \subseteq \mathbb{P}(\operatorname{Sym}(n, \mathbb{C}))$. A general linear space $\mathbb{P}(\mathcal{A}) \subseteq \mathbb{P}(\operatorname{Sym}(n, \mathbb{C}))$ intersects $V(\det)$ transversally, thus there are no accidental singularities. The locus of $\operatorname{corank} \geq 2$ matrices has

codimension 3 in $\mathbb{P}(\text{Sym}(n, \mathbb{C}))$ and degree $\binom{n+1}{3}$ [HT]. As its intersection with $\mathbb{P}(\mathcal{A})$ consists of the essential singularities of F , the first statement follows.

For the second statement one must show that a general essential singularity is a node. This can be done with arguments similar to the ones in the proof of Proposition 1.3. \square

Cossac studied these general linear symmetric surfaces in degree 4 taking up ideas of Cayley. In particular, he pointed out their connection to Enriques surfaces [Co]. In order to examine the essential singularities further we localize our definitions.

Definition 1.6 *A local symmetric matrix representation of a power series $f \in \mathbb{C}[[x_1, \dots, x_N]]$ is a symmetric matrix $M \in \text{Sym}(r, \mathbb{C}[[x_1, \dots, x_N]])$ with $\det M = f$. Two matrix representations M and M' are equivalent if there exists a $T \in \text{GL}(r, \mathbb{C}[[x_1, \dots, x_N]])$ such that $M' = T^t M T$. A matrix representation M is essential if $\text{corank } M(0) \geq 2$ and reduced if $M(0) = 0$.*

If one considers the equation of the power series f only up to a choice of holomorphic coordinates, it is convenient to extend the above definition of equivalence by allowing changes of coordinates as well. It is enough to consider only reduced matrix representations due to the following well-known lemma.

Lemma 1.7 *Any local symmetric matrix representation M of a power series $f \in \mathbb{C}[[x_1, \dots, x_N]]$ is equivalent to*

$$\begin{pmatrix} \tilde{M} & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix},$$

where \tilde{M} is a reduced local symmetric matrix representation of f .

Not every singularity has an essential local symmetric matrix representation. For the *ADE*-singularities we have the following

Theorem 1.8 *The surface singularities A_{2k} , E_6 , E_8 have no essential local symmetric matrix representation. The reduced essential symmetric matrix rep-*

representations for A_{2k+1} , D_{2k} , D_{2k+1} , E_7 are up to equivalence:

singularity X	equation	matrix representation M	$l(X)$
A_{2k+1} (A_{2k+1}^\bullet)	$-x^2 + y^2 - z^{2k+2}$	$\begin{pmatrix} y + z^{k+1} & x \\ x & y - z^{k+1} \end{pmatrix}$	$k + 1$
D_{2k} (D_{2k}^\bullet)	$-x^2 + y^2z - z^{2k-1}$	$\begin{pmatrix} z & x \\ x & y^2 - z^{2k-2} \end{pmatrix}$	2
D_{2k} (D_{2k}^\pm)		$\begin{pmatrix} y \pm z^{k-1} & x \\ x & z(y \mp z^{k-1}) \end{pmatrix}$	k
D_{2k+1} (D_{2k+1}^\bullet)	$-x^2 + y^2z - z^{2k}$	$\begin{pmatrix} z & x \\ x & y^2 - z^{2k-1} \end{pmatrix}$	2
E_7 (E_7^\bullet)	$-x^2 + z^3 + zy^3$	$\begin{pmatrix} z & x \\ x & z^2 + y^3 \end{pmatrix}$	3

The symbols in brackets in the first column denote the specific matrix representation of the singularity from now on. The last column gives the length of the first Fitting ideal, F_1M , of the matrix representation of the singularity, which is here the ideal generated by the entries of the matrix.

Proof. Let M be a local symmetric matrix representation of an ADE -singularity, which is given by the equation $f = \det M$. We set $R = \mathbb{C}[[x, y, z]]/(f)$. Then $\hat{M} = \text{coker } M$ is a maximal Cohen–Macaulay module of rank 1 [Yo, Chap. 7]. Due to the symmetry of M , we obtain a surjection $\hat{M} \rightarrow \text{Hom}_R(\hat{M}, R)$. Such a module \hat{M} is called a contact module. It determines the matrix M up to equivalence ([KU, §2] or [M–B, 3.34]). Over the local ring of an ADE -surface singularity there exists only a finite number of irreducible modules. This was proven by Auslander as follows: Recall that for each of the ADE -surface singularities there exists a group $G \subset \text{GL}(2, \mathbb{C})$ such that the invariant subring $\mathbb{C}[[x, y]]^G$ is isomorphic to the local ring R of the singularity. Auslander exhibited a bijection between these irreducible modules and the irreducible representations of G [Yo, Chap. 10].

Since a contact module has rank 1, we are only interested in the irreducible rank 1 modules, not isomorphic to R . There are k for A_k , 3 for D_k , 2 for E_6 , 1 for E_7 , and none for E_8 [Yo, p. 95]. This already proves the claim for D_{2k} , E_7 , and E_8 . For the other singularities one uses the Auslander’s bijection to work out the following representation matrices for the irreducible modules of

rank 1, besides the ones that occur in the above table:

singularity	standard equation	representation matrix
A_k	$-xy + z^{k+1}$	$M_i = \begin{pmatrix} z^i & y \\ x & z^{k+1-i} \end{pmatrix}$ for $1 \leq i \leq k$
D_{2k+1}	$-x^2 + zy^2 + z^{2k}$	$M_{\pm} = \begin{pmatrix} z^k \pm x & yz \\ y & z^k \mp x \end{pmatrix}$
E_6	$x^2 - y^3 - z^4$	$M_{\pm} = \begin{pmatrix} x \pm z^2 & y^2 \\ y & x \mp z^2 \end{pmatrix}$

Kleiman and Ulrich showed that if an R -module of rank 1 represented by an $r \times r$ -matrix M is a contact module then there exists a matrix $T \in \text{GL}(r, \mathbb{C}[[x, y, z]])$ such that TM is symmetric [KU, 2.2]. As we are dealing only with 2×2 -matrices, this condition is very easy to check. Let

$$T = \begin{pmatrix} g_1 & g_2 \\ f_2 & f_1 \end{pmatrix} \quad \text{with } g_1 f_1 - g_2 f_2 \in \mathbb{C}[[x, y, z]]^*.$$

For A_k the symmetry of TM_i is equivalent to

$$x f_1 + z^i f_2 = y g_1 + z^{k+1-i} g_2.$$

Clearly, we have $f_1(0) = g_1(0) = 0$, which implies $f_2(0) \cdot g_2(0) \neq 0$ for T to be invertible. Therefore, $i = k + 1 - i$, i.e., $k + 1$ is even and $i = (k + 1)/2$. Hence, there can be no contact module for A_{2k} and only one for A_{2k+1} . Completely analogous arguments work for the matrices M_{\pm} for D_{2k+1} and E_6 .

The computation of the length of the Fitting ideals is simple. Denoting $S := \mathbb{C}[[x, y, z]]$ we have

$$\begin{aligned} l(A_{2k+1}^{\bullet}) &= \dim S/(x, y + z^{k+1}, y - z^{k+1}) &= \dim S/(x, y, z^{k+1}) &= k + 1 \\ l(D_{2k}^{\bullet}) &= \dim S/(x, z, y^2 - z^{2k-2}) &= \dim S/(x, z, y^2) &= 2 \\ l(D_{2k}^{\pm}) &= \dim S/(x, y \pm z^{k-1}, z(y \mp z^{k-1})) &= \dim S/(x, z^k, y \pm z^{k-1}) &= k \\ l(D_{2k+1}^{\bullet}) &= \dim S/(x, z, y^2 - z^{2k-1}) &= \dim S/(x, z, y^2) &= 2 \\ l(E_7^{\bullet}) &= \dim S/(x, z, z^2 + y^3) &= \dim S/(x, z, y^3) &= 3 \end{aligned}$$

□

Often it does not make much sense to distinguish between the representations D_{2k}^+ and D_{2k}^- because the automorphism of the local ring of the singularity induced by $x \mapsto -x, y \mapsto -y$ swaps them. The above theorem restricts the possible combinations of essential singularities on a linear symmetric surface severely:

Corollary 1.9 *Let F be a linear symmetric determinantal surface of degree n in \mathbb{P}^3 whose essential singularities X_1, \dots, X_t are ADE -singularities. Then $X_i \in \{A_{2k+1}^\bullet, D_{2k+1}^\bullet, D_{2k}^\bullet, D_{2k}^\pm, E_7^\bullet\}$ and*

$$\sum_{i=1}^t l(X_i) = \binom{n+1}{3}.$$

Proof. This follows similarly to Proposition 1.5. We view F as $\mathbb{P}(\mathcal{A}) \cap V(\det) \subset \mathbb{P}(\text{Sym}(n, \mathbb{C}))$. Let I_i be the vanishing ideal of symmetric matrices of corank $\geq i$. Then we find the essential singularities as the intersection of $\mathbb{P}(\mathcal{A})$ and $V(I_2)$, thus the sum of their intersection multiplicity is $\deg V(I_2) = \binom{n+1}{3}$. From the above theorem we see that the essential ADE -singularities appear only at corank 2 matrices and never at matrices of higher corank. As $V(I_2)$ is smooth outside $V(I_3)$, the local intersection multiplicities of $\mathbb{P}(\mathcal{A})$ and $V(I_2)$ can be found by computing locally the length of the sum of the ideal I_2 and the vanishing ideal of $\mathbb{P}(\mathcal{A})$, i.e., by computing locally the length of the first Fitting ideal of matrix representation. This was done for the various singularities in the above theorem. \square

Remark. We will see later that linear symmetric cubics and quartics cannot have D_{2k+1} -singularities. However, here is a quintic with an essential D_5 -singularity at $x = y = z = 0$, showing that essential D_{2k+1} -singularities are in fact possible.

$$\begin{pmatrix} 0 & 665x & -2y+z & 3y+z & 2y+4z \\ 665x & 2y & -2771x & 6606x & 7138x \\ -2y+z & -2771x & 26y-6z & 0 & 4z+w \\ 3y+z & 6606x & 0 & w & 0 \\ 2y+4z & 7138x & 4z+w & 0 & 224y+136z \end{pmatrix}$$

A linear symmetric representation of $F \subset \mathbb{P}^3$ is closely related to the contact surfaces of F ; a surface G is a contact surface if the intersection $G \cap F$ is twice a curve C . These partially classical ideas, which are connected with the Hilbert–Burch theorem, were recently refined by Beauville [Be], Catanese [C], Eisenbud [Ei2], Kleiman and Ulrich [KU], and Meyer–Brandis [M–B]. The next few pages are devoted to extend Catanese’s results for even sets of nodes to sets of ADE -singularities.

While studying contact surfaces one also encounters nonlinear symmetric matrices, thus the following definition will be useful.

Definition 1.10 *A symmetric matrix $M = (m_{ij}) \in \text{Sym}(r, \mathbb{C}[x_0, \dots, x_N])$ is homogeneous if all its entries are homogeneous polynomials and $\deg m_{ii} + \deg m_{jj} = 2 \deg m_{ij}$ for all $i, j = 1, \dots, r$. The degree of M is $\deg M := (d_1, d_2, \dots, d_r)$, where $d_i := \deg m_{ii}$. By permutation of the rows and columns one can achieve that $d_1 \leq d_2 \leq \dots \leq d_r$. For a homogeneous matrix M the determinant $F = \det M$ is a homogeneous polynomial of degree $n = \sum_{i=1}^r d_i$. Such an F is called a symmetric (determinantal) hypersurface.*

M is linear if and only if $d_1 = \dots = d_r = 1$. A consequence of the homogeneity of M is that the adjoint matrix, $\text{adj } M$, of M is homogeneous, too. From the

adjoint matrix one obtains contact surfaces. Various versions of the following well-known lemma have appeared in the literature, we repeat the proof for the reader's convenience.

Lemma 1.11 *Let $F = \det M$ be a symmetric surface in \mathbb{P}^3 and m^{ii} a diagonal entry of $\text{adj } M$. Assume that no component of $\text{div}_F(m^{ii})$ is contained in the essential singular locus of F . Then $\text{div}_F(m^{ii}) = 2C$, where C is a Cartier divisor outside the essential singularities of F .*

Proof. The proof is based on the Laplace identity ([KU, 2.4] or [C, (1.3)]) which states

$$Fm^{ik,jl} = m^{kj}m^{il} - m^{kl}m^{ij},$$

where m^{ij} are the entries of the adjoint matrix and $m^{ik,jl}$ is $(-1)^{i+j+k+l}$ -times the determinant of the matrix M with the rows i, k and columns j, l deleted. Setting $k = j, l = i$, we have $m^{ii}m^{jj} = (m^{ij})^2$ modulo F , thus

$$\text{div}_F(m^{ii}) + \text{div}_F(m^{jj}) = 2\text{div}_F(m^{ij}). \quad (*)$$

This formula also implies that at the zero locus of $m^{11} = \dots = m^{rr} = 0$ on F all m^{ij} and with them $\text{adj } M$ vanish. Therefore, the divisors $\text{div}_F(m^{ii})$ cannot have a common component outside the essential singular locus, hence $(*)$ shows that all components in $\text{div}_F(m^{ii})$ occur with even multiplicity. Finally, $m^{ii}m^{jj} = (m^{ij})^2 \pmod{F}$ shows that C is Cartier outside the essential singularities. \square

If one uses instead of only M all equivalent matrix representations of F , one obtains a whole system of contact surfaces [M-B, §2.1]. From now on we restrict our attention to symmetric surfaces whose essential singularities are *ADE*-singularities. To understand their contact surfaces it is important to examine the local symmetric *ADE*-singularities, found in Theorem 1.8.

Definition 1.12 *Let $X \in \{A_{2k+1}^\bullet, D_k^\bullet, D_{2k}^\pm, E_7^\bullet\}$ be one of the essential symmetric surface singularities with equation $f = \det M$. The Fitting cycle of X on the minimal resolution $\pi : \tilde{X} \rightarrow X$ is defined as*

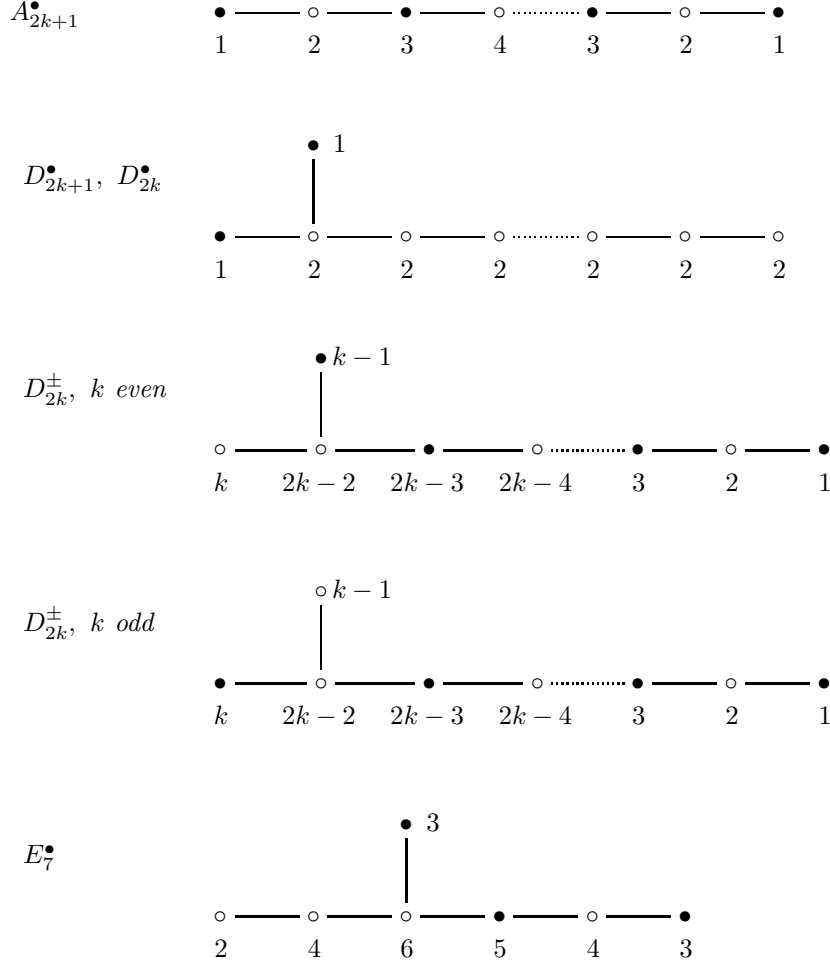
$$Z_X := \text{gcd}\{\text{div}_{\tilde{X}}(\pi^*g) \mid \text{for all } g \in F_1M\}.$$

*Let g be a local contact surface induced by M , for example one of the main corank 1 minors. The parity diagram of X is the minimal resolution graph G_X of X where the vertices are marked as follows: A vertex of G is drawn as \bullet if the corresponding curve occurs with odd multiplicity in the total transform π^*g of g , otherwise it is drawn as \circ .*

The generalized Laplace identity [M-B, 2.2] implies that the parity diagrams are the same for equivalent matrix representations and thus well-defined. Let us compute them.

Proposition 1.13 *The essential symmetric surface *ADE*-singularities have the following parity diagrams and Fitting cycles: (The multiplicity of an exceptional rational curve in the Fitting cycle is noted near the vertex representing*

this curve in the Dynkin diagram.)



In particular, the number of the \bullet -vertices in the parity diagram is the length of the first Fitting ideal of the matrix representation and the self-intersection number of the Fitting cycle is -2 times the length of the Fitting ideal, i.e., $(Z_X)^2 = -2l(X)$. Further, $Z_X \cdot E \leq 0$ for any exceptional curve E .

Proof. Because of Theorem 1.8 we need only to resolve the singularities while keeping track of the divisors given by the matrix entries. Such a task is traditionally left to the interested reader. \square

We return to the global situation.

Definition 1.14 Let $F \subset \mathbb{P}^3$ be a surface and $P = \{p_1, \dots, p_t\} \subset F$ a set of singular points of type A_{2k+1} , D_k , or E_7 on F . To each of this singularities assign an essential symmetric surface ADE-singularity symbol of the same underlying type, i.e., for A_{2k+1} , D_{2k+1} , and E_7 one uses A_{2k+1}^{\bullet} , D_{2k+1}^{\bullet} , and E_7^{\bullet} respectively, but for D_{2k} one may choose between D_{2k}^{\bullet} and D_{2k}^{\pm} . Let $X = \{X_1, \dots, X_t\}$

be the resulting set. Further, let $\pi : \tilde{F} \rightarrow F$ be the minimal resolution of F in these points and H the pullback of a hyperplane divisor of F to \tilde{F} .

The set X is said to be even if the divisor $\delta H + \sum_{i=1}^t Z_{X_i}$ for some $\delta \in \{0, 1\}$ is divisible by 2 in $\text{Pic}(\tilde{F})$.

X is strictly even if $\delta = 0$, otherwise weakly even.

The set X is called (linearly) symmetric if there is a (linearly) homogeneous symmetric matrix M with $F = \det M$ such that X is precisely the set of essential symmetric singularities of F .

Note that in the case of a symmetric set of ADE -singularities the pullbacks of the entries of the adjoint matrix of M define the cycle $\sum_{i=1}^t Z_{X_i}$ schemethetically by the definition of the Fitting cycles.

Proposition 1.15 *A symmetric set of ADE -singularities is even.*

Proof. Let $F = \det M$ be the surface which has X as essential singularities and G a contact surface given by a main corank 1 minor of M . Set $l = \deg G$ and $C = \frac{1}{2} \text{div}_F(G)$. Pulling G back to the minimal resolution $\pi : \tilde{F} \rightarrow F$ we find $\pi^*G = 2\tilde{C} + D$, where \tilde{C} is the strict transform of C and D a divisor supported on the exceptional set. By the definition of the Fitting cycle $D - \sum_{i=1}^t Z_{X_i}$ is effective as well. Further, from Proposition 1.13 we see that for all singularities the parity of the multiplicity of the exceptional rational curves in the Fitting cycle is the same as the one in the pullback of a contact surface, thus $D - \sum_{i=1}^t Z_{X_i}$ is divisible by 2, say $D - \sum_{i=1}^t Z_{X_i} = 2B$. Altogether we have with $\delta = l - 2\lfloor l/2 \rfloor$

$$\begin{aligned} \text{div}_{\tilde{F}} \pi^*G &= lH = \sum_{i=1}^t Z_{X_i} + 2(\tilde{C} + B) \\ \implies \sum_{i=1}^t Z_{X_i} + \delta H &= 2\left(\left\lfloor \frac{l}{2} \right\rfloor H - \tilde{C} - B\right), \end{aligned}$$

i.e., $\sum_{i=1}^t Z_{X_i} + \delta H$ is divisible by 2 in $\text{Pic}(\tilde{F})$. \square

We want to ensure the existence of contact surfaces for an even set of ADE -singularities with the same properties as G in the above proof.

Proposition 1.16 *Let X be an even set of ADE -singularities on a surface $F \subset \mathbb{P}^3$ and $\pi : \tilde{F} \rightarrow F$ the minimal resolution of F in these singular points. Then there exists a surface $G \subset \mathbb{P}^3$ such that its pullback divisor $\text{div}_{\tilde{F}} \pi^*G$ on \tilde{F} contains the Fitting cycles Z_{X_i} for $X_i \in X$ and the effective divisor $\text{div}_{\tilde{F}} \pi^*G - \sum_{i=1}^t Z_{X_i} \in \text{Pic}(\tilde{F})$ is divisible by 2.*

Proof. The proof is the same as the second half of [C, 2.6], we repeat it here because it is short and helps to understand the rest of the section. Let L be a divisor such that $2L = \sum_{i=1}^t Z_{X_i} + \delta H$. Choose l such that $lH - L$ is linearly equivalent to an effective divisor \tilde{C} . Then $(2l - \delta)H = 2\tilde{C} + \sum_{i=1}^t Z_{X_i}$; hence, there exists a surface of degree $2l - \delta$ with the required properties. \square

From now on the theory of the even sets of ADE -singularities is the same as Catanese's theory of even nodes [C, 2.16–2.23]. We repeat the statements, but leave out the proofs if they are identical with the ones in the node case.

Definition 1.17 Let X be an even set of ADE-singularities on F . The order of X is the smallest degree of a surface with the same properties as G in the above proposition.

Let S be the graded ring $\mathbb{C}[x, y, z, w]$ and $L \in \text{Pic}(\tilde{F})$ such that $2L = \delta H + \sum_{i=1}^t Z_{X_i}$. The associated graded S -module of X is

$$R^- = \bigoplus_{l=0}^{\infty} H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(lH - L)) = \bigoplus_{l=0}^{\infty} H^0(F, (\pi_* \mathcal{O}_{\tilde{F}}(-L))(l)).$$

Note that if $w \in H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(lH - L))$ and $w' \in H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(l'H - L))$ then $ww' \in H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}((l+l')H - 2L)) = H^0(F, \mathcal{O}_F(l+l'-\delta))$. In particular, if l is the smallest number for which $R_l^- \neq 0$ then X has order $2l - \delta$ by Proposition 1.16.

Lemma 1.18 $H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(lH - L)) \cong H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}((l - \delta)H + L))$.

Proof. (Compare [C, 2.15].) From the long exact cohomology sequence associated to

$$0 \rightarrow \mathcal{O}_{\tilde{F}}(lH - L) \rightarrow \mathcal{O}_{\tilde{F}}((l - \delta)H + L) \rightarrow \bigoplus_{i=0}^t \mathcal{O}_{Z_{X_i}}(L) \rightarrow 0$$

we see that it is enough to show that the cohomology group $H^0(Z_{X_i}, \mathcal{O}_{Z_{X_i}}(L))$ vanishes. If there exists a section $s \in H^0(Z_{X_i}, \mathcal{O}_{Z_{X_i}}(L))$ then $s^2 \in H^0(Z_{X_i}, \mathcal{O}_{Z_{X_i}}(2L)) = H^0(Z_{X_i}, \mathcal{O}_{Z_{X_i}}(Z_{X_i} + \delta H + \sum_{j \neq i} Z_{X_j}))$, but the last homology group is zero by [R, Ex. 4.14]. \square

Theorem 1.19 If X is a symmetric set of ADE-singularities on a reduced surface $F = \det M$, then the associated module R^- is a Cohen-Macaulay S -module.

More precisely, if $\deg M = (d_1, \dots, d_r)$, set $k_i = (n + \delta - d_i)/2$, $l_j = (n + \delta + d_j)/2$, where $n = \deg F$ and $\delta = n - d_i \pmod{2}$. Then there exists a minimal set of generators w_1, \dots, w_r of R^- of degrees k_1, \dots, k_r such that $w_i w_j = m^{ij}$, where $(m^{ij}) = \text{adj } M$. Moreover R^- admits the minimal free resolution

$$0 \rightarrow \bigoplus_{j=1}^r S[-l_j] \xrightarrow{(m^{ij})} \bigoplus_{i=1}^r S[-k_i] \xrightarrow{(w_j)} R^- \rightarrow 0.$$

The order of X is $n - \max\{d_i\}$.

Theorem 1.20 Let F be an irreducible and reduced surface of degree n and X an even set of ADE-singularities on F . Then the following conditions are equivalent:

- X is symmetric.
- Let w_1, \dots, w_r be a minimal set of homogeneous generators for the S -module R^- . Set $m^{ij} = w_i w_j \in \bigoplus_{l=0}^{\infty} H(F, \mathcal{O}(l)) = S/(F)$. Then $\det(m^{ij})$ is a nonzero polynomial of degree $n(r - 1)$.
- R^- is a Cohen-Macaulay S -module.

- $H^1(\tilde{F}, \mathcal{O}_{\tilde{F}}(lH - L)) = 0 \quad \forall l \in \mathbb{Z}$.

Catanese's construction of the symmetric matrix is such that none of the matrix entries is a nonzero constant, because the set of generators of R^- was chosen to be minimal.

Proposition 1.21 *Let F be a surface of degree n with an even set X of ADE-singularities.*

If $l(X) := \sum_{i=1}^t l(X_i) \leq \binom{n+1}{3}$, then X has order $\leq n - 1$.

If $l(X) = \binom{n+1}{3}$ and $n = \delta \pmod{2}$, then n is divisible by 8.

Proof. (Following [C, 2.21].) By the remark after the definition 1.17 it suffices to show that $h^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(lH - L)) \neq 0$ for $2l - \delta \geq n - 1$ or even only for $2l \geq n - 1$, observing that the order of X is an element of $2l - \delta + 2\mathbb{N}$. By Serre duality and Lemma 1.18

$$h^2(\tilde{F}, \mathcal{O}_{\tilde{F}}(lH - L)) = h^0(\tilde{F}, \mathcal{O}_{\tilde{F}}((n-4-l)H + L)) = h^0(\tilde{F}, \mathcal{O}_{\tilde{F}}((n-4-l+\delta)H - L)).$$

Since $n-4-l+\delta \leq l-3+\delta < l$, we get $h^2(\tilde{F}, \mathcal{O}_{\tilde{F}}(lH - L)) \leq h^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(lH - L))$, and it is enough to show $\chi(\tilde{F}, \mathcal{O}_{\tilde{F}}(lH - L)) > 0$. Using $(\sum Z_{X_i})^2 = -2l(X)$ from Proposition 1.13 we get from Riemann-Roch:

$$\begin{aligned} \chi(\tilde{F}, \mathcal{O}_{\tilde{F}}(lH - L)) &= \chi(\tilde{F}, \mathcal{O}_{\tilde{F}}) + \frac{1}{2}(lH - L)(lH - L - (n-4)H) \\ &= \chi(F, \mathcal{O}_F) + \frac{1}{2} \left((l - \frac{\delta}{2})H - \frac{1}{2} \sum Z_{X_i} \right) \left((l - n + 4 - \frac{\delta}{2})H - \frac{1}{2} \sum Z_{X_i} \right) \\ &= 1 + \binom{n-1}{3} + \frac{1}{2} \left((l - \frac{\delta}{2}) \left(l - n + 4 - \frac{\delta}{2} \right) n - \frac{1}{2} l(X) \right). \end{aligned}$$

It is not hard to see that this term is positive for $2r \geq n - 1$ and $l(X) \leq \binom{n+1}{3}$. For further reference we note that for $l(X) = \binom{n+1}{3}$ and $n - 1 = \delta \pmod{2}$ one finds

$$\chi(\tilde{F}, \mathcal{O}_{\tilde{F}}(\lfloor \frac{n}{2} \rfloor H - L)) = n \quad \text{and} \quad \chi(\tilde{F}, \mathcal{O}_{\tilde{F}}((\lfloor \frac{n}{2} \rfloor - 1)H - L)) = 0.$$

For $l(X) = \binom{n+1}{3}$ and $n = \delta \pmod{2}$ we get

$$\chi(\tilde{F}, \mathcal{O}_{\tilde{F}}((\lfloor \frac{n}{2} \rfloor - 1)H - L)) = \frac{3n}{8} \in \mathbb{Z},$$

showing that n is divisible by 8. □

Theorem 1.22 *Let X be an even set of ADE-singularities on a reduced surface $F \subset \mathbb{P}^3$ of degree n : Then X is linearly symmetric if and only if X has length $\binom{n+1}{3}$ and order $n - 1$.*

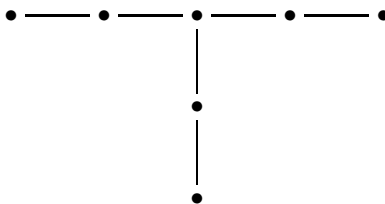
Proof. If X is linearly symmetric, its order is $n - 1$ by Theorem 1.19 and its length was computed in Corollary 1.9. Alternatively, one can compute the length with the arguments in the proof of the above Proposition using Theorem 1.20. For the nontrivial reverse implication of the theorem we refer to Catanese's proof of [C, 2.23]. □

Catanese showed by example that in general the hypothesis on the order of X cannot be dropped.

2 Cubics

Before we study the determinantal cubics we recall the following beautiful theorem about cubics in \mathbb{P}^3 , see Bruce and Wall [BW] or Looijenga [L].

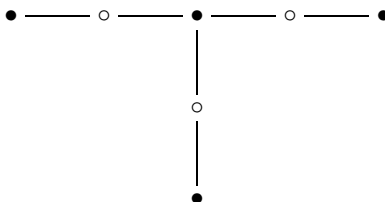
Theorem 2.1 *The combinations of singularities which can occur on a normal cubic surface in \mathbb{P}^3 are precisely the subgraphs of \tilde{E}_6*



which one obtains by removing some of the points. The nonnormal cubics are the cones over plane cubic curves, the reducible cubics, and two special irreducible types.

We want to prove a similar statement for linear symmetric cubics. As all plane cubics have a linear symmetric matrix representation (Appendix A), we focus on nondegenerate linear symmetric representations of the cubic surfaces first.

Theorem 2.2 *There are three nondegenerate linear symmetric determinantal cubics with isolated singularities. Their combinations of the singularities are given by the subgraphs of \tilde{E}_6*



which are obtained by removing some — but at least one — of the white dots. They all have unique matrix representations up to equivalence.

In addition, of the nonnormal cubics both special irreducible types, the smooth quadric with a tangent plane, the quadric cone with a transversal plane, and the double plane with an additional plane are nondegenerate linear symmetric cubics. All their nondegenerate linear symmetric matrix representations are unique.

Including the degenerate matrix representations and with them all cubic cones, we immediately obtain the following corollary.

Corollary 2.3 *There are four types of linear symmetric determinantal cubics with isolated singularities. Their combinations of the singularities are given by the subgraphs of the above marked Dynkin diagram \tilde{E}_6 which are obtained by removing some of the white dots. The cubics with an elliptic singularity have three matrix representations up to equivalence, the other cubics only one.*

In addition, all nonnormal cubics with the exception of the smooth quadric with a transversal plane are linear symmetric cubics.

Proof of the Theorem 2.2. (Following the outline in [CD, Prop. 0.5.5], where only the normal cubics are considered.) A nondegenerate linear symmetric representation is determined up to equivalence by choosing a four-dimensional linear subspace $\mathcal{A} \subset \text{Sym}(3, \mathbb{C})$, see the discussion near Definition 1.1. Now there is a nondegenerate symmetric bilinear form on $\text{Sym}(r, \mathbb{C})$ given by

$$\langle , \rangle : \text{Sym}(r, \mathbb{C}) \times \text{Sym}(r, \mathbb{C}) \longrightarrow \mathbb{C}, (A, B) \longmapsto \text{tr}(A \cdot B) = \sum_{i,j=1}^r a_{ij}b_{ij},$$

where tr denotes the trace. Therefore, instead of choosing a four-dimensional linear subspace $\mathcal{A} \subset \text{Sym}(3, \mathbb{C})$, we may choose dually a two-dimensional linear subspace $\mathcal{A}^\perp \subset \text{Sym}(3, \mathbb{C})$. There is only a finite number of these pencils of \mathcal{A}^\perp . This can be extracted from [G, 12.6] where these pencils together with a choice of basis are classified. However, using the identification of a symmetric 3×3 -matrix modulo \mathbb{C}^* with a quadric in \mathbb{P}^2 , we can also view $\mathcal{A}^\perp \subset \text{Sym}(3, \mathbb{C})$ as a pencil of quadrics in \mathbb{P}^2 . Then one can see that prescribing the intersection type of two general members of this pencil determines the pencil up to a choice of coordinates. From there one can compute the corresponding determinantal cubic. We will give one example of this and summarize the remaining cases in a table.

Let us assume that two quadrics of the pencil intersect with multiplicities 1, 1, and 2. We choose coordinates such that $(0 : 0 : 1)$ and $(0 : 1 : 0)$ are the simple intersection points and $(1 : 0 : 0)$ is the point where the quadrics intersect with multiplicity 2, i.e., they have a common tangent. This tangent cannot pass through $(0 : 0 : 1)$ or $(0 : 1 : 0)$, because otherwise it would intersect every quadric of the pencil with multiplicity $2 + 1 = 3$, i.e., it would be a component of every quadric by Bezout's theorem. Thus by a further adaption of coordinates we may assume that the tangent is spanned by $(1 : 0 : 0)$ and $(1 : 1 : 1)$. Let $(r : s : t)$ be the coordinates on \mathbb{P}^2 . Then a quadric q passing through $(1 : 0 : 0)$, $(0 : 1 : 0)$, and $(0 : 0 : 1)$ has the form $ars + brt + cst$. Its tangent in the point $(1 : 0 : 0)$ is given by $\text{grad}_{(1,0,0)}q = (0, a, b)$; hence, passing through $(1 : 1 : 1)$ implies $a = -b$. Therefore, the pencil of quadrics is spanned by $2r(s - t)$ and $2st$. These correspond to the symmetric matrices

$$\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

which therefore span \mathcal{A}^\perp . From this a basis of \mathcal{A} can be easily computed as

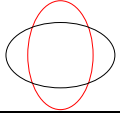
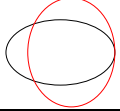
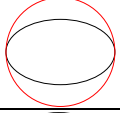
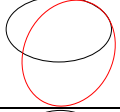
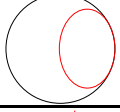
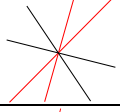
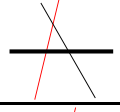
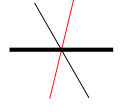
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and the equation of the cubic is

$$F = \det \begin{pmatrix} w & z & z \\ z & x & 0 \\ z & 0 & y \end{pmatrix} = wxy - xz^2 - yz^2.$$

It is easy to see that the singularities of F are the two A_1 -singularities at $(0 : 1 : 0 : 0)$ and $(0 : 0 : 1 : 0)$ and an A_3 -singularity at $(1 : 0 : 0 : 0)$.

We summarize all cases in the following table. Its first column describes the pencil of quadrics. If it contains only numbers, we consider the pencil whose general member is smooth and two of those intersect with multiplicities given by the numbers. \square

description of pencil	pencil of quadrics	two general members	F	description of cubic
(1, 1, 1, 1)	$r(s - t)$ $t(r - s)$		$wxy + wxz + wyz + xyz$	cubic with $4A_1$
(2, 1, 1)	$r(s - t)$ st		$wxy - xz^2 - yz^2$	cubic with $2A_1 + A_3$
(2, 2)	r^2 st		$-xy^2 - wz^2$	irreducible type I
(3, 1)	$2r^2 - 2st$ rt		$wxz - z^3 - xy^2$	cubic with $A_5 + A_1$
(4)	$2s^2 - 2rt$ r^2		$-w^3 + 2wxy - x^2z$	irreducible type II
all quadrics singular; no fixed line	r^2 s^2		$-x(wx - 2yz)$	nondeg. quadric + tangent plane
fixed line; pencil with center outside line	rt st		$y(wx - z^2)$	quadric cone + transversal plane
fixed line; pencil with center on line	r^2 rs		$-wy^2$	double plane + plane

Looking at the table one notices that whenever the cubic F has only isolated singularities, these singularities are precisely the singularities of the quartic Q which is the union of the two smooth members of the pencil of the quadrics given by \mathcal{A} . We are going to explain this amazing fact.

Recall that a plane A_{2k+1} -singularity is defined to be the intersection of two smooth branches intersecting with multiplicity k . Thus knowing the singularities of the quartic Q , which is the union of the two smooth quadrics C_1 and C_2 , is the same as knowing the intersection multiplicities of $C_1 \cap C_2$, whose sum is 4. Now we embed \mathbb{P}^2 via the Veronese embedding

$$v : \mathbb{P}^2 \longrightarrow \mathcal{V} \subset \mathbb{P}^5 = \mathbb{P}(\text{Sym}(3, \mathbb{C})), [x] \longmapsto [x \cdot x^t].$$

as the Veronese surface \mathcal{V} into \mathbb{P}^5 . Then the quadrics C_1 and C_2 are pullbacks of two hyperplanes H_1 and H_2 of \mathbb{P}^5 . The intersection multiplicities of $C_1 \cap C_2$ are the same as the intersection multiplicities of the curves $H_1 \cap \mathcal{V}$ and $H_2 \cap \mathcal{V}$ on the Veronese surface \mathcal{V} by the projection formula. These are also the intersection multiplicities of the Veronese surface \mathcal{V} and the 3-plane $H_1 \cap H_2 = \mathbb{P}(\mathcal{A})$. Denoting the affine coordinate ring of $\text{Sym}^2(3, \mathbb{C})$ by $\mathbb{C}[x_0, \dots, x_5]$, they can be computed as the vector space dimensions of the ring

$$\mathbb{C}[x_0, \dots, x_5]/(I(\mathcal{V}) + I(H_1) + I(H_2))$$

localized at the corresponding points of $\mathbb{P}(\text{Sym}(3, \mathbb{C}))$. Because H_1 and H_2 are linear and the ideal of the Veronese surface is given by the 2×2 -minors of the general symmetric matrix, the above ring is isomorphic to

$$\mathbb{C}[w, x, y, z]/(2 \times 2\text{-minors of } M),$$

where M is the matrix representation of $F \in \mathbb{C}[w, x, y, z]$.

To determine the singularities of $F = \det M$, we project F from a general smooth point of F . Then it is classically known that the singularities of F are stably equivalent to the singularities of the branch curve of the projection. Let us recall the proof. If

$$F(w, x, y, z) = w^2 g_1(x, y, z) + w g_2(x, y, z) + g_3(x, y, w)$$

with $\deg g_i = i$ and $g_1 \neq 0$, then $(1 : 0 : 0 : 0)$ is a smooth point of F and the branch curve G of the projection is

$$G = g_2^2 - 4g_1 g_3.$$

The stable equivalence between the points of F and G can be seen from

$$F/g_1 = \left(w + \frac{g_2}{2g_1}\right)^2 - \frac{1}{4g_1^2}G.$$

Now we apply this to our $F = \det M$. We know a priori that F has at least four A_1 -singularities or worse in terms of the sum of the Milnor numbers; thus the branch curve G has these singularities as well and will be a reducible quartic.

We will show that it is the union of two quadrics. We choose coordinates such that the general projection point is $(1 : 0 : 0 : 0)$ and

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{then} \quad M = \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & w + f_{22} & f_{23} \\ f_{13} & f_{23} & w + f_{33} \end{pmatrix}$$

with $f_{ij} \in \mathbb{C}[x, y, z]$ linear. We denote the adjoint matrix of M by $\text{adj } M = (m^{ij})$. Then

$$F = w^2 g_1 + w g_2 + g_3 \quad \text{with} \quad g_1 = f_{11}, \quad g_2 = m_{w=0}^{22} + m_{w=0}^{33}, \quad g_3 = \det M_{w=0},$$

where the index $w = 0$ stands for setting w equal to zero in the polynomial resp. matrix. By the determinantal formula of Laplace [KU, 2.4]

$$F \cdot f_{11} = m^{22} m^{33} - (m^{23})^2 \quad \implies \quad g_1 g_3 = m_{w=0}^{22} m_{w=0}^{33} - (m_{w=0}^{23})^2$$

and

$$\begin{aligned} G &= g_2^2 - 4g_1 g_3 = (m_{w=0}^{22} + m_{w=0}^{33})^2 - 4m_{w=0}^{22} m_{w=0}^{33} + 4(m_{w=0}^{23})^2 \\ &= (m^{22} - m^{33} + 2\sqrt{-1}m^{23})(m^{22} - m^{33} - 2\sqrt{-1}m^{23}), \end{aligned}$$

using $m_{w=0}^{22} - m_{w=0}^{33} = m^{22} - m^{33}$ and $m_{w=0}^{23} = m^{23}$. Hence, G is the union of the quadric cones $\tilde{C}_1 = V(m^{22} - m^{33} + 2\sqrt{-1}m^{23})$ and $\tilde{C}_2 = V(m^{22} - m^{33} - 2\sqrt{-1}m^{23})$ with vertex $(1 : 0 : 0 : 0)$. We consider them as plane curves and compute their intersection multiplicities. They are given by the vector space dimensions of the ring

$$\mathbb{C}[x, y, z]/(m^{22} - m^{33} \pm 2\sqrt{-1}m^{23}) = \mathbb{C}[x, y, z]/(m^{22} - m^{33}, m^{23})$$

localized at the appropriate points. Since

$$(m^{22} - m^{33}, m^{23}) + (m^{11} = g_1 w + m_{w=0}^{11}) \subseteq (2 \times 2\text{-minors of } M)$$

and the sum of the intersection multiplicities is 4 in all cases, the intersection multiplicities of $C_1 \cap C_2$, $V \cap \mathbb{P}(\mathcal{A})$, and $\tilde{C}_1 \cap \tilde{C}_2$ are equal at corresponding points! Further, by what we said in the beginning the intersection multiplicities of $\tilde{C}_1 \cap \tilde{C}_2$ would determine the singularities of the branch curve if we knew that \tilde{C}_1 and \tilde{C}_2 are smooth, which we do not. However, the singularities will get only worse if \tilde{C}_1 or \tilde{C}_2 are singular, and we can at least conclude that the singularities of the branch curve, which are also the singularities of the cubic F , are equal to or worse than the singularities of the quartic $Q = C_1 + C_2$, that we started with. But the combination of the singularities for $C_1 + C_2$ are $4A_1$, $2A_1 + A_3$, $2A_3$, $A_5 + A_1$, and A_7 and these combinations are all extremal combinations of isolated singularities on a normal cubic — with the exception of $2A_3$ which is impossible — by the classification of cubics [BW]. Therefore, the singularities of F are in fact the singularities of $C_1 + C_2$ if F is normal.

3 Quartics

The methods of studying a normal quartic in \mathbb{P}^3 depend on whether its resolution is a $K3$ -surface or a rational surface. If the quartic has only rational

double points, its resolution is a $K3$ -surface. In this case Urabe and Yang used Torelli type theorems for $K3$ -surfaces to list all possible combinations of rational singularities. If the normal quartic surface possesses a nonrational double point or a triple point, the quartic is rational and can be examined by studying the projection of the quartic from this singular point. Degtyarev used this to list all possible combinations of singularities in this case. The proof also yields a method for producing equations of these quartics, in contrast to this for most possible combinations of rational singularities equations of the respective quartics are unknown. In the next three subsections we will adapt all this to the case of linear symmetric quartics.

If a quartic surface has a quadruple point, it is a cone over a plane curve. As any plane curve can be represented by a linear symmetric matrix [Be, 4.4], the same holds for any such quartic surface, and we will not discuss this case further.

3.1 Linear symmetric quartics with only rational double points

Urabe and Yang examined the question which combinations of rational double points can occur on a quartic at all [U1, U2, Y]. The general idea is not to study the quartic in \mathbb{P}^3 directly, but its minimal desingularization Y , which is a $K3$ -surface. For general facts about $K3$ -surfaces see [BPV, VIII]; we recall only the following: For all $K3$ -surfaces the second cohomology group $H^2(Y, \mathbb{Z})$ is a free abelian group of rank 22. Together with the intersection form it is the unique unimodular even lattice of signature $(3, 19)$, which is $Q(-E_8) \oplus Q(-E_8) \oplus \mathbb{H} \oplus \mathbb{H} \oplus \mathbb{H}$. Here, \oplus denotes the orthogonal direct sum, $Q(-E_8)$ the rank 8 lattice whose bilinear form is given by the Dynkin graph E_8 with sign-reversed weights, and \mathbb{H} the hyperbolic plane $\mathbb{H} = \mathbb{Z}u + \mathbb{Z}v$, where $-$ writing the symmetric bilinear form as multiplication $- u^2 = v^2 = 0$ and $u \cdot v = 1$. Due to $H^1(Y, \mathcal{O}) = 0$, the Picard group $\text{Pic}(Y)$ injects into $H^2(Y, \mathbb{Z})$ and is in fact a primitive subgroup there, i.e. $H^2(Y, \mathbb{Z})/\text{Pic}(Y)$ is torsion free.

Using Torelli type theorems for $K3$ -surfaces and work of Saint-Donat, Urabe proved the following

Theorem 3.1 ([U1, Theorem 1.15]) *Let $G = \sum a_k A_k + \sum b_l D_l + \sum c_m E_m$ be a Dynkin graph with components of type A , D , or E only. The following conditions are equivalent:*

1. *There is a quartic surface in \mathbb{P}^3 with only rational double points as singularities, the combination of singularities corresponding to G .*
2. *Let $Q = Q(G)$ be the lattice of type G . Let $\Lambda := Q(-E_8) \oplus Q(-E_8) \oplus \mathbb{H} \oplus \mathbb{H} \oplus \mathbb{H}$ denote the unimodular even lattice with signature $(3, 19)$. The lattice $S = \mathbb{Z}H \oplus Q$ ($H^2 = 4$, orthogonal direct sum) has an embedding $S \subseteq \Lambda$ satisfying the following conditions (a) and (b). Let $\tilde{S} = \{x \in \Lambda \mid mx \in S \text{ for some } m \in \mathbb{Z} \setminus \{0\}\}$ denote the primitive hull of S in Λ .*

(a) *If $\eta \in \tilde{S}$, $\eta \cdot H = 0$, and $\eta^2 = -2$, then $\eta \in Q$.*

(b) *\tilde{S} does not contain any element u with $u^2 = 0$ and $u \cdot H = 2$.*

The sum $\mu := \sum a_k k + \sum b_l l + \sum c_m m$ is called the *Milnor number* of G or X . For quartic surfaces one always has $\mu \leq 19$.

Condition (a) ensures that there are only the expected singularities G on the quartic; condition (b) that the linear system given by H induces an embedding into \mathbb{P}^3 .

By this theorem Urabe reduced the question of the existence of a quartic with a given combination of singularities to a purely lattice theoretic problem. We want a similar theorem for our situation and start by providing the Dynkin graph with additional information.

Definition 3.2 A parity Dynkin graph G is a formal sum of the following marked Dynkin diagrams:

- The essential parity Dynkin diagrams, which are the marked Dynkin diagrams of Proposition 1.13. (We do not distinguish between D_{2k}^+ and D_{2k}^- or between D_4^\bullet and D_4^\pm .)
- The accidental parity Dynkin diagrams, which are the Dynkin diagrams of A_k, D_k, E_6, E_7, E_8 with every vertex drawn as \circ . They are denoted by $A_k^\circ, D_k^\circ, E_6^\circ, E_7^\circ, E_8^\circ$.

The number of vertices of G is the Milnor number $\mu(G)$ of G and the number of \bullet -vertices is the length $l(G)$ of G .

To a linear symmetric surface with only rational singularities we assign a parity Dynkin diagram whose components correspond to the singularities in the obvious way: for the essential singularities we use the correspondence of Proposition 1.13 and to the accidental singularities we assign the corresponding accidental Dynkin diagrams.

Every parity Dynkin diagram comes with a special divisor in corresponding lattice:

Definition 3.3 The lattice $Q(G)$ of a parity Dynkin graph has a canonical basis given by the vertices of the graph G . The parity divisor D_G is the sum of the \bullet -vertices.

Now we can state the extension of Urabe's Theorem for linear symmetric quartics:

Theorem 3.4 Let G be a parity Dynkin graph. The following conditions are equivalent:

1. There is a linear symmetric quartic in \mathbb{P}^3 with only rational double points as singularities, the combination of singularities corresponding to G .
2. Let G satisfy the condition 2 described in the above Theorem and in addition:
 - (c) The length of G is 10 and $\frac{1}{2}H + \frac{1}{2}D_G \in \tilde{S}$, where D_G is the parity divisor of G .

Proof. In Urabe's correspondence between the lattices and the quartics, the primitive lattices \tilde{S} correspond to the Picard group of the minimal resolution \tilde{F} of the quartic. Now on a linear symmetric quartic F the essential singularities form an even set X of ADE -singularities of length 10 by Proposition 1.15 and Theorem 1.22. Let G be the parity Dynkin graph of F . Clearly, by the definitions $l(G) = l(X)$ and $H + D_G$ is divisible by 2 in $\text{Pic}(\tilde{F}) = \tilde{S}$ precisely if $H + \sum Z_{X_i}$ is. Therefore, condition (c) holds.

Starting with a parity Dynkin graph with the properties (a) – (c) Urabe's Theorem yields a quartic F with an even set of ADE -singularities of length 10. Let \tilde{F} be the minimal resolution of F . (This time for all singularities of F , but this makes no difference for the statements of Section 1.) By Theorem 1.22 the quartic F is linearly symmetric if the order of X is 3. By Proposition 1.21 X is weakly even and we only need to show that the order of X is not 1. Setting $L = \frac{1}{2}(H + \sum Z_{X_i}) \in \text{Pic}(\tilde{F})$ and using the remark after Definition 1.17 this is equivalent to $H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(H - L)) = 0$, i.e., we need to show that $H - L = \frac{1}{2}(H - \sum Z_{X_i}) \in \text{Pic}(\tilde{F})$ is not effective.

Assume that $H - L$ is effective. Then we can decompose it into $\sum_{j=1}^s C_j + \sum_k B_k$ where the C_j, B_k are irreducible curves with $H \cdot C_j > 0$ and $H \cdot B_k = 0$. Recall that for any curve C on a $K3$ -surface $C^2 \geq -2$ and C^2 is divisible by 2 [BPV, VIII (3.6)]. Because Q is a negative definite lattice and $B_k \in \mathbb{Q} \otimes Q$, we get $B_k^2 = -2$ and $B_k \in Q$ by condition (a). We claim that there are at most two curves C_j , i.e., $s \leq 2$. Write $C_j = a_j H + \tilde{C}_j \in \mathbb{Q}H \oplus \mathbb{Q} \otimes Q$, then $H \cdot C_j = 4a_j \in \mathbb{N}$ thus $a_j = n_j/4$ for some $n_j \in \mathbb{N}$. From $\sum C_j = \frac{1}{2}H \pmod{\mathbb{Q} \otimes Q}$ we find either $s = 1$ and $a_1 = \frac{1}{2}$ or $s = 2$ and $a_1 = a_2 = \frac{1}{4}$. It is not difficult to obtain contradictions for $s = 1$ or $s = 2$ and $C_1 \neq C_2$ by completely elementary calculations with divisors, but the $C_1 = C_2$ case seems inaccessible by these simple methods. Hence, we recall more lattice theory.

The primitive hull \tilde{S} of S will always lie in $S^* = \text{Hom}(S, \mathbb{Z}) \subset \mathbb{Q} \otimes S$, thus $\tilde{S}/S \subseteq S^*/S$. The finite group S^*/S is well known. If $G = \sum X_i$ is the decomposition of the parity Dynkin graph G into the parity Dynkin diagrams, then $S^*/S = \mathbb{Z}/4\mathbb{Z} \oplus \bigoplus_i Q(-X_i)^*/Q(-X_i)$, where the first summand is generated by $H/4$ and $Q(-X_i)^*/Q(-X_i)$ depends only on the underlying Dynkin diagram and is isomorphic to $\mathbb{Z}/(k+1)\mathbb{Z}$ for A_k , $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ for D_{2k} , $\mathbb{Z}/4\mathbb{Z}$ for D_{2k+1} , $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$, 0 for E_6, E_7, E_8 respectively [U3, 1.3]. Define for $D \in Q^*$

$$m(D) := \max\{(D + B)^2 \mid B \in Q\}.$$

Because the intersection form is negative definite, $m(D) < 0$ for $D \notin Q$. These numbers were computed by Urabe [U3, 1.3]. In particular, he found $m(\frac{1}{2}D_{X_i}) = -\frac{1}{2}l(X_i)$ for the parity divisors of the singularities X_i . Since $\mathbb{Q} \otimes Q$ is the orthogonal sum of the $\mathbb{Q} \otimes Q(X_i)$, we get $m(\frac{1}{2}D_G) = \sum m(\frac{1}{2}D_{X_i}) = -\frac{1}{2} \sum l(X_i) = -5$.

Now if $s = 1$ then $C_1 = H - L - \sum B_k = \frac{1}{2}H + \frac{1}{2}D_G + B$ for some $B \in Q$ and

$$C_1^2 = (\frac{1}{2}H + \frac{1}{2}D_G + B)^2 = (\frac{1}{2}H)^2 + (\frac{1}{2}D_G + B)^2 \leq 1 + m(\frac{1}{2}D_G) = 1 - 5 = -4,$$

contradicting $C_1^2 \geq -2$.

If $s = 2$ then write

$$C_j = \frac{1}{4}H + \sum_i C_{j, X_i} \quad \text{with } C_{j, X_i} \in \mathbb{Q} \otimes Q(X_i).$$

We see from $C_1 + C_2 = \frac{1}{2}H + \frac{1}{2}D_G \bmod Q$ that $C_{1,X_i} + C_{2,X_i} = \frac{1}{2}D_{X_i} \bmod Q$. Further, we find the estimates

$$\begin{aligned} C_j^2 &\leq (\tfrac{1}{4}H)^2 + \sum_i m(C_{j,X_i}) = \tfrac{1}{4} + \sum_i m(C_{j,X_i}) \\ C_1^2 + C_2^2 &\leq \tfrac{1}{2} + \sum_i (m(C_{1,X_i}) + m(C_{2,X_i})). \end{aligned}$$

A small computation using Urabe's values for the function m shows

$$m(C_{1,X_i}) + m(C_{2,X_i}) \leq m(\tfrac{1}{2}D_{X_i}) = -\tfrac{1}{2}l(X_i)$$

for any C_{j,X_i} with $C_{1,X_i} + C_{2,X_i} = \frac{1}{2}D_{X_i} \bmod Q(X_i)$. This implies $C_1^2 + C_2^2 \leq \frac{1}{2} - 5 = -4\frac{1}{2}$ and hence $C_1^2 < -2$ or $C_2^2 < -2$, which yields the required contradiction. \square

From Urabe's Theorem it follows immediately that if there exists a quartic with Dynkin graph G then one can find a quartic for any complete subgraph $G' \subset G$. For linear symmetric quartics a similar statement holds:

Definition 3.5 *A parity Dynkin subgraph G' of a parity Dynkin graph G is a complete subgraph $G' \subset G$ which contains all the \bullet -vertices of G , i.e., $l(G') = l(G)$.*

Corollary 3.6 (Parity splitting principle) *If there exists a linear symmetric quartic with parity Dynkin graph G , then there exists a linear symmetric quartic for any parity Dynkin subgraph G' of G .*

Proof. Because of $D_{G'} = D_G \in \mathbb{Q}H \oplus \mathbb{Q} \otimes \mathbb{Q}(G')$ we can use $\mathbb{Z}H \oplus \mathbb{Q}(G') \subseteq \mathbb{Z}L \oplus \mathbb{Q}(G) \subseteq \Lambda$ for the embedding required in the theorem. \square

This parity splitting principle has amazing consequences which we state in the following summarizing theorem.

Theorem 3.7 *Let G be the parity Dynkin graph of a linear symmetric quartic with only rational double points, then the following holds:*

1. $10 \leq \mu(G) \leq 19$ and $l(G) = 10$.
2. G is a union of the parity Dynkin diagrams A_{2k+1}^\bullet , D_{2k}^\pm , and A_1° .

In particular, the parity Dynkin graph G is determined by its underlying Dynkin graph.

Proof. $l(G) = 10$ was stated in Theorem 3.4 and $\mu(G) \leq 19$ holds for any quartic. By Proposition 1.3 the only possible accidental singularity on a linear symmetric quartic is an A_1 -singularity. Proposition 1.13 already says that there are no essential A_{2k} , E_6 , and E_8 singularities. Further, for the parity Dynkin diagrams D_{2k+1}^\bullet , D_{2k}^\bullet — except for $D_4^\bullet = D_4^\pm$ — there exist parity splittings which have an accidental A_l° , $l \geq 2$, parity Dynkin diagram as a component, contradicting the parity splitting principle. \square

Urabe used his Theorem to give a short list of so-called basic Dynkin graphs and define two kinds of transformations for Dynkin graphs such that after applying two transformations to a basic Dynkin graph the resulting graph is a possible

combination of rational singularities on a quartic [U1, U2]. This produced a long list of possible combinations of singularities on a quartic. Unfortunately, these operations are not compatible with our new condition (c). However, this long list of combinations of singularities was not complete as Urabe noted himself in [U2, 3]. There he also remarked that each Dynkin graph G can be checked individually by a tedious computation using Nikulin's lattice theory [N]. Yang wrote a computer program which precisely does this [Y]. Yang was so kind to send his program to the author. The modification to incorporate condition (c) is not difficult. The output of the program can be summarized as follows:

Theorem 3.8 *For linear symmetric quartics with only rational double points only the following parity Dynkin graphs or their parity splittings occur: (Only the underlying Dynkin graphs are listed as they determine the parity Dynkin diagrams by Theorem 3.7.)*

$D_{18} + A_1$	$D_{14} + A_5$	$D_{14} + A_3 + 2A_1$
$D_{12} + D_6 + A_1$	$D_{12} + A_5 + 2A_1$	$D_{10} + D_8 + A_1$
$D_{10} + D_6 + A_3$	$D_{10} + A_9$	$D_{10} + A_7 + 2A_1$
$D_{10} + A_5 + A_3 + A_1$	$2D_8 + 3A_1$	$D_8 + D_6 + D_4 + A_1$
$D_8 + D_6 + A_5$	$D_8 + D_6 + A_3 + 2A_1$	$D_8 + D_4 + A_5 + 2A_1$
$D_8 + A_9 + 2A_1$	$D_8 + A_5 + A_3 + 3A_1$	$3D_6 + A_1$
$2D_6 + D_4 + 3A_1$	$2D_6 + A_7$	$2D_6 + A_5 + 2A_1$
$D_6 + 2D_4 + A_3 + 2A_1$	$D_6 + D_4 + A_5 + A_3 + A_1$	$D_6 + A_{13}$
$D_6 + A_9 + A_3 + A_1$	$D_6 + A_7 + A_5 + A_1$	$4D_4 + 3A_1$
$D_4 + A_9 + A_5 + A_1$	$D_4 + 2A_5 + A_3 + 2A_1$	A_{19}
$A_{17} + 2A_1$	$A_{15} + A_3 + A_1$	$A_{13} + A_5 + A_1$
$A_{11} + A_7 + A_1$	$A_{11} + A_5 + 3A_1$	$A_{11} + 2A_3 + 2A_1$
$2A_9 + A_1$	$A_9 + A_7 + 3A_1$	$2A_7 + A_3 + 2A_1$
$3A_5 + 4A_1$	$4D_4 + 2A_1$	$16A_1$

The Theorem shows that the possible combinations of singularities on a linear symmetric determinantal quartic are far less than the one of a general quartic, where one has 27 pages of combinations for the Milnor numbers 19, 18, and 17 alone and most combinations for the Milnor numbers 16 and 15 and below that all combinations are possible [Y]. However, one might have hoped for even less possible combinations.

Without the use of the program it is not clear why one needs only the parity diagrams of Milnor number 19 and $4D_4^\pm + 2A_1^\bullet$ as well as $10A_1^\bullet + 6A_1^\circ$ as starting points for the parity splitting process.

Example. In general it is difficult to find an explicit matrix representation for the combinations of rational singularities we determined above. However, with some tricks and enough computing power one finds the following matrix representation

$$\begin{pmatrix} x & iy & iy/2 & y/2 - iz \\ iy & x & y/2 + iz & iy/2 \\ iy/2 & y/2 + iz & w + ix + 3iy/2 & i(-2x + y - 4z)/4 \\ y/2 - iz & iy/2 & i(-2x + y - 4z)/4 & w - ix - 3iy/2 \end{pmatrix}$$

of the unique quartic with an A_{19} -singularity found by Kato and Naruki [KN].

In the following sections, when the quartic has nonsimple singular point, we will study the quartic by projecting it from a singular point. The referee suggested that this might also be possible in the above case due to a result of Cayley [Co, 2.4.3]: We project the quartic from one of its simple singular double points. Let D be the branch locus of this projection and C be the projection of the tangent cone of the this singular point. Cayley proved that for a linear symmetric quartic the sextic curve B is a union of two cubics and has contact to the conic C . Enumerating all possible intersection configurations of the cubics and the conic, especially with respect to their singularities, will lead to the above classification. This might even yield equations for the quartics.

3.2 Linear symmetric Quartics with a nonsimple double point

As soon as the normal quartic acquires a nonsimple double point, it is no longer a K3-surface, but a rational surface. Hence the techniques of the last section cannot be used to study this case. Degtyarev studies these quartics by projecting them from their worst singularity onto a plane [D]. We will use his extensive study of quartic equations to obtain the following

Theorem 3.9 *Only the following combinations of double points occur on a rational linear symmetric quartic with at most double points:*

$$\begin{aligned} & X_{1,0} + A_3 + \{X_{1,0}, D_6 + A_1, D_4 + 3A_1, A_7, 2A_3 + A_1, 6A_1\}, \\ & X_{1,2} + A_3 + \{D_6, D_4 + 2A_1\}, \{X_{1,2} + A_1, X_{1,4}\} + \{D_6 + A_1, A_7, 2A_3 + A_1\}, \\ & X_{1,4} + A_3 + \{A_3 + A_1, A_3\}, X_{1,6} + A_3 + A_1, X_{1,8} + A_3, \\ & Y_{2,2}^1 + A_5 + A_1, Y_{2,2}^1 + 2A_1 + \{D_4, 4A_1\}, Y_{2,4}^1 + 2A_1 + \{2A_1, A_1\}, \\ & Y_{4,4}^1 + 2A_1 + \{A_1, \emptyset\}, Y_{2,6}^1 + 2A_1. \end{aligned}$$

Hereby, one has to choose one element out of the sets to get a valid expression, and the A_{2k+1} and D_{2k} singularities may be splitted in the same manner as A_{2k+1}^\bullet and D_{2k}^\pm in the section before.

Proof. To apply the results of Degtyarev, we need explicit equations. Let us assume that $M = wA_0 + xA_1 + yA_2 + zA_3$, $F = \det M$, and the worst singular point is at $p = (1 : 0 : 0 : 0)$. The rank of A_0 is two by Proposition 1.3 and the obvious fact that the multiplicity of F at p is equal or higher than the corank of A_0 . We can choose a basis of \mathbb{C}^4 such that

$$A_0 = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{E}_2 \end{pmatrix} \quad \text{with} \quad \tilde{E}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If we use a 2×2 -blocking for M ,

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^t & M_{22} \end{pmatrix},$$

the quadric part of F in p is given by $-\det M_{11}$. Since we are still free to choose an arbitrary basis in $\text{span}\{e_1, e_2\}$ resp. $\text{span}\{x, y, z\}$, we may think of M_{11} as given by a linear subspace in $\mathbb{P}(\text{Sym}(2, \mathbb{C})) \cong \mathbb{P}^2$. The matrices of rank 1 form a

smooth conic C in this \mathbb{P}^2 , and the linear spaces inside this \mathbb{P}^2 are characterized by their intersection with this conic [H, 10]. We get the following list:

subspace	normal form of M_{11}	$\det M_{11}$
\mathbb{P}^2	$\begin{pmatrix} x & z \\ z & y \end{pmatrix}$	$xy - z^2$
secant of C	$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$	xy
tangent to C	$\begin{pmatrix} 0 & x \\ x & y \end{pmatrix}$	$-x^2$
point outside C	$\begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}$	$-x^2$
point on C	$\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}$	0
\emptyset	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	0

In the last two cases we get nonisolated singularities by Lemma 1.4. In the first two cases we get A_k -singularities by the classification of singularities [AGV, 16.2]. We want to show that in the third case only D_k -singularities occur. We write

$$M = \begin{pmatrix} 0 & x & f_{13} & f_{14} \\ x & y & f_{23} & f_{24} \\ f_{13} & f_{23} & f_{33} & w + f_{34} \\ f_{14} & f_{24} & w + f_{34} & f_{44} \end{pmatrix} \quad \text{with } f_{ij} \in \mathbb{C}[x, y, z].$$

After a base change of the type $e_3 \mapsto e_3 - \lambda_3 e_1 - \mu_3 e_2$ and $e_4 \mapsto e_4 - \lambda_4 e_1 - \mu_4 e_2$, we may assume $f_{13}, f_{14}, f_{23}, f_{24} \in \mathbb{C}[y, z]$. Setting $w = 1$, computing the determinant and performing the substitution $x \rightarrow x - x f_{34} + f_{14} f_{23} + f_{13} f_{24}$, the equation of F starts with

$$x^2 + 2y f_{13} f_{14} + \dots$$

Because of Lemma 1.4 the linear polynomials f_{13} and f_{14} are linear independent. Thus F has a D_k -singularity in p [AGV, 16.2].

Therefore, the fourth case is the only case where nonsimple double points may occur. We have

$$M = \begin{pmatrix} 0 & x & f_{13} & f_{14} \\ x & 0 & f_{23} & f_{24} \\ f_{13} & f_{23} & f_{33} & w + f_{34} \\ f_{14} & f_{24} & w + f_{34} & f_{44} \end{pmatrix} \quad \text{with } f_{ij} \in \mathbb{C}[x, y, z].$$

The surface F is given as $F = w^2 x^2 + wxP + Q$ with

$$P = 2x f_{34} - 2(f_{13} f_{24} + f_{14} f_{23})$$

$$Q = x^2(f_{34}^2 - f_{33} f_{44}) + 2x(f_{13} f_{23} f_{44} + f_{14} f_{24} f_{33} - f_{13} f_{24} f_{34} - f_{14} f_{23} f_{34}) + (f_{13} f_{24} - f_{14} f_{23})^2.$$

The branch curve of the canonical projection of F from p is – besides the x^2 -factor

$$D = P^2 - 4Q = 4(x f_{33} - 2f_{13} f_{23})(x f_{44} - 2f_{14} f_{24}) = 4C_1 C_2,$$

a union of two conics C_1, C_2 . Note that there is no restriction on the equation of the conics, since we can choose the f_{ij} arbitrary so far. Further, let the line L be the projected tangent cone $V(x)$ of F in p .

According to Degtyarev [D, §2], p will be an isolated singularity only if D is smooth at $L \cap Q$. Note that D cannot contain L because of the linear independence of the linear forms x, f_{13}, f_{14} resp. x, f_{23}, f_{24} (Lemma 1.4). This excludes singularities of the type N [D, §3]. F has the following singularities:

- To each singular point of D not lying on L there corresponds a singular point of F stably equivalent to it. In particular, the curve D cannot have a multiple component for a normal surface F .
- To each s -fold point of $Q \cap L$ not on D there corresponds an exceptional singular point of F of type A_{s-1} .
- The type of the singularity of F at p can be read off the following table. The first column describes the intersection configuration of L and D where 1 and 2 stands for a transversal resp. tangential intersection at a smooth point of D and A_k for a transversal intersection at an A_k -singularity of D . If $D \cap L$ and $Q \cap L$ have a common multiple point, its multiplicity in $Q \cap L$ is denoted by q , otherwise we set $q = 1$. The case of two common double points is written loosely as $q = (2, 2)$.

$D \cap L$	q	
$(1, 1, 1, 1)$	–	$X_{1,0}$
$(2, 1, 1)$	q	$X_{1,q}$
$(2, 2)$	q	$Y_{1,q}$
$(2, 2)$	$(2, 2)$	$Y_{2,2}^1$
$(A_k, 1, 1)$	–	$X_{1,k+1}$
$(A_k, 2)$	q	$Y_{k+1,q}^1$
(A_k, A_l)	–	$Y_{k+1,l+1}^1$

We apply this to our case. We have

$$D \cap L = V(f_{13}f_{14}f_{23}f_{24}, x).$$

By Lemma 1.4 the linear forms x, f_{13}, f_{14} resp. x, f_{23}, f_{24} are linear independent; hence D and L may only intersect in the configuration given in the above table. Note that

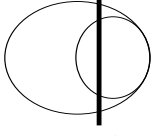
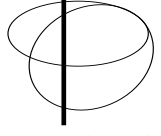
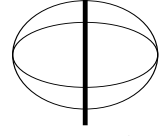
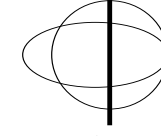
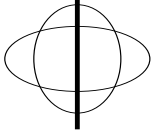
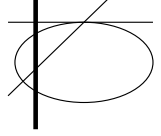
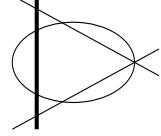
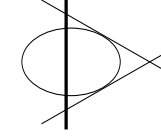
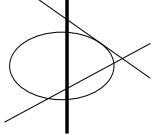
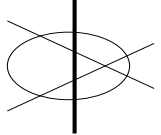
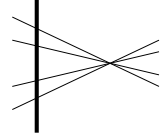
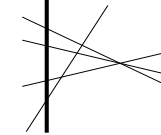
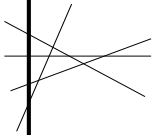
$$Q \cap L = V((f_{13}f_{24} - f_{14}f_{23})^2, x);$$

thus the exceptional singularities are of type A_1 or A_3 if the corresponding multiple points do not lie on D . We treat the case of the different intersection configuration of D and L separately.

Case (1,1,1,1). Our main singularity is an $X_{1,0}$. Since $f_{13}, f_{14}, f_{23}, f_{24}$ have pairwise distinct zeros on L , $Q \cap L$ and $D \cap L$ cannot have a common multiple zero, thus we can have either an A_3 or two A_1 s as exceptional singularities. Further, we can change (f_{14}, f_{24}) to $(\lambda f_{14}, \lambda^{-1} f_{24})$ with $\lambda \in \mathbb{C}^*$ without changing the equation of D , but $Q \cap L$ changes to $V((f_{13}f_{24} - \lambda^2 f_{14}f_{23})^2, x)$. This restricted pencil for $\lambda^2 \in \mathbb{C}^*$ contains a quadruple point, because the complete

pencil with $\lambda^2 \in \mathbb{P}^1$ does and we can exclude $\lambda = 0, \infty$. Therefore, we can always have two A_1 as well as an A_3 as exceptional singularities.

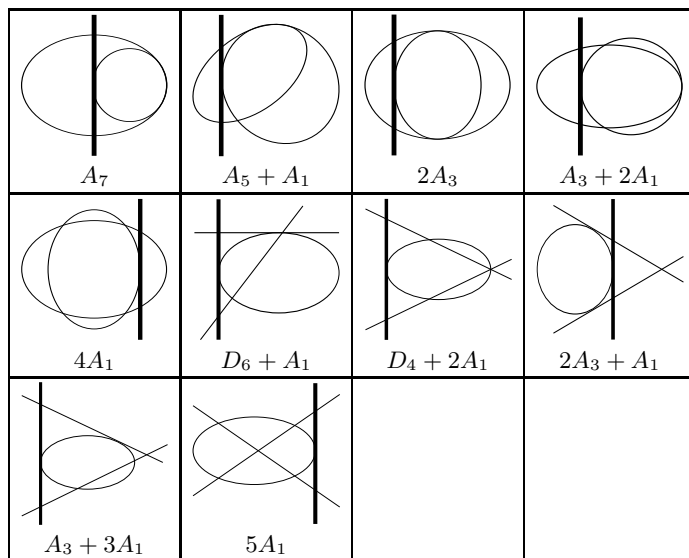
Finally, in the following table we sketch all singularities which can occur on a quartic D which is the union of two conics and list which singularities — apart from the exceptional ones — the surface F has in the corresponding case. The fat line represents L .

			
$X_{1,0} + A_7$	$X_{1,0} + A_5 + A_1$	$X_{1,0} + 2A_3$	$X_{1,0} + A_3 + 2A_1$
			
$X_{1,0} + 4A_1$	$X_{1,0} + D_6 + A_1$	$X_{1,0} + D_4 + 2A_1$	$X_{1,0} + 2A_3 + A_1$
			
$X_{1,0} + A_3 + 3A_1$	$X_{1,0} + 5A_1$	$2X_{1,0}$	$X_{1,0} + D_4 + 3A_1$
			
$X_{1,0} + 6A_1$			

Case (2,1,1). Since L intersects D tangentially, one of the two conics C_1, C_2 — say C_1 — must be smooth and f_{13} and f_{23} are proportional modulo x , i.e., there exist $\alpha \in \mathbb{C}^*, \beta \in \mathbb{C}$ with $f_{23} = \alpha f_{13} + \beta x$. It follows that

$$D \cap L = V(f_{13}^2 f_{14} f_{24}, x) \quad \text{and} \quad Q \cap L = V(f_{13}^2 (f_{24} - \alpha f_{14})^2, x)$$

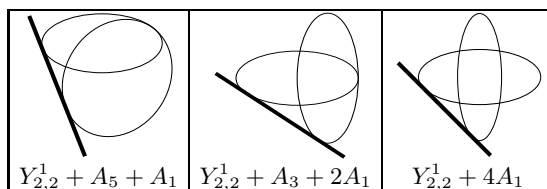
have a common double point. This point may become a quadruple point of $Q \cap L$ and thus the main singularity is either an $X_{1,4}$ or an $X_{1,2}$ in the latter case there exist an exceptional singularity of type A_1 . With a similar argument as in the case before, the condition that $Q \cap L$ has a quadruple point is seen to be independent of the equation of D . We list the possible singularities F has apart from $X_{1,4}$ or $X_{1,2} + A_1$ in dependence of the shape of D :



Case (2,2). Because of the two tangential intersections of L and D , both conics C_1, C_2 must be smooth and f_{13} and f_{23} as well as f_{14} and f_{24} are proportional modulo x ; hence

$$D \cap L = V(f_{13}^2 f_{14}^2, x) \quad \text{and} \quad Q \cap L = V(f_{13}^2 f_{14}^2, x)$$

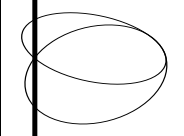
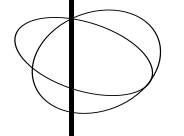
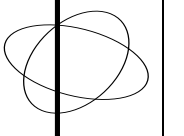
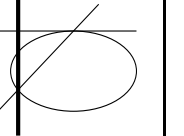
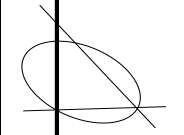
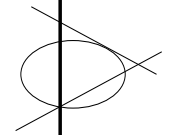
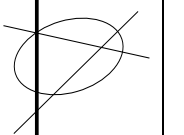
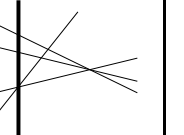
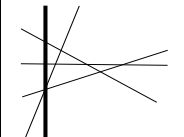
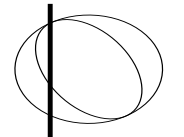
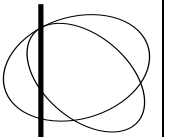
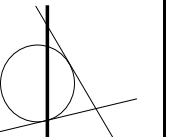
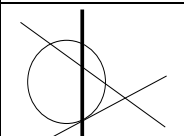
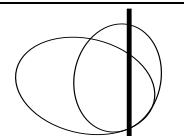
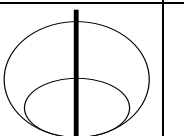
are the same divisor with two double points. Therefore, our main singularity is a $Y_{2,2}^1$ and there are no exceptional singularities. The singularities of F in dependence of the shape of D can be read off the following table:



Case $(A_k, 1, 1)$. Here D has an A_k -singularity on L where k is necessarily odd. Its two branches belong to C_1 and C_2 . Namely, if both branches belong to C_1 , i.e., f_{13} and f_{23} are proportional modulo x , then the singular point of D would belong to $Q \cap L$, and F would have a nonisolated singularity. Remembering that f_{13} and f_{14} resp. f_{23} and f_{24} are also not proportional modulo x , we find that $f_{24} = \alpha f_{13} + \beta x$ for some $\alpha \in \mathbb{C}^*, \beta \in \mathbb{C}$ (or the same with the indices 1 and 2 exchanged) and

$$D \cap L = V(f_{13}^2 f_{14} f_{23}, x) \quad \text{and} \quad Q \cap L = V((\alpha f_{13}^2 - f_{14} f_{23})^2, x).$$

Thus $D \cap L$ and $Q \cap L$ cannot have a common multiple point. Our usual argument that $Q \cap L$ may have two double points as well as a quadruple point without changing the equation of D can be adapted to this case as well. Thus we can always have one A_3 as well as two A_1 as exceptional singularities. It remains to list all the singularities of F apart from the exceptional ones depending on the shape of D :

 $X_{1,2} + A_5$	 $X_{1,2} + A_3 + A_1$	 $X_{1,2} + 3A_1$	 $X_{1,2} + D_6$
 $X_{1,2} + D_4 + A_1$	 $X_{1,2} + A_3 + 2A_1$	 $X_{1,2} + 4A_1$	 $X_{1,2} + D_4 + 2A_1$
 $X_{1,2} + 5A_1$	 $X_{1,4} + A_3$	 $X_{1,4} + 2A_1$	 $X_{1,4} + A_3 + A_1$
 $X_{1,4} + 3A_1$	 $X_{1,6} + A_1$	 $X_{1,8}$	

Cases $(A_k, 2)$ and (A_k, A_l) . We have seen above that an A_k -singularity of D on L can only occur as the intersection of both C_1 and C_2 ; hence, no further tangential intersection of L and C_1 or C_2 is possible, i.e., the case $(A_k, 2)$ does not occur. In the (A_k, A_l) -case we obtain $f_{23} = \alpha_1 f_{14} + \beta_1 x$ and $f_{24} = \alpha_2 f_{13} + \beta_2 x$ for some $\alpha_1, \alpha_2 \in \mathbb{C}^*$ and $\beta_1, \beta_2 \in \mathbb{C}$; thus

$$D \cap L = V(f_{13}^2 f_{14}^2, x) \quad \text{and}$$

$$Q \cap L = V((\sqrt{\alpha_2} f_{13} + \sqrt{\alpha_1} f_{14})^2 (\sqrt{\alpha_2} f_{13} - \sqrt{\alpha_1} f_{14})^2, x).$$

Therefore $Q \cap L$ has always two double points outside $D \cap L$, i.e., we have two exceptional A_1 -singularities. We list the remaining singularities of F according to the shape of D :

□

3.3 Linear symmetric quartics with a triple point

Similar to the above case we obtain

Theorem 3.10 *Only the following combinations of singularities occur on a normal linear symmetric quartic with a triple point:*

$$\begin{aligned}
& T_{3,3,3} + A_{11}, T_{3,3,5} + A_9, T_{3,3,7} + A_7, T_{3,3,9} + A_5, T_{3,3,11} + A_3, T_{3,3,13} + A_1, \\
& T_{3,3,15}, T_{3,5,5} + A_5 + A_1, T_{3,5,7} + A_5, T_{3,5,9} + 2A_1, T_{3,5,11} + A_1, T_{3,7,7} + A_3, \\
& T_{3,7,9} + A_1, T_{3,7,11}, T_{5,5,5} + 3A_1, T_{5,5,7} + 2A_1, T_{5,7,7} + A_1, T_{7,7,7}, \\
& T_{3,3,4} + A_{11}, T_{3,4,4} + A_3 + A_7, T_{4,4,4} + 3A_3, \\
& Q_{11} + A_9, S_{1,0} + A_5 + A_1, S_{1,2}^\# + A_5, S_{1,4}^\# + 2A_1, \text{ and } S_{1,6}^\# + A_1
\end{aligned}$$

Hereby, the A_{2k+1} -singularities can be splitted in the same manner as A_{2k+1}^\bullet before.

Proof. Let $p = (1 : 0 : 0)$ be the triple point of the quartic F . From the first part of the proof of Theorem 3.9 it follows that the rank of A_0 is 1; hence, we choose a basis of \mathbb{C}^4 such that

$$A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

in a (3,1)-blocking. Then the expansion of $F = \det(wA_0 + xA_1 + yA_2 + zA_3)$ with respect to w is $F = wP + Q$ where Q is the determinant of the matrix $A_{123} = xA_1 + yA_2 + zA_3$ and P is the upper left 3×3 -minor of the same matrix; hence, we consider A_{123} as a matrix representation of the curve Q , and P is one of the contact curves of Q , i.e., all intersection multiplicities are even. In order to determine the singularities of F we quote the following results of Degtyarev [D, §4]:

- The point p is an isolated singularity of F only if P and Q have no common singularities.

- Apart from the triple point the normal surface F has only A_{r-1} -singularities which are in one-to-one correspondence with points of r -fold intersection of P and Q at smooth points of P .
- The type of the double point of F is determined as follows: For a singular point S_i of P let r_i be the intersection multiplicity of P and Q in S_i , then we have the following correspondence

singularities of P	triple point of F
—	$P_8 = T_{3,3,3}$
A_1	$P_{5+q_1} = T_{3,3,q_1}$, where $q_i = \max\{4, 3 + r_i\}$
$2A_1$	$R_{q_1,q_2} = T_{3,q_1,q_2}$
$3A_1$	T_{q_1,q_2,q_3}
A_2	Q_{10}, Q_{11}, Q_{12} for $r_1 = 0, 2$, and 3 respectively.
A_3	$S_{11}, S_{12}, S_{1,0}$ for $r_1 = 0, 2$, and 4 respectively.
	$S_{1,r_1-4}^\#$ for $r_1 > 4$.

Now we have to analyze our linear symmetric quartic F for the different possibilities of P . For the case of a smooth cubic P we can use abstract arguments involving the Jacobian of P and the theory of contact curves; for singular P we have to analyze the equations using the determinantal representations of P in Appendix A.

Case: P smooth cubic. Since P and Q have contact, the results of Degtyarev say that F has a $P_8 = T_{3,3,3}$ singularity and a combination of A_k -singularities, which is a splitting of A_{11} in the usual way. To show that any such splitting is possible, we have to prove that for any partition $\sum m_i = 6$, $m_i \in \mathbb{N}$, of 6 , there exists a quartic Q which intersects P with the multiplicities $(2m_i)$. We compute in the Jacobian of the smooth cubic P , which is isomorphic to P . We think of the Jacobian as a complex torus given as the warp around of a parallelogram inside \mathbb{C} determined by the numbers $1, \tau \in \mathbb{C} \setminus \mathbb{R}$. Clearly, we can find pairwise distinct points q_i in the interior of the parallelogram given by $\frac{1}{2}, \frac{\tau}{2}$ such that $\sum m_i q_i = \frac{\tau+1}{2}$ in \mathbb{C} . Then $\sum 2m_i q_i = 0$ in the Jacobian of P , i.e., $\sum 2m_i q_i$ is a principal divisor. Due to our choice of the q_i , all proper nontrivial subcombinations $\sum n_i q_i$, $0 \leq n_i \leq m_i$, are not principal. Since a plane cubic is projectively normal and $\deg \sum 2m_i q_i = 3 \cdot 4$, there is a quartic Q with $P \cap Q = \sum 2m_i q_i$. Now it remains to show that there is a linear symmetric matrix representation of Q such that the top 3×3 -minor of this matrix is P , because we can obtain a matrix representation of F from this matrix by adding w to the bottom right entry. Since P is smooth, we find a self-linked ideal I with respect to (P, Q) , i.e., $(P, Q) : I = I$ [M-B, Prop. 4.3]. Note that I does not contain a quadric polynomial. Namely, if $G \in I$ with $\deg G = 2$ then $G^2 \in (P, Q)$, i.e., $G^2 = \lambda Q + LP$ with $\lambda \in \mathbb{C}^*$ and $L \in \mathbb{C}[x, y, z]_1$. Hence, we would have $2G \cap P = Q \cap P$ and $G \cap P$ would be a principle subdivisor of $Q \cap P$ on P , which is impossible by the construction of Q . By [Be, 2.4] or [M-B, §4] such a self-linked ideal induces a linear symmetric matrix representation of Q with a contact cubic P . After a change of basis we may assume that P is the upper left 3×3 -minor of this matrix. In fact, knowing that such a matrix exists, it can be easily constructed by Dixon's method [Di].

Case: P nodal cubic. From Appendix A we know that up to a choice of basis there are only two different linear symmetric matrix representations of a nodal

cubic $P = x^3 + y^3 + xyz$, i.e., for a matrix M with $F = \det M$ we may assume that

$$M = \begin{pmatrix} -y & 0 & x & a_y y + a_z z \\ 0 & -x & y & b_y y + b_z z \\ x & y & z & c_y y + c_z z \\ a_y y + a_z z & b_y y + b_z z & c_y y + c_z z & w + f \end{pmatrix} \quad \text{or}$$

$$M' = \begin{pmatrix} -y & \frac{1}{2}z & x & a_y y + a_z z \\ \frac{1}{2}z & -x & y & b_y y + b_z z \\ x & y & 0 & c_y y + c_z z \\ a_y y + a_z z & b_y y + b_z z & c_y y + c_z z & w + f \end{pmatrix},$$

where $f \in \mathbb{C}[x, y, z]$. The variable x was eliminated from the last column by adding a suitable multiples of the first three columns. Now one can compute Q and Q' and obtains that $(a_z, b_z) \neq 0$ resp. $c_z \neq 0$ for Q resp. Q' to be smooth at the singular point $(0 : 0 : 1)$ of P . To compute the intersection multiplicities of P and the quartics, we choose the parameterization $(s : t) \mapsto (-s^2 t : st^2 : (t - s)(s^2 + st + t^2))$ of P which maps $(1 : 0)$ and $(0 : 1)$ to the singular point of P . Plugging it into the quartics we get

$$st(a_z s^5 - c_z s^4 t + (b_z - a_y)s^3 t^2 + (c_y - a_z)s^2 t^3 + (c_z - b_y)st^4 - b_z t^5)^2$$

resp.

$$\left(-\frac{1}{2}c_z s^6 + b_z s^5 t + (a_z + \frac{1}{2}c_y)s^4 t^2 - b_y s^3 t^3 - (b_z + a_y)s^2 t^4 - (a_z - \frac{1}{2}c_y)st^5 + \frac{1}{2}c_z t^6\right)^2.$$

In the first case the polynomial of degree 5 is arbitrary apart from $(a_z, b_z) \neq 0$, and we can distribute its zeros arbitrarily with the exception that we cannot have zeros at $(1 : 0)$ and $(0 : 1)$ at the same time. In the second case the sextic polynomial can also have any combination of multiple zeros, but none of the zeros can be at the points that map to the singular point of P because of $c_z \neq 0$. Therefore, by Degtyarev's results we get the following possible combinations of singularities together with the usual splitting of the A -singularities:

$$T_{3,3,5+2k} + A_{9-2k}, \text{ for } k \in \{0, \dots, 4\}, T_{3,3,15}, \text{ and } T_{3,3,4} + A_{11}.$$

Case: P smooth quadric+secant. Again there are two linear symmetric matrix representations of $P = x(x^2 + yz)$, thus we may assume that

$$M = \begin{pmatrix} y & 0 & x & a_y y + a_z z \\ 0 & -x & 0 & b_y y + b_z z \\ x & 0 & -z & c_y y + c_z z \\ a_y y + a_z z & b_y y + b_z z & c_y y + c_z z & w + f \end{pmatrix} \quad \text{or}$$

$$M' = \begin{pmatrix} 0 & y & x & a_y y + a_z z \\ y & -x & \frac{1}{2}z & b_y y + b_z z \\ x & \frac{1}{2}z & 0 & c_y y + c_z z \\ a_y y + a_z z & b_y y + b_z z & c_y y + c_z z & w + f \end{pmatrix}.$$

The singularities of P are $(0 : 1 : 0)$ and $(0 : 0 : 1)$. Since Q resp. Q' must be smooth at these points, we find that $(a_z, b_z) \neq 0$ and $(b_y, c_y) \neq 0$ resp. $a_z \neq 0$

and $c_y \neq 0$. We parameterize the secant of P by $(s : t) \mapsto (0 : s : t)$ and the quadric by $(s : t) \mapsto (st : -s^2 : t^2)$, thus in both cases $(1 : 0)$ and $(0 : 1)$ map to the singular points of P . To compute the intersection multiplicities of P and the quartics Q and Q' , we pull the quartics back via these parameterizations and obtain

$$st(b_ys + b_zt)^2 \quad \text{and} \quad -st(c_ys^3 + a_ys^2t - c_zst^2 - a_zt^3)^2$$

resp.

$$(c_ys^2 + (c_z - \frac{1}{2}a_y)st - \frac{1}{2}a_zt^2)^2 \quad \text{and}$$

$$(c_ys^4 + b_ys^3t - (c_z + \frac{1}{2}a_y)s^2t^2 - b_zst^3 + \frac{1}{2}a_zt^4)^2.$$

In the first case we can distribute the zeros arbitrarily with the exception that not both, the linear and the cubic polynomial, have zeros at points which are mapped to the same singular point of P . In the second case we cannot have zeros at the points that are mapped to the singular points of P , but any combination of multiple zeros can occur; hence F can have the following combinations of singularities with the usual splitting of the A -singularities:

$$T_{3,5,5} + A_5 + A_1, T_{3,5,7} + A_5, T_{3,5,9} + 2A_1, T_{3,5,11} + A_1, T_{3,7,7} + A_3,$$

$$T_{3,7,9} + A_1, T_{3,7,11}, \text{ and } T_{3,4,4} + A_3 + A_7.$$

Case: P three noncongruent lines. This is the last case where there are two nonequivalent linear symmetric matrix representations of the cubic. We take P as xyz and may assume that

$$M = \begin{pmatrix} x & 0 & 0 & a_yy + a_zz \\ 0 & y & 0 & b_xx + b_zz \\ 0 & 0 & z & c_xx + c_yy \\ a_yy + a_zz & b_xx + b_zz & c_xx + c_yy & w + f \end{pmatrix} \quad \text{or}$$

$$M' = \begin{pmatrix} 0 & x & \frac{1}{2}y & a_zz \\ x & 0 & z & b_yy + b_zz \\ \frac{1}{2}y & z & 0 & c_xx + c_yy + c_zz \\ a_zz & b_yy + b_zz & c_xx + c_yy + c_zz & w + f \end{pmatrix}.$$

In order for Q resp. Q' to be smooth at the singular points of P , we must have $(a_z, b_z) \neq 0$, $(a_y, c_y) \neq 0$, and $(b_x, c_x) \neq 0$ resp. $a_z \neq 0$, $b_y \neq 0$, and $c_x \neq 0$. We use the parameterizations $(s : t) \mapsto (0 : s : t)$, $(s : t) \mapsto (t : 0 : s)$, and $(s : t) \mapsto (s : t : 0)$, which map $(1 : 0)$ and $(0 : 1)$ to the singular points of P . Pulling Q and Q' back via these mappings gives

$$-st(a_ys + a_zt)^2, \quad -st(b_zs + b_xt)^2, \quad \text{and} \quad -st(c_xs + c_yt)^2$$

resp.

$$(\frac{1}{2}b_ys^2 + \frac{1}{2}b_zst - a_zt^2)^2, \quad (a_zs^2 - c_zst - c_xt^2)^2, \quad \text{and} \quad (c_xs^2 + c_yst - \frac{1}{2}b_yt^2)^2.$$

The above inequalities imply in the first case that the linear forms can only contribute to the intersection multiplicities of $P \cap Q$ at different singular points

of P and in the second case that the quadrics cannot have zeros at the points that map to the singular points of P . Therefore, F can have the following combinations of singularities with the usual splitting of the A -singularities:

$$T_{5,5,5} + 3A_1, T_{5,5,7} + 2A_1, T_{5,7,7} + A_1, T_{7,7,7}, \text{ and } T_{4,4,4} + 3A_3.$$

Case: P cuspidal cubic. Up to a choice of coordinates there is only one representation of $P = x^3 + yz^2$ as a linear symmetric determinant; hence we may assume that

$$M = \begin{pmatrix} -y & 0 & x & a_y y + a_z z \\ 0 & -x & z & b_y y + b_z z \\ x & z & 0 & c_y y + c_z z \\ a_y y + a_z z & b_y y + b_z z & c_y y + c_z z & w + f \end{pmatrix}.$$

The condition that Q has no singular point at the singular point $(0 : 1 : 0)$ of P turns out to be $c_y \neq 0$. Plugging the parameterization $(s : t) \mapsto (t^2 s : -s^3 : t^3)$ of P into Q gives

$$t^2 (c_y s^5 + b_y s^4 t - a_y s^3 t^2 - c_z s^2 t^3 - b_z s t^4 + a_z t^5)^2.$$

Because of $c_y \neq 0$, the intersection multiplicity of P and Q is always 2 in the singular point of P , otherwise the quintic is arbitrary; hence F can have $Q_{11} + A_9$ as singularities as well as any combination of singularities obtained by splitting A_9 in the usual way.

Case: P smooth quadric+tangent. As there is only one linear symmetric matrix representation of $P = z(x^2 + yz)$, we may assume that

$$M = \begin{pmatrix} y & x & 0 & a_y y + a_z z \\ x & -z & 0 & b_y y + b_z z \\ 0 & 0 & -z & c_x x + c_y y \\ a_y y + a_z z & b_y y + b_z z & c_x x + c_y y & w + f \end{pmatrix}.$$

The meeting point of the quartic and the tangent is the singular point $(0 : 1 : 0)$ of P . For Q to be nonsingular at this point means $b_y^2 + c_y^2 \neq 0$. To compute the intersection multiplicities of P and Q , we parameterize the line and the quadric of P by $(s : t) \mapsto (t : s : 0)$ and $(s : t) \mapsto (st : s^2 : -t^2)$. Both parameterizations map $(1 : 0)$ to the singular point of P . Pulling Q back via these parameterizations yields

$$t^2 (c_y s + c_x t)^2 \text{ and } -t^2 (b_y s^3 - a_y s^2 t - b_z s t^2 + a_z t^3)^2.$$

Since $b_y^2 + c_y^2 \neq 0$, not both terms can contribute further to the intersection multiplicities of P and Q at the singular point of F at the same time; therefore F can have the following combinations of singularities with the usual splitting of the A -singularities:

$$S_{1,0} + A_5 + A_1, S_{1,2}^\# + A_5, S_{1,4}^\# + 2A_1, \text{ and } S_{1,6}^\# + A_1.$$

Case: P three congruent lines, double line+line, triple line, empty set. All these cases lead to nonnormal quartics. Since a linear symmetric matrix representation \tilde{M} of three congruent lines involves only the variables x and y , the rank of a 4×4 -matrix M with \tilde{M} in the upper left corner is only 2 along the line $\{x = y = 0\}$. Therefore, F is singular along this line. For the remaining cases one can apply Lemma 1.4 after a reshuffling of coordinates. \square

A Linear symmetric matrix representations of plane cubics

Finding linear symmetric matrix representations of plane cubics is a classical problem. The three representations of a smooth cubic were found by Hesse [He]. The matrix representations of the singular cubics are scattered throughout the literature. Most of them were computed by Barth [B] and Meyer-Brandis [M-B]. The case of the empty cubic is a special case of Atkinson [A]. For the following complete list the representation matrices of the singular cubics were computed using the straight-forward method of Barth or Taussky [T]. The remarkable fact is that a reduced singular cubic has two nonequivalent representations if it has only A_1 -singularities and only one representation if it has another singularity. In the last column the number of accidental singularities is written down, i.e., the number of points of \mathbb{P}^2 where the matrix has only rank 1. This number distinguishes the two representations of the cubics with A_1 -singularities.

cubic	equation	representation	number of rank 1 parameter ranges
smooth	$x^3 + y^3 + z^3 - \lambda xyz$	$\frac{-1}{\mu} \begin{pmatrix} \mu x & z & y \\ z & \mu y & x \\ y & x & \mu z \end{pmatrix}$	$\frac{0}{\mu^2 + 2\mu^{-1} = \lambda}$
nodal	$x^3 + y^3 + xyz$	$\begin{pmatrix} -y & \frac{1}{2}z & x \\ \frac{1}{2}z & -x & y \\ x & y & 0 \end{pmatrix}$	0
		$\begin{pmatrix} -y & 0 & x \\ 0 & -x & y \\ x & y & z \end{pmatrix}$	1
quadric+secant	$x(x^2 + yz)$	$\begin{pmatrix} 0 & y & x \\ y & -x & \frac{1}{2}z \\ x & \frac{1}{2}z & 0 \end{pmatrix}$	0
		$\begin{pmatrix} y & 0 & x \\ 0 & -x & 0 \\ x & 0 & -z \end{pmatrix}$	2
3 lines	xyz	$\begin{pmatrix} 0 & x & \frac{1}{2}y \\ x & 0 & z \\ \frac{1}{2}y & z & 0 \end{pmatrix}$	0
		$\begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}$	3
cuspidal	$x^3 + yz^2$	$\begin{pmatrix} -y & 0 & x \\ 0 & -x & z \\ x & z & 0 \end{pmatrix}$	1
quadric+tangent	$z(x^2 + yz)$	$\begin{pmatrix} y & x & 0 \\ x & -z & 0 \\ 0 & 0 & -z \end{pmatrix}$	1
3 congruent lines	$x(x^2 + y^2)$	$\begin{pmatrix} 0 & y & x \\ y & -x & \frac{1}{2}y \\ x & \frac{1}{2}y & 0 \end{pmatrix}$	1
double line+line	x^2y	$\begin{pmatrix} az & x & bz \\ x & 0 & 0 \\ bz & 0 & -y \end{pmatrix}$	$\frac{1 \text{ or line}}{a, b \in \mathbb{C}}$
triple line	x^3	$\begin{pmatrix} az & by & x \\ by & -x & 0 \\ x & 0 & 0 \end{pmatrix}$	$\frac{0}{a, b \in \{0, 1\}}$
empty cubic	\emptyset	$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{pmatrix}$	---
empty cubic	\emptyset	$\begin{pmatrix} * & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}$	---

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