Developable Varieties of Gauss rank 2

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Abstract

We classify projective varieties whose Gauss map has a two dimensional image. A global geometric description of the seven types is given.
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For an n-dimensional projective variety $X \subset \mathbb{P}^N$, its rational Gauss map

$$\gamma : X \dashrightarrow \mathbb{G}(n, N), \quad x \mapsto \mathbb{T}_x X,$$

assigns to every smooth point x of X its embedded tangent space $\mathbb{T}_x X$ as a point of the respective Grassmannian. In general, the dimension of the image of the Gauss map, the Gauss rank r, is n; if this is not the case, X is called developable.

The developable varieties of Gauss rank 1 are classically known [4, 6, 5]; they are the osculating scrolls, i.e. the union of the osculating k-planes over a curve, or cones over curves or osculating scrolls. For developable varieties of higher Gauss rank, Griffiths and Harris showed that they are a union of two-dimensional cones or tangent scrolls [6]. Akivis and Goldberg improved this result by showing that any developable variety is foliated by cones, osculating scrolls, or developable hypersurfaces [1]. Their paper also contains a historical account and an extensive bibliography. Akivis, Goldberg and Landsberg gave a global geometric description of developable hypersurfaces of Gauss rank 2 in $\mathbb{P}^4$ [3]. Five different types were found. Here we extend their result to developable varieties of Gauss rank 2 in general. There will be seven types. Before we describe them, we recall some facts.

The fundamental result about developable varieties is that the general fiber F of the Gauss map is a linear space of dimension $d = n - r$. On each of these fibers, there exists a hypersurface of degree r, called the focal hypersurface, along which X is singular. The closure of the union of the these focal hypersurfaces is the focal variety X_f of X. The developable varieties of Gauss rank 2 will be classified by the codimension c of the focal variety in X and the rank of s of the focal quadric in the general Gauss fiber. Only seven pairs (c, s) are possible: $(1, s)$, $(2, s)$, and $(3, 1)$ with $s \in \{1, 2, 3\}$.

From the differential of the Gauss map γ, one obtains the second fundamental form

$$\mathrm{II}_x : \mathrm{Sym}^2 T_x X \longrightarrow N_x$$

which is a symmetric bilinear map from the tangent space $T_x X \cong \mathbb{T}_x X / x$ to the normal space. The singular locus of II_x is precisely the Gauss fiber F through x.

A k-plane $L \subset \mathbb{T}_x X$ is called asymptotic if it contains the Gauss fiber and $\mathrm{I\!I}_x$ vanishes on L. Further, a k-plane $L \subset \mathbb{T}_x X$ and a $(n-k+d)$-plane $M \subset \mathbb{T}_x X$ are conjugate if both contain the Gauss fiber, together they span the tangent space, and $\mathrm{I\!I}_x(L, M) = 0$. We will call L a conjugate plane if there exists a unique plane M which is conjugate to it. Finally, we will sometimes view the second fundamental form dually as a map $\mathrm{I\!I}_x^* : N_x^* \to \mathrm{Sym}^2 T_x X^*$; thus we may speak of the linear system of quadrics of the second fundamental form.

Now we can state the result of this article:

Structure Theorem. *Let $X \subset \mathbb{P}^N$ be a developable variety of Gauss rank 2. Then X is of one of the seven following types distinguished by the pairs (c, s) where c is the codimension of the focal variety in X and s is the rank of a general focal quadric.*

Type (1,1) *Here the focal variety X_f of X has Gauss rank 2. It has a unique asymptotic d-plane in a general tangent space and X is the union of those. If X_f is developable, it is of type (1,1) itself or of type (3,1).*

Type (1,2) *The focal variety X_f has a $(d+1)$-dimensional component Y of Gauss rank 2, which has two conjugate d-planes in a general tangent space. X is the union of a family of those. If Y is developable, it is of type (1,2), (2,2), or (3,1).*

Type (1,3) *X is a hypersurface inside a $\mathbb{P}^{n+1} \subseteq \mathbb{P}^N$. The focal variety $X_f \subset \mathbb{P}^{n+1}$ has Gauss rank 3. The pencil of quadrics of its second fundamental form is generated by a quadric of rank 3 and a double tangent plane to this quadric. The variety X can be recovered from X_f as the union of these d-dimensional double tangent planes.*

Type (2,1) *The geometry of X can be described in the following way. There exist*

1. *two curves C_1 and C_2 in $\mathbb{P}^N$, possibly $C_1 = C_2$, and a correspondence between them*
2. *two numbers $0 \leq a_1, a_2 \leq d$ with $1 \leq a_1 + a_2 \leq d$*
3. *a linear subspace $L \subset \mathbb{P}^N$ of dimension $d - 1 - a_1 - a_2$*

such that X is the union of the one-dimensional family of $(d+1)$–planes spanned by L and the a_k-th osculating planes to C_k at corresponding points for $k = 1, 2$.

Type (2,2) *X is a cone over the join of two curves C_1 and C_2, possibly $C_1 = C_2$.*

Type (2,3) *X is the secant variety of the Veronese surface or a cone over it.*

Type (3,1) *X is a cone over a nondevelopable surface.*

For the developable varieties of type (1,1) the recursion in the above description can be solved; for the ones of type (1,2) this is only partially possible, see section 2.

The developable varieties of type $(1,3)$ and $(2,3)$ cannot exist in $\mathbb{P}^4$; thus the specialization of the remaining five types were found in the classification of the developable varieties in $\mathbb{P}^4$ by Akivis, Goldberg, and Landsberg [3]. The description of the varieties of type (2,1) as the union of a one-dimensional family of 2-planes containing the tangent lines to a curve was only a conjecture there.

After the finishing of this article, Goldberg pointed out to the author the paper [10] of Savel'ev, which contains — without a proof — a description of Gauss rank 2 varieties which are not hypersurfaces, i.e the types with $s \leq 2$. However, the varieties of type (2,1) are described there only as a union of a one-dimensional family of $(n-1)$-dimensional cones with vertices of dimension less than or equal to $[(n-1)/2]$ such that the union of these vertices is a developable variety or a cone. Savel'ev does not discuss focal varieties, and it appears that he approaches the problem through the analysis of the second fundamental form.

Let us consider varieties that are smooth outside the hyperplane at infinity, i.e. complete developable manifolds in $\mathbb{C}^N$. Then only two types are possible: The cones of type (3,1) with vertices in the hyperplane at infinity and the varieties of type (2,1) where $a_1 = 0$ and C_2 as well as L lie in the hyperplane at infinity [9, 11, 12].

The above geometric descriptions can be used to construct developable varieties of the respective types; for the types (1,1), (1,2), (1,3) one first has to construct a variety Y, which has the properties described above for the focal variety X_f and then take the union of the appropriate d-planes. By using the Cartan test, one can show that the local constructions of Y depends on two functions of two variables for the types (1,1) and (1,2) and on $d+3$ functions of two variables for the type (1,3). In contrast to this, the construction of developable varieties with a 2-codimensional focal variety depend only on several functions of one variable or on several constants in case of type (2,3).

Finally, on a "moduli space" of developable varieties of Gauss rank 2, the dimension of the focal variety and the rank of the generic focal quadric will be lower semi-continuous functions. Therefore, the general developable hypersurface of Gauss rank 2 in $\mathbb{P}^N$, $N \geq 5$, will be of type (1,3) and in $\mathbb{P}^4$ of type (1,2). Further, the general developable variety of Gauss rank 2 which is not a hypersurface will be of type (1,2) as well.

1 Classification

We examine the developable varieties by using Cartan's moving frame method. Here we recall some facts, in order to fix the notations. For a complete introduction see section 4 and 5 of [8] or [2]. On the projective space $\mathbb{P}^N$, we have the bundle of projective frames $\mathcal{F}$, consisting of bases $(e_0, \ldots, e_N)$ of $\mathbb{C}^{N+1}$. The infinitesimal motion of the frame is described by

$$de_A = \omega_A^0 e_0 + \ldots + \omega_A^N e_N \quad \text{for } 0 \leq A \leq N,$$

where the ω_A^B are the Maurer-Cartan 1-forms on $\mathrm{GL}(\mathbb{C}^{N+1})$, which fulfill the Maurer-Cartan equation

$$d\omega_B^A = -\omega_C^A \wedge \omega_B^C.$$

To study the geometry of a developable variety, we work only on the submanifold of the projective frame bundle where the general frame has the following properties:

$\{e_0\}$	is a point of X,
$\{e_0, \ldots, e_d\}$	is the Gauss fiber F of X through $\{e_0\}$,
$\{e_0, \ldots, e_n\}$	is the tangent space of X in $\{e_0\}$.

Our adaptions to the geometry of X have the effect that

$$\begin{aligned} de_0 &= \omega^0 e_0 + \omega^\delta e_\delta + \omega^i e_i \\ de_\delta &= \omega^0_\delta e_0 + \omega^\varepsilon_\delta e_\varepsilon + \omega^i_\delta e_i \\ de_i &= \omega^0_i e_0 + \omega^\delta_i e_\delta + \omega^j_i e_j + \omega^\mu_i e_\mu \end{aligned}$$

with $\omega^A := \omega^A_0$ and the index ranges $1 \le \delta, \varepsilon \le d$, $d+1 \le i, j \le n$, and $n+1 \le \mu \le N$.

Differentiating $\omega^\mu = 0$ resp. $\omega^\mu_\delta = 0$ and using the Cartan lemma, we find functions $q^\mu_{ij}, a^i_{\delta j}$ such that

$$\omega^\mu_i = q^\mu_{ij}\omega^j, \qquad \omega^i_\delta = a^i_{\delta j}\omega^j$$

and q^μ_{ij} as well as $q^\mu_{ik}a^k_{\delta j}$ are symmetric in i, j. Setting $Q^\mu = (q^\mu_{ij})$, $A_\delta = (a^i_{\delta j})$, $\mathcal{Q} = \operatorname{span}\{Q^\mu\}$, and $\mathcal{A} = \{A_0 = \mathrm{id}, A_\delta\}$, this means that QA is symmetric for $Q \in \mathcal{Q}$ and $A \in \mathcal{A}$ [2, 4.1]

The second fundamental form of X in $x = \{e_0\}$ is the second differential of e_0 modulo the tangent space

$$\mathrm{I\!I}_x = d^2 e_0 = \omega^\mu_i \omega^i e_\mu = q^\mu_{ij}\omega^i\omega^j e_\mu \mod \{e_0, \ldots, e_n\}.$$

Since the second fundamental form is essentially the differential of the Gauss map, the singular locus of $\mathrm{I\!I}$ is the Gauss fiber $F = \{e_0, \ldots, e_d\}$.

Now we recall that X is singular along a hypersurface in a Gauss fiber F. Let $e = \lambda^0 e_0 + \lambda^\delta e_\delta \in F \subset X$ be a point of the fiber. We determine the tangent space of X at e. Since $F \subset X$, we have $F \subset \mathbb{T}_e X$; so, we may compute modulo F

$$de = (\lambda^0\omega^i + \lambda^\delta\omega^i_\delta)e_i = (\lambda^0\delta^i_j + \lambda^\delta a^i_{\delta j}\omega^j)e_i \mod \{e_0, \ldots, e_d\}.$$

The point $e \in X$ is smooth iff $\mathbb{T}_e X = \mathbb{T}_x X$. This will be the case if the matrix $\lambda^0 E_r + \lambda^\delta A_\delta$ is invertible. Note that this is a local computation; hence, the point e may in fact be a singular point if it is a point of self–intersection of X. On the other hand, points $e = \lambda^0 e_0 + \lambda^\delta e_\delta \in F$ with $\det(\lambda^0 E_r + \lambda^\delta A_\delta) = 0$ will always be singular in X. They form the degree r *focal hypersurface* of the Gauss fiber F. The closure of the union of the focal hypersurfaces is the *focal variety* $X_f \subseteq \operatorname{Sing} X$ of X.

Now we fix a general point x of a developable variety of Gauss rank 2 and study the linear systems $\mathcal{A}$ and $\mathcal{Q}$ with the help of linear algebra.

Proposition 1 *Let $\mathcal{A}$ be a linear system of endomorphisms of $\mathbb{C}^2$ which contains the identity and $\mathcal{Q}$ a linear system of symmetric bilinear forms of $\mathbb{C}^2$ with*

1. the bilinear form $Q(\,\cdot\,, A(\,\cdot\,))$ is symmetric for every $A \in \mathcal{A}$ and $Q \in \mathcal{Q}$,

2. $\operatorname{Sing} \mathcal{Q} := \{v \in \mathbb{C}^2 \mid Q(v, \mathbb{C}^2) = 0 \ \forall Q \in \mathcal{Q}\} = 0$.

Then, up to a choice of basis of $\mathbb{C}^2$, there exist the following four possibilities for the linear system $\mathcal{A}$

	$\mathcal{A}$ generated by	*$\mathcal{Q}$ generated by*
0)	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ [*, arbitrary*]
1)	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left[, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]$
2)	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left[, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]$
3)	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Here, the square brackets indicate optional matrices.

Proof. The first case is characterized by $\dim \mathcal{A} = 1$. So let us assume that $\dim \mathcal{A} = 2$ and all matrices of $\mathcal{A}$ have only one eigenvalue, then for a matrix $A_1' \in \mathcal{A} \setminus \mathbb{C}E_2$ there exists a basis of $\mathbb{C}^2$ such that

$$A_1' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}; \text{ hence } A_1 := A_1' - \lambda E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{A}.$$

The symmetry of QA_1 for $Q \in \mathcal{Q}$ implies

$$Q = \begin{pmatrix} 0 & q_{12} \\ q_{12} & q_{22} \end{pmatrix}.$$

Condition 2 ensures the existence of a matrix $Q \in \mathcal{Q}$ with $q_{12} \neq 0$. By a scaling of this matrix, we get $q_{12} = 1$. A transformation of the basis of $\mathbb{C}^2$ by $e_2 \mapsto e_2 - (q_{22}/2)e_1$ takes Q to

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and leaves A_1 fixed.

Now let us assume that $\dim \mathcal{A} = 2$ and $\mathcal{A}$ contains a matrix A_1 with two different eigenvalues. Then there exists a basis of $\mathbb{C}^2$ such that A_1 is a diagonal matrix; through scaling and subtracting a multiple of E_2, we can achieve that

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

For a matrix

$$Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix} \in \mathcal{Q},$$

the symmetry of QA_1 implies $q_{12} = 0$, i.e.

$$Q = \begin{pmatrix} q_{11} & 0 \\ 0 & q_{22} \end{pmatrix} \in \mathcal{Q}.$$

This and condition 2 imply the existence of a nonsingular matrix in the linear system $\mathcal{Q}$. A scaling of coordinates leaves E_2 and A_1 fixed and may be chosen such that this nonsingular matrix is taken to

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We turn to the case of $\dim \mathcal{A} = 3$. Since a three dimensional subspace $\mathcal{A}$ inside the four dimensional space $\mathrm{End}(\mathbb{C}^2)$ cannot lie inside the nondegenerate quadric of matrices with only one eigenvalue, there exists an $A_1 \in \mathcal{A}$ with two eigenvalues; hence, we may assume that we are in the situation at the end of the preceeding case. The symmetry of

$$QA = A \quad \text{for} \;\; Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \;\; A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathcal{A}$$

implies $a_{12} = a_{21}$, thus

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Further, the symmetry of

$$Q \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{for} \;\; Q = \begin{pmatrix} q_{11} & 0 \\ 0 & q_{22} \end{pmatrix} \in \mathcal{Q}$$

yields $q_{11} = q_{22}$.

Finally, it is easy to see that the case of $\dim \mathcal{A} = 4$ is impossible. $\square$

Now let us return to the examination of the geometry of a developable variety X of Gauss rank two. By the above proposition there are four possibilities for the linear system $\mathcal{A}$ of a general fiber of X. We will have to treat these four cases separately.

Case $\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$.

We will show that a developable variety with such a linear system $\mathcal{A}$ is a cone over a surface with a $(d-1)$–dimensional vertex. Its focal variety is the cone vertex.

We can adapt the frame such that $A_\delta = 0$; hence, $\omega^i_\delta = 0$ and the focal hypersurface of the Gauss fiber F is the two-fold linear space $\{e_1, \dots, e_d\}$. Differentiating the equations $\omega^i_\delta = 0$, we get

$$0 = d\omega^{n-1}_\delta = -\omega^{n-1} \wedge \omega^0_\delta \quad \text{and} \quad 0 = d\omega^n_\delta = -\omega^n \wedge \omega^0_\delta.$$

Since ω^{n-1} and ω^n are linearly independent, this yields $\omega^0_\delta = 0$. Therefore,

$$de_\delta = \omega^\varepsilon_\delta e_\varepsilon = 0 \mod \{e_1, \dots, e_d\},$$

and the $(d-1)$–dimensional linear subspace $L = \{e_1, \dots, e_d\}$ will be the fixed focal hypersurface of the general Gauss fiber of X; thus X is a cone over L.

Case $\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$.

We adapt the frame such that

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \ldots = A_d = 0$$

and

$$Q^{n+1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Q^{n+2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } 0, \quad Q^m = 0 \quad \text{for } m \geq n+3.$$

It is understood that all terms with indices greater than N are ignored in the computation. We see that the focal hypersurface of a general Gauss fiber is given by

$$\det(\lambda^0 A_0 + \lambda^1 A_1) = (\lambda^0)^2,$$

i.e. it is a quadric of rank 1, namely the double $(d-1)$–plane $\{e_1, \ldots, e_d\}$.

Our adaptions of the frame imply that with the index range $2 \leq \varrho \leq d$

$$\omega_1^{n-1} = \omega^n, \quad \omega_\varrho^{n-1} = 0, \quad \omega_\delta^n = 0, \quad \omega_{n-1}^{n+1} = \omega^n, \quad \text{and } \omega_{n-1}^{n+2} = 0.$$

The point e_1 is a general point of the focal variety X_f. The tangent space of X_f at e_1 is the image of

$$de_1 = \omega_1^0 e_0 + \omega_1^\varrho e_\varrho + \omega_1^{n-1} e_{n-1} \mod \{e_1\}.$$

Depending on whether $\dim X_f = d+1$ or d, the forms $\omega_1^0, \omega_1^\varrho, \omega_1^{n-1}$ will be linearly independent or not. Before treating these cases separately, we will derive an equality which is useful in both cases. We differentiate $\omega_1^n = 0$ to get

$$\begin{aligned} 0 = d\omega_1^n &= -\omega^n \wedge \omega_1^0 - \omega_{n-1}^n \wedge \omega_1^{n-1} = \omega_1^{n-1} \wedge (\omega_{n-1}^n - \omega_1^0) \\ \Rightarrow \omega_{n-1}^n &= \omega_1^0 + f\omega_1^{n-1} \quad \text{for a suitable function } f. \end{aligned}$$

Subcase: $\dim X_f = d+1$, i.e. X is of type (1,1).

We will show that in this case X is the union of the unique asymptotic d–planes of the focal variety X_f, which has Gauss rank 2.

The second fundamental form of X_f at e_1 is

$$\begin{aligned} \mathrm{I\!I}_{X_f, e_1} = d^2 e_1 &= (\omega_1^0 \omega^n + \omega_1^{n-1} \omega_{n-1}^n) e_n + \omega_1^{n-1} \omega_{n-1}^{n+1} e_{n+1} \\ &= (2\omega_1^0 \omega_1^{n-1} + f(\omega_1^{n-1})^2) e_n + (\omega_1^{n-1})^2 e_{n+1} \mod \{e_0, \ldots, e_{n-1}\}, \end{aligned}$$

where we used $\omega_{n-1}^{n+1} = \omega^n = \omega_1^{n-1}$ and $\omega_{n-1}^n = \omega_1^0 + f\omega_1^{n-1}$. The singular locus of $\mathrm{I\!I}_{X_f, e_1}$ is $\{e_1, \ldots, e_d\}$; thus X_f has Gauss rank 2. Its second fundamental form is given in the basis $(\omega_1^0, \omega_1^{n-1})$ by

$$\tilde{Q}^n = \begin{pmatrix} 0 & 1 \\ 1 & f \end{pmatrix} \quad \text{and} \quad \tilde{Q}^{n+1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix};$$

hence, $\{e_0, \ldots, e_d\}$ is the unique asymptotic d–plane in the tangent space of X_f. However, it is also the Gauss fiber of X, and the union of all these planes will be X itself.

If X_f is developable, then its linear system $\tilde{\mathcal{A}}$ is either of type 0) or of type 1) in the table of Proposition 1, because of the form of the matrices $\tilde{Q}^n$ and $\tilde{Q}^{n+1}$. We will see in the next subsection that the asymptotic d-planes of a $(d+1)$–dimensional variety of type (2,1) are contained in the variety itself; thus X_f can only be of type (1,1) or (3,1).

Subcase: $\dim X_f = d$, i.e. X is of type (2,1).

We want to show that X is the union of $(n-1)$-planes.

From $\dim X_f = d$ and knowing that X_f is the union of the $(d-1)$-planes $\{e_1, \ldots, e_d\}$, we see that the forms ω_1^0 and $\omega_1^{n-1} = \omega^n$ appearing in the expression for de_1 are linearly dependent; thus $\omega_1^0 = g\omega^n$. Together with the equation $\omega_{n-1}^n = \omega_1^0 + f\omega_1^{n-1}$, it follows that $\omega_{n-1}^n = (f+g)\omega^n$.

On X itself, we consider the distribution given by $\omega^n = 0$. Since

$$\begin{aligned} d\omega^n &= -\omega^n \wedge \omega^0 - \omega_{n-1}^n \wedge \omega^{n-1} - \omega_n^n \wedge \omega^n \\ &= -\omega^n \wedge \omega^0 - (f+g)\omega^n \wedge \omega^{n-1} - \omega_n^n \wedge \omega^n = 0 \mod \{\omega^n\}, \end{aligned}$$

this distribution is completely integrable by the theorem of Frobenius.

Let L be an integral manifold of this distribution. We want to show that L is a $(d+1)$–plane. It is enough to show that the second fundamental form of L vanishes. On L we have

$$\begin{aligned} de_0 &= \omega^\delta e_\delta + \omega^{n-1} e_{n-1} \mod \{e_0, \omega^n\} \\ \mathrm{II}_{L,e_0} &= d^2 e_0 = \omega^{n-1}(\omega_{n-1}^n e_n + \omega_{n-1}^{n+1} e_{n+1}) \\ &= \omega^{n-1}((f+g)\omega^n e_n + \omega^n e_{n+1}) = 0 \mod \{e_0, \ldots, e_{n-1}, \omega^n\}. \end{aligned}$$

This shows that X is the union of a one-dimensional family of $(d+1)$–planes. From [9, Sec 4] or [5, 2.3.5] it follows that X has the structure stated in the introduction.

Case $\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}$.

We adjust the frame such that

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \ldots = A_d = 0.$$

Then the focal hypersurface of the Gauss fiber F is given by

$$0 = \det(\lambda^0 A_0 + \lambda^1 A_1) = \lambda^0(\lambda^0 + \lambda^1),$$

i.e. it is a quadric of rank 2, consisting of the $(d-1)$-planes $\{e_1, \ldots, e_d\}$ and $\{e_0 - e_1, e_2, \ldots, e_d\}$. Letting the Gauss fibers vary, we obtain two subvarieties $\mathcal{H}_1, \mathcal{H}_2 \subseteq \mathbb{G}(d-1, N)$ after a closing procedure. It is possible that $\mathcal{H}_1 = \mathcal{H}_2$. Defining

$$Y_k = \bigcup_{L \in \mathcal{H}_k} L \subseteq \mathbb{P}_N,$$

the focal variety X_f is the union of Y_1 and Y_2. With such a linear system $\mathcal{A}$, the variety X cannot be a cone; therefore, the dimension of $\mathcal{H}_k$ is 1 or 2 and that of X_f is d or $d+1$.

Subcase: $\dim X_f = d+1$, i.e. X is of type (1,2).

We may assume that $\dim Y_1 = d+1$. Then we will show that X is the union of conjugate d–planes of the variety Y_1 of Gauss rank 2.

Note that due to our adaptions of the frame e_1 is a general point of Y_1 and with the index range $2 \leq \varrho \leq d$,

$$\omega_1^{n-1} = \omega^{n-1}, \quad \omega_\varrho^{n-1} = 0, \quad \text{and } \omega_\delta^n = 0.$$

The tangent space of Y_1 at the point e_1 is the image of

$$de_1 = \omega_1^0 e_0 + \omega_1^\varrho e_\varrho + \omega_1^{n-1} e_{n-1} \mod \{e_1\}.$$

By dimension counting, $T_{e_1} Y_1 = \{e_0, \ldots, e_{n-1}\}$ and the forms $\omega_1^0, \omega_1^\varrho, \omega_1^{n-1}$ are linearly independent basis forms on Y_1. By Proposition 1 we can adapt the frame further, such that the second fundamental form of X is given by

$$Q^{n+1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q^{n+2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } 0, \quad Q^m = 0 \quad \text{for } m \geq n+3,$$

in particular $\omega_{n-1}^{n+1} = \omega^{n-1}$ and $\omega_{n-1}^{n+2} = 0$. Again, it is understood that all terms with indices greater than N are ignored in the computation. The second fundamental form of Y_1 is

$$\begin{aligned} \mathrm{I\!I}_{Y_1,e_1} = d^2 e_1 &= (\omega_1^0 \omega^n + \omega_1^{n-1} \omega_{n-1}^n) e_n + \omega_1^{n-1} \omega_{n-1}^{n+1} e_{n+1} \\ &= (\omega_1^0 \omega^n + \omega_1^{n-1} \omega_{n-1}^n) e_n + (\omega_1^{n-1})^2 e_{n+1} \mod \{e_0, \ldots, e_{n-1}\}. \end{aligned}$$

We want to express ω^n and ω_{n-1}^n in the basis forms of Y_1. Differentiating $\omega_1^n = 0$ yields

$$\begin{aligned} &0 = d\omega_1^n = -\omega^n \wedge \omega_1^0 - \omega_{n-1}^n \wedge \omega_1^{n-1} \\ &\Rightarrow \quad \omega^n = f\omega_1^0 + g\omega_1^{n-1}, \quad \omega_{n-1}^n = g\omega_1^0 + h\omega_1^{n-1}. \end{aligned}$$

We have $f \neq 0$, because ω^n and $\omega_1^{n-1} = \omega^{n-1}$ are linearly independent. Now the second fundamental form reads

$$\mathrm{I\!I}_{Y_1,e_1} = (f(\omega_1^0)^2 + 2g\omega_1^0\omega_1^{n-1} + h(\omega_1^{n-1})^2) e_n + (\omega_1^{n-1})^2 e_{n+1} \mod \{e_0, \ldots, e_{n-1}\}.$$

Clearly, its singular locus is $\{e_1, \ldots, e_d\}$; thus Y_1 has Gauss rank 2, and its second fundamental form is in terms of $(\omega_1^0, \omega_1^{n-1})$

$$\tilde{Q}^n = \begin{pmatrix} f & g \\ g & h \end{pmatrix}, \qquad \tilde{Q}^{n+1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The linear space $L = \{e_0, e_1, \ldots, e_d\}$ is conjugate to $\{-ge_0 + fe_{n-1}, e_1, \ldots, e_d\}$, and these two linear spaces are the only conjugate d-planes. Finally, because the linear space L is precisely a Gauss fiber of X, the union of all these spaces will be X.

If X_f is developable, then its linear system $\tilde{\mathcal{A}}$ is either of type 0) or of type 2) in the table of Proposition 1, because of the form of the matrices $\tilde{Q}^n$ and $\tilde{Q}^{n+1}$. Therefore, X_f can only be of type (1,2), (2,2), or (3,1).

Using $\omega_\varrho^{n-1} = 0$, we describe the linear system $\tilde{\mathcal{A}}$ more precisely as

$$\tilde{\mathcal{A}} = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}.$$

Thus $\{e_2, \dots, e_d\}$ is a component of the focal hyperquadric in a general tangent plane of X_f. Let Z be the component of the focal variety X_{ff} of X_f which is the union of these linear spaces. Due to $\omega_\varrho^{n-1} = \omega_\varrho^n = 0$, the tangent variety of Z is contained in X. However, it will not always be equal to it. If X_f is of type (3,1) or (2,2) then Z is a linear space resp. a cone over a curve, thus $\operatorname{Tan} Z \neq X$. Even if X_f is of type (1,2) it might happen that $\operatorname{Tan} Z \neq X$, namely if $\dim Z < \dim X_{ff} = \dim X - 2$.

Subcase: $\dim X_f = d$, i.e. X is of type (2,2).

In this case X is a cone over the join of two curves.

We start proving this by showing that $\dim \mathcal{H}_1 = \dim \mathcal{H}_2 = 1$. Assume $\dim \mathcal{H}_k = 2$. Then, since $\dim Y_k = d$, each point $y \in Y_k$ lies on a one-dimensional family of $(d-1)$–planes. By dimension reasons, the union of these is Y_k itself; thus Y_k is a cone over y. Since Y_k is cone over all its points, it is a d–plane.

Now if $\dim \mathcal{H}_1 = \dim \mathcal{H}_2 = 2$, then we consider the focal hypersurface $L_1 \cup L_2$, $L_k \in \mathcal{H}_k$, of a Gauss fiber F. L_1 and L_2 intersect in a $(d-2)$–plane, thus the two d–planes Y_1 and Y_2 intersect in at least a $(d-2)$–plane, too. Hence, Y_1 and Y_2 span at most a $(d+2)$–plane inside $\mathbb{P}^N$. Since by construction X is contained in the span of Y_1 and Y_2, X must be a $(d+2)$–plane, which is impossible.

Now assume that $\dim \mathcal{H}_1 = 2$, but $\dim \mathcal{H}_2 = 1$. Let $L_2 \in \mathcal{H}_2$, then there is a one-dimensional subvariety $\mathcal{F}'$ of Gauss fibers of X that contain L_2. To this family corresponds a one-dimensional family $\mathcal{H}_1'$ of $\mathcal{H}_1$, such that the Gauss fibers of $\mathcal{F}'$ are spanned by L_2 and $L_1 \in \mathcal{H}_1'$. Since the $L_1 \in \mathcal{H}_1'$ sweep out the d–plane Y_1, the variety X contains the linear space $M_{L_2} = L_2 \vee Y_1$ spanned by L_2 and Y_1. Thus X, as the union of all M_{L_2}, $L_2 \in \mathcal{H}_2$, is a cone over the d–plane Y_1; in particular, it has Gauss rank less or equal to 1, which is impossible.

Up to now, we have shown that $\dim \mathcal{H}_1 = \dim \mathcal{H}_2 = 1$. By dimension reasons the two-dimensional family of Gauss fibers is given by

$$L_1 \vee L_2, \quad L_k \in \mathcal{H}_k.$$

We want to show that $L_1 \cap L_2$ is a fixed $(d-2)$–plane for all $L_k \in \mathcal{H}_k$. If that is not the case, let $L_2, L_2' \in \mathcal{H}_2$ be general $(d-1)$–planes. Looking at the dimension, we see that

$$L_1 = (L_1 \cap L_2) \vee (L_1 \cap L_2') \quad \text{for nearly all } L_1 \in \mathcal{H}_1$$

Thus L_1 and hence Y_1 is contained in $L_2 \vee L_2'$. Because of $\dim L_1 \cap L_2 = \dim L_1 \cap L_2' = d-2$, the space $L_2 \vee L_2'$ is at most a $(d+1)$–plane. The same argument with $\mathcal{H}_1$ and $\mathcal{H}_2$ exchanged implies that Y_1 and Y_2 are contained in the same $(d+1)$–plane. By the above remark about the Gauss fibers of X, it follows that X is contained in this $(d+1)$–plane, which is impossible.

Now let L be the fixed intersection of $L_1 \in \mathcal{H}_1$ and $L_2 \in \mathcal{H}_2$. Then there exist curves C_1 and C_2 such that

$$\mathcal{H}_1 = \{p \vee L \mid p \in C_1\} \quad \text{and} \quad \mathcal{H}_2 = \{q \vee L \mid q \in C_2\}.$$

Therefore, the Gauss fibers of X are

$$p \vee q \vee L \quad \text{with} \;\; p \in C_1,\; q \in C_2,$$

and we recognize X as the cone over the join of C_1 and C_2 with vertex L.

Case $\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$.

We adapt the frame such that

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_3 = \ldots = A_d = 0$$

and

$$Q^{n+1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q^m = 0 \quad \text{for } n+2 \leq m \leq N.$$

Since the linear system of quadrics of the second fundamental form of X is generated by just one quadric, X is a hypersurface inside a $\mathbb{P}^{n+1} \subseteq \mathbb{P}^N$ by the Segre theorem [2, 2.5.3]. We will assume that $\mathbb{P}^{n+1} = \mathbb{P}^N$.

The focal hypersurface is

$$\det(\lambda^0 A_0 + \lambda^1 A_1 + \lambda^2 A_2) = \det \begin{pmatrix} \lambda^0 + \lambda^1 & \lambda^2 \\ \lambda^2 & \lambda^0 \end{pmatrix} = (\lambda^0)^2 + \lambda^0\lambda^1 - (\lambda^2)^2,$$

i.e. it is a quadric of rank 3.

Note that the general matrix of $\mathcal{A}$ is diagonalizable, and that this is even true for those matrices of $\mathcal{A}$ whose determinate vanishes; thus A_1 is a general matrix under those with vanishing determinate, and e_1 is a general point of X_f.

Our adaptions of the frame imply that with the index range $3 \leq \varrho \leq d$

$$\omega_1^{n-1} = \omega^{n-1}, \quad \omega_1^n = 0, \quad \omega_2^{n-1} = \omega^n, \quad \omega_2^n = \omega^{n-1}, \quad \omega_\varrho^{n-1} = \omega_\varrho^n = 0,$$
$$\omega_{n-1}^{n+1} = \omega^{n-1}, \quad \text{and} \quad \omega_n^{n+1} = \omega^n.$$

Again, we treat the cases of $\dim X_f = d+1$ or d separately.

Subcase: $\dim X_f = d+1$, i.e. X is of type (1,3).

We will show that X_f has Gauss rank 3 and the pencil of quadrics of the second fundamental form is generated by a quadric of rank 3 and a double tangent plane to this quadric. The original variety X can be recovered from X_f as the union of these double tangent planes.

The tangent space of X_f at the general point $e_1 \in X_f$ is the image of

$$de_1 = \omega_1^0 e_0 + \omega_1^2 e_2 + \omega_1^\varrho e_\varrho + \omega_1^{n-1} e_{n-1} \quad \text{mod } \{e_1\}.$$

Since it has dimension $d+1$ the forms $\omega_1^0, \omega_1^2, \omega_1^\varrho, \omega_1^{n-1}$ must be linearly independent. We find for the second fundamental form

$$\mathrm{I\!I}_{X_f,e_1} = d^2 e_1 = (\omega_1^0\omega^n + \omega_1^2\omega_2^n + \omega_1^{n-1}\omega_{n-1}^n)e_n + \omega_1^{n-1}\omega_{n-1}^{n+1}e_{n+1} \mod \{e_0, \dots, e_{n-1}\}.$$

Knowing $\omega_{n-1}^{n+1} = \omega_2^n = \omega^{n-1} = \omega_1^{n-1}$, it remains to express ω^n and ω_{n-1}^n in terms of $\omega_1^0, \omega_1^2, \omega_1^\varrho, \omega_1^{n-1}$. We differentiate $\omega_1^n = 0$ and find

$$\begin{aligned}
0 = d\omega_1^n &= -\omega^n \wedge \omega_1^0 - \omega_2^n \wedge \omega_1^2 - \omega_{n-1}^n \wedge \omega_1^{n-1} \\
&= \omega_1^{n-1} \wedge (\omega_{n-1}^n - \omega_1^2) + \omega_1^0 \wedge \omega^n \\
\Rightarrow \omega_{n-1}^n - \omega_1^2 &= f\omega_1^{n-1} + g\omega_1^0 \\
\omega^n &= g\omega_1^{n-1} + h\omega_1^0 \quad \text{for suitable functions } \ f, g, h.
\end{aligned}$$

Note that $h \neq 0$, because ω^n and $\omega_1^{n-1} = \omega^{n-1}$ are linearly independent. Using this, the second fundamental form is

$$\mathrm{I\!I}_{X_f,e_1} = (2\omega_1^2\omega_1^{n-1} + 2g\omega_1^0\omega_1^{n-1} + h(\omega_1^0)^2 + f(\omega_1^{n-1})^2)e_n + (\omega_1^{n-1})^2 e_{n+1} \mod \{e_0, \dots, e_{n-1}\}.$$

Thus the singular locus of $\mathrm{I\!I}$ is $\{e_1, e_\varrho\}$, and in the basis $(\omega_1^2, \omega_1^0, \omega_1^{n-1})$ the second fundamental form is given by

$$\tilde{Q}^n = \begin{pmatrix} 0 & 0 & 1 \\ 0 & h & g \\ 1 & g & f \end{pmatrix}, \qquad \tilde{Q}^{n+1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The zero locus of the quadric $\tilde{Q}^{n+1}$, the double plane $\{e_0, \dots, e_d\}$, is tangent to the quadric $\tilde{Q}^n$ at the points $\{e_1, \dots, e_d\}$. Since $\{e_0, \dots, e_d\}$ is a Gauss fiber of X, the union of these will be X as well. Finally, we remark that $\tilde{Q}^n$ and $\tilde{Q}^{n+1}$ can be brought into the normal form

$$\tilde{Q}^n = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \tilde{Q}^{n+1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

by a basis transformation in $(\omega_1^2, \omega_1^0, \omega_1^{n-1})$ and subtracting a multiple of $\tilde{Q}^{n+1}$ from $\tilde{Q}^n$.

Subcase: $\dim X_f = d$, i.e. X is of type (2,3).

We claim that the hypersurface X is a cone over the secant variety of the Veronese surface. Since the dual of the secant variety of the Veronese surface in $\mathbb{P}^5$ is the Veronese surface in the dual space $(\mathbb{P}^5)^*$, this is equivalent to the fact that X^* is the Veronese surface. By [6, 6c] in the version of [2, 8.4], it is enough to show that

1. X^* has a 5-dimensional second osculating space
2. for any two points $x, y \in X^*$, the intersection of the projective tangent spaces $\mathbb{T}_x X^*$ and $\mathbb{T}_y X^*$ is nonempty.

Let e^B be the frame dual to e_A, then from $e^B(e_A) = \delta^B_A$ one obtains easily [8, 7.4]

$$de^B = -\omega^B_C e^C.$$

The point e^{n+1} is a general point of X^*, and the tangent space of X^* in e^{n+1} is the image of

$$de^{n+1} = -\omega^{n+1}_{n-1} e^{n-1} - \omega^{n+1}_n e^n = -\omega^{n-1} e^{n-1} - \omega^n e^n \mod \{e^{n+1}\}.$$

Since ω^{n-1} and ω^n are linearly independent, X^* has dimension 2. Its second fundamental form is

$$\begin{aligned} \mathrm{I\!I}_{X^*, e^{n+1}} = d^2 e^{n+1} &= \omega^{n-1}(\omega^{n-1} e^0 + \omega^{n-1}_1 e^1 + \omega^{n-1}_2 e^2) + \omega^n(\omega^n e^0 + \omega^n_2 e^2) \\ &= (\omega^n)^2 e^0 + (\omega^{n-1})^2 (e_0 + e_1) + 2\omega^{n-1}\omega^n e^2 \mod \{e^{n-1}, e^n, e^{n+1}\}; \end{aligned}$$

thus the second osculating space of X^* is 5-dimensional.

The dual of the tangent space $\mathbb{T}_{e^{n+1}} X^* = \{e^{n-1}, e^n, e^{n+1}\}$ is the Gauss fiber $F = \{e_0, \ldots, e_d\}$ of X. Hence, instead of showing that any two tangent spaces of X^* intersect nontrivially, we may show that any two Gauss fibers of X do not span $\mathbb{P}^{n+1}$. Let $F \subset X$ be a Gauss fiber of X and C its focal hypersurface. Take a one-dimensional family of Gauss fibers $\mathcal{F}' \subset \mathcal{F}$. By dimension reasons, the union of the focal hypersurfaces C' of the Gauss fibers $F' \in \mathcal{F}'$ is the whole X_f. In particular, it covers C; therefore,

$$\dim C \cap C' \geq \dim C - 1 = d - 2.$$

Then we have $\dim F \cap F' \geq d - 2$ as well, and $\dim F \vee F' \leq d + 2$ implies that F and F' cannot span the whole $\mathbb{P}^{n+1}$.

2 Constructions

In this section we want to show that the descriptions of the developable varieties given above can be used to construct them.

Clearly, cones over nondevelopable surfaces or the secant variety of the Veronese surface are developable varieties of Gauss rank 2.

That a union of a one-dimensional family of $(n-1)$-planes as described for developable varieties of type (2,1) yields a developable variety of Gauss rank 2 was proven in [9, Sec 4] and [5, 2.3.5]. Hereby, the curves must be chosen sufficiently general, otherwise X might have only Gauss rank 1 or 0; analogous remarks will be true for all the following constructions.

Developable varieties of type (2,2) are the join of two curves and a linear space. That such a join is developable follows from Terracini's Lemma [13, II.1.10].

Terracini's Lemma *The tangent space of the join variety $X = Y_1 \# \ldots \# Y_m$ at a general point $x \in \overline{y_1 \cdots y_m}$ with $y_i \in Y_i$ is*

$$\mathbb{T}_x X = \mathbb{T}_{y_1} Y_1 \vee \cdots \vee \mathbb{T}_{y_m} Y_m.$$

Terracini's Lemma even implies that the join X of the a_1–th osculating scroll, $C_1^{(a_1)}$, of a curve C_1, the a_2–th osculating scroll, $C_2^{(a_2)}$, of a curve C_2, and a linear space L is developable of Gauss rank 2. Its focal variety X_f is the union of $C_1^{(a_1-1)}\#C_2^{(a_2)}\#L$ and $C_1^{(a_1)}\#C_2^{(a_2-1)}\#L$. Hence, X is of type (1,2) if $a_1+a_2\geq 1$.

It remains to study the developable varieties whose focal varieties have only codimension one. Three-dimensional developable varieties of the types (1,1) and (1,2) were already constructed in [3].

We want to construct developable varieties of type (1,2). Let $Y\subset\mathbb{P}^N$ be a $(d+1)$-dimensional variety of Gauss rank 2 with conjugate d-planes. We claim that the union of these d-planes has Gauss rank 2. We choose a moving frame $(e_0,\dots,e_N)$ such that

$\{e_0\}$	is a point of Y,
$\{e_0,\dots,e_{d-1}\}=\{e_0,e_\varepsilon\}$	is the Gauss fiber of Y through $\{e_0\}$,
$\{e_0,\dots,e_{d+1}\}$	is the tangent space of Y,
$\{e_0,\dots,e_d\}$	is the d–plane conjugate to $\{e_0,\dots,e_{d-1},e_{d+1}\}$,
$\{e_0,\dots,e_{d+3}\}$	is the second osculating space of Y.

Due to our assumptions on the conjugate d-planes, the second fundamental form may be given by

$$Q^{d+2}=\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad Q^{d+3}=\begin{pmatrix}0&0\\0&1\end{pmatrix},$$

in particular $\omega_d^{d+3}=0$.

Now e_d is a general point of X, and the tangent space of X is the image of

$$de_d=\omega_d^0e_0+\omega_d^\varepsilon e_\varepsilon+\omega_d^{d+1}e_{d+1}+\omega_d^{d+2}e_{d+2}\mod\{e_d\}.$$

Since X has dimension $n=d+2$, the forms $\{\omega_d^0,\omega_d^\varepsilon,\omega_d^{d+1},\omega_d^{d+2}\}$ are the basis forms of X. The second fundamental form of X is with the index range $d+3\leq\mu\leq N$

$$\mathrm{II}_{X,e_d}=(\omega_d^{d+1}\omega_{d+1}^\mu+\omega_d^{d+2}\omega_{d+2}^\mu)e_\mu\mod\{e_0,\dots,e_n\}.$$

Differentiating $\omega_d^{d+3}=0$ yields

$$\begin{aligned}\omega_{d+1}^{d+3}&=f\omega_d^{d+1}+g\omega_d^{d+2}\\ \omega_{d+2}^{d+3}&=g\omega_d^{d+1}+h\omega_d^{d+2}\end{aligned}$$

with $f\neq0$ since $\omega_{d+1}^{d+3}=\omega^{d+1}$ and $\omega_d^{d+2}=\omega^d$ are linearly independent. Further, differentiating $\omega_d^\nu=0$ for $d+4\leq\nu\leq N$ yields

$$0=d\omega_d^\nu=-\omega_{d+2}^\nu\wedge\omega_d^{d+2}\quad\Rightarrow\quad\omega_{d+2}^\nu=h^\nu\omega_d^{d+2}.$$

Therefore, the second fundamental form of X is

$$\mathrm{II}_{X,e_d}=(f(\omega_d^{d+1})^2+2g\omega_d^{d+1}\omega_d^{d+2}+h(\omega_d^{d+2})^2)e_{d+3}+h^\nu(\omega_d^{d+2})^2e_\nu\mod\{e_0,\dots,e_n\},$$

thus X has Gauss rank 2.

If X is not a hypersurface, then the system of quadrics of II_{X,e_d} is two-dimensional by Segre's theorem [2, 2.5.3] and $\{e_0, \ldots, e_{d+1}\} = \mathbb{T}_{e_0}Y$ is a conjugate $(d+1)$-plane of X. Taking again the union of the these conjugate planes, we obtain a developable variety of type (1,2), which is simply the tangent variety of Y. Thus we may replace the twice iterated process of taking the union of the correctly chosen conjugated planes by taking the union of the tangent spaces of the original variety Y. On the other hand, we could have constructed a developable variety Z by taking the union of the other conjugated planes of X. Then the question is if there is a variety Y' with $Z = \operatorname{Tan} Y'$. The remarks at the end of the analysis of the varieties of type (1,2) indicate that this might not be the case if the component of the focal variety X_f which is not Y has a smaller dimension than Y.

The construction of a developable variety of type (1,1) is very similar. Let Y be a $(d+1)$-dimensional developable variety of Gauss rank 2 with unique asymptotic d-planes, which do not lie in Y. Further, let X be the union of these d-planes. To prove that X has Gauss rank 2, we choose a moving frame adapted like above with the exception that $\{e_0, \ldots, e_d\}$ is now the asymptotic d-plane. This time we may assume that the second fundamental form is given by

$$Q^{d+2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Q^{d+3} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Completely analogous to the above case, we obtain as the second fundamental form of X

$$\mathrm{II}_{X,e_d} = (2\omega_d^{d+1}\omega_d^{d+2} + g(\omega_d^{d+2})^2)e_{d+3} + f^\nu(\omega_d^{d+2})^2 e_\nu \mod \{e_0, \ldots, e_n\}.$$

Therefore, X has Gauss rank 2. If X is not a hypersurface, then $\{e_0, \ldots, e_{d+1}\}$, the tangent space of Y, is the unique asymptotic $(d+1)$–plane in the general tangent space of X. The union of those, the tangent variety of Y, will be developable of Gauss rank 2. Thus we may replace the twice iterated process of taking the union of the asymptotic planes by taking the union of the tangent spaces of the original variety Y.

Now we state the result for higher iteration. Note that since the pencil of quadrics of the second fundamental form is generated by $\{\omega^d\omega^{d+1}, (\omega^{d+1})^2\}$ in the asymptotic case and by $\{(\omega^d)^2, (\omega^{d+1})^2\}$ in the conjugate case, the polynomials of degree k of the k–th fundamental form $\mathbb{FF}^{(k)}$ lie in $\{\omega^d(\omega^{d+1})^{k-1}, (\omega^{d+1})^k\}$ resp. $\{(\omega^d)^k, (\omega^{d+1})^k\}$ by the prolongation property [8, 4.2.4] and formulas like [2, 2.51]. Hence, the dimension of the k–th osculating tangent space $\mathbb{T}^{(k)}Y$ is at most the dimension of the $(k-1)$–th osculating tangent space plus 2. Thus if E denotes the asymptotic resp. a conjugate d–plane of Y, then the dimensions of the linear spaces in the following sequence are increasing by at most one:

$$E \subseteq \mathbb{T}Y \subseteq \mathrm{II}(E, TY) + \mathbb{T}Y \subseteq \mathbb{T}Y^{(2)} \subseteq \mathbb{FF}^{(3)}(E, TY, TY) + \mathbb{T}Y^{(2)} \subseteq \mathbb{T}Y^{(3)} \subseteq \ldots.$$

By computations similar to the above one, one can show that iterating the process of taking the union of the asymptotic (correctly chosen conjugate) planes k–times results in the same variety as taking the union of the k–th linear spaces in the above sequence of the variety Y.

Finally, we construct varieties of type (1,3). Let $Y \subset \mathbb{P}^{n+1}$ be a developable variety of dimension $d+1 = n-1$ and Gauss rank 3, whose pencil of quadrics of the second fundamental form is generated by a quadric of rank 3 and a double tangent plane to it. We will show that the union of these d-dimensional double tangent planes is a n-dimensional developable variety X of Gauss rank 2.

We choose a frame $(e_0, \ldots, e_{n+1})$ such that

$$\begin{array}{ll} \{e_0\} & \text{is a point of } Y, \\ \{e_0, \ldots, e_{d-2}\} = \{e_0, e_\varepsilon\} & \text{is the Gauss fiber of } Y \text{ through } \{e_0\}, \\ \{e_0, \ldots, e_{d+1}\} & \text{is the tangent space of } Y. \end{array}$$

We may assume that the quadric of rank 3 inside the tangent space of Y is given by $2x_{d-1}x_{d+1} + (x_d)^2$, and the double tangent plane is tangent at the point e_{d-1}, then the second fundamental fundamental form of X is given by

$$Q^{d+2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad Q^{d+3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

in particular $\omega_{d-1}^{d+3} = \omega_d^{d+3} = 0$.

The point e_d is a general point of the double tangent plane and hence of X. We compute the tangent space of X as the image of

$$de_d = \omega_d^0 e_0 + \omega_d^\varepsilon e_\varepsilon + \omega_d^{d-1} e_{d-1} + \omega_d^{d+1} e_{d+1} + \omega_d^{d+2} e_{d+2} \mod \{e_d\};$$

thus X has dimension $n = d+2$ and not $d+3$ as a naive dimension count suggests. Its second fundamental form is

$$\mathrm{II}_{X,e_d} = (\omega_d^{d+1}\omega_{d+1}^{d+3} + \omega_d^{d+2}\omega_{d+2}^{d+3})e_{d+3} \mod \{e_0, \ldots, e_n\}.$$

Differentiating $\omega_d^{d+3} = 0$ yields

$$\begin{aligned} 0 = d\omega_d^{d+3} &= -\omega_{d+1}^{d+3} \wedge \omega_d^{d+1} - \omega_{d+2}^{d+3} \wedge \omega_d^{d+2} \\ \Rightarrow \quad & \omega_{d+1}^{d+3}, \omega_{d+2}^{d+3} = 0 \mod \{\omega_d^{d+1}, \omega_d^{d+2}\}. \end{aligned}$$

Hence II_{X,e_d} can be expressed in the forms $\omega_d^{d+1}, \omega_d^{d+2}$ alone, and therefore X has Gauss rank 2.

Finally, we make some remarks about the degree of freedom one has in the local construction of developable varieties of Gauss rank 2 whose focal varieties have codimension one. Using the Cartan test [2, 7], one can compute the degree of freedom in the construction of varieties Y with the properties described above, from which we obtained developable varieties X with $X_f \supseteq Y$. This has been done for a surface Y with conjugate directions in [2, p. 85]. The extention to developable varieties of Gauss rank 2 is easy. The case where Y is a developable variety of Gauss rank 2 with asymptotic planes is nearly the same. In both cases we find that the construction depends on two functions of two variables. Proving the existence of the variety Y needed in the construction of a developable variety of type $(1,3)$, is analogous, but computationally more difficult. Here one finds that the construction of Y depends on $d+3$ functions of two variables.

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