Mathematisches Institut
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WS 2018/19
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Sheet 2

Stochastic Processes and Stochastic Analysis II
Exercise sheet

Please gather in groups of 2 people to hand in your answers.

Exercise 4: *(6 points)* Let $(B_t)_{t\geq 0}$ and $(C_t)_{t\geq 0}$ be two independent one-dimensional Brownian motions. For each $t \geq 0$ we define the processes

$$Y_t = \int_0^t e^{C_t - C_s} dB_s$$

and

$$W_t = \int_0^t \frac{Y_s}{\sqrt{1+Y_s^2}} dC_s + \int_0^t \frac{1}{\sqrt{1+Y_s^2}} dB_s.$$

(a) Compute $\langle W \rangle_t$ for every $t \geq 0$.
(b) Compute dM_t for $M_t = \exp(i\lambda W_t + \frac{\lambda^2 t}{2})$ and every $t \geq 0$.
(c) Show that $(W_t)_{t\geq 0}$ is a Brownian motion.

Exercise 5: *(8 points)* Let $Y = (Y_t)_{t\geq 0}$ and $W = (W_t)_{t\geq 0}$ be as in Exercise 4.
(a) Show that Y solves the stochastic initial value problem

$$dY_t = \frac{Y_t}{2} dt + \sqrt{1+Y_t^2} dW_t, \quad Y_0 = 0.$$

(b) Compute dX_t for $X_t = \sinh W_t$, $t \geq 0$.
(c) Show that $X_t = Y_t$ for every $t \geq 0$ holds with probability one.

Exercise 6: *(6 points)* For a probability space $(\Omega, \mathcal{A}, P)$ consider a process X given by

$$X_t = x + B_t + \int_0^t f(s) ds, \quad t \geq 0,$$

with $x \in \mathbb{R}$, $B = (B_t)_{t\geq 0}$ a one-dimensional Brownian motion and $f : \mathbb{R}_+ \to \mathbb{R}$ is continuous. Define

$$\tau_y^+(X) := \inf\{t \geq 0 : X_t > y\}, \quad \tau_y^-(X) := \inf\{t \geq 0 : X_t < y\}.$$

(a) Show that the measure Q given by

$$\frac{dQ}{dP} = \exp\Big(- \int_0^1 f(s) dB_s - \frac{1}{2} \int_0^1 f^2(s) ds \Big)$$

defines a probability measure on $(\Omega, \mathcal{A})$.

(b) Construct a stochastic process $Y = (Y_t)_{t \geq 0}$ which is a Brownian motion with respect to Q.

(c) Show that $P\big(\tau_x^+(X) = \tau_x^-(X) = 0\big) = 1$.
<u>Hint:</u> Recall that $P\big(\tau_0^+(B) = \tau_0^-(B) = 0\big) = 1$.

Hand in on: Thursday, 01.11.2018
Revision: Tuesday, 06.11.2018 in Exercise lesson