Mathematisches Institut
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WS 2018/19
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Sheet 3

Stochastic Processes and Stochastic Analysis II
Exercise sheet

Please gather in groups of 2 people to hand in your answers.

Consider a one-dimensional Brownian motion $B = (B_t)_{t\geq 0}$ on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. The aim of the following exercises is to calculate for every $a \geq 0$, $b > 0$ the exact value of

$$I(a,b) = \mathbb{E}\Big[\exp\Big(-aB_t^2 - \frac{b^2}{2}\int_0^t B_s^2 ds\Big)\Big].$$

Exercise 7: *(4 points)*

(a) Show that the process $V = (V_t)_{t\geq 0}$ given by

$$V_t = e^{-bt}V_0 + \int_0^t e^{-b(t-s)}dB_s$$

solves the stochastic differential equation

$$dV_t = -bV_t dt + dB_t.$$

(b) Let X be a standard normally distributed random variable. Determine every $\alpha \in \mathbb{R}$ such that $\mathbb{E}[e^{\alpha X^2}]$ exists and compute its exact value in this case.

Exercise 8: *(8 points)*

(a) Determine a process $G = (G_t)_{t\geq 0}$ such that $Z = (Z_t)_{t\geq 0}$ given by

$$Z_t = \exp\Big(-b\int_0^t B_s dB_s - \int_0^t G_s ds\Big), \quad t \geq 0,$$

is a local martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$.

(b) Show that Z is a martingale in this case and compute its expectation.

(c) Conclude that

$$I(a,b) = \mathbb{E}\Big[Z_t \exp\Big((\frac{b}{2} - a)B_t^2\Big)\Big]\exp(-\frac{bt}{2}).$$

Exercise 9: *(8 points)* Consider the probability measure Q on $(\Omega, \mathcal{F})$ given by

$$dQ_{|\mathcal{F}_t} = Z_t dP_{|\mathcal{F}_t}, \quad \forall t \geq 0,$$

where Z is the martingale in Exercise 8. Define the process $C = (C_t)_{t\geq 0}$ for all $t \geq 0$ as

$$C_t = B_t + b\int_0^t B_s ds.$$

(a) Show that for every $t \geq 0$

$$B_t = \int_0^t e^{b(s-t)} dC_s.$$

(b) Conclude that for every $a \geq 0$, $b > 0$

$$I(a,b) = \Big(\cosh(bt) + \frac{2a}{b}\sinh(bt)\Big)^{-\frac{1}{2}}.$$

Hand in on: Thursday, 08.11.2018
Revision: Tuesday, 13.11.2018 in Exercise lesson