Mathematisches Institut
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WS 2018/19
08.11.2018
Sheet 4

Stochastic Processes and Stochastic Analysis II
Exercise sheet

Please gather in groups of 2 people to hand in your answers.

Exercise 10: *(6 points)* Let $B = (B_t)_{t\geq 0}$ be a one-dimensional Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and $b_1, b_2, \sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ Lipschitz-continuous functions. Assume that $X = (X_t)_{t\geq 0}$ is the solution to

$$dX_t = b_1(t, X_t)dt + \sigma(t, X_t)dB_t.$$

Show that, under suitable assumptions to be precised, there exists a probability measure Q on $(\Omega, \mathcal{F})$ such that X is the solution to

$$dX_t = b_2(t, X_t)dt + \sigma(t, X_t)d\tilde{B}_t,$$

where $\tilde{B} = (\tilde{B}_t)_{t\geq 0}$ is a Brownian motion with respect to Q.

Exercise 11: *(6 points)* Let U be the uniform distribution on $\partial B(0,1) = \{x \in \mathbb{R}^d : \|x\| = 1\}$.

(a) Show that U is rotationally invariant.

(b) Show that U is the unique probability measure on $\partial B(0,1)$ that is rotationally invariant.

Exercise 12: *(8 points)* Let $B = (B_t)_{t\geq 0}$ be a one-dimensional Brownian motion started at $x \in D = (-1, 1)$ and consider the stopping time $\tau = \inf\{t \geq 0 : B_t \notin D\}$. Define

$$v(t, x) = \mathbb{E}[f(t + \sqrt{\tau}Z, B_\tau)],$$

where $f(t, x) = e^x \cos t$ and Z is a standard normally distributed random variable independent of B.

(a) Show that for every $t \geq 0$ and $x \in D$

$$v(t, x) = f(t, x).$$

<u>Hint:</u> Use the optional stopping theorem.

(b) Assume that Y has a Cauchy distribution with density

$$p(y) = \frac{1}{\pi(1 + y^2)}$$

and is independent of B, Z. Show that $u(t, x) := \mathbb{E}[v(tY, x)]$ satisfies the wave equation

$$\frac{\partial^2}{\partial t^2} u(t, x) = \frac{\partial^2}{\partial x^2} u(t, x).$$

<u>Hint:</u> Recall that $\mathbb{E}[e^{isY}] = e^{-|s|}$ for all $s \in \mathbb{R}$.

Hand in on: Thursday, 15.11.2018
Revision: Tuesday, 20.11.2018 in Exercise lesson