Mathematisches Institut
der Heinrich-Heine-Universität
Düsseldorf
Jun.-Prof. Dr. C. Rojas-Molina / Dr. E. Sönmez

WS 2018/19
15.11.2018
Sheet 5

Stochastic Processes and Stochastic Analysis II
Exercise sheet

Please gather in groups of 2 people to hand in your answers.

Exercise 13: *(6 points)* Consider the Dirichlet problem in dimension $d = 1$ on $U = (a, b) \subseteq \mathbb{R}$ with $a < b$.

(a) Determine every harmonic function f on U with $f(a) = 0$ and $f(b) = 1$.

(b) Let $(B_t)_{t\geq 0}$ be a one-dimensional Brownian motion and $x_0 \in (a, b)$. Use the solution of the Dirichlet problems to compute the probability $P(x_0 + B_t$ hits b before $a)$.

Exercise 14: *(10 points)* For $d = 2$ let g be a continuous function on $\partial B(0, 1)$, where each $z \in \partial B(0, 1)$ is identified with $z = e^{i\theta}$, $\theta \in [0, 2\pi)$. Consider the harmonic function

$$h_\theta(x) = \frac{1 - \|x\|_2^2}{2\pi\|e^{i\theta} - x\|_2^2}$$

on $B(0, 1) \setminus \{0\}$ and define $f : B(0, 1) \to \mathbb{R}$ by

$$f(x) = \int_0^{2\pi} g(e^{i\theta})h_\theta(x)d\theta.$$

(a) Use the mean value property to show that f is harmonic on $B(0, 1) \setminus \{0\}$.

(b) Show that f can be extended continuously to $\overline{B(0, 1)}$.
<u>Hint:</u> First show the assertion in the case $g \equiv 1$.

(c) Let $x_0 \in \mathbb{R}^2$ with $\|x_0\|_2 < 1$, $T = \inf\{t \geq 0 : \|x_0 + B_t\|_2 = 1\}$ and $(B_t)_{t\geq 0}$ a two-dimensional Brownian motion. Show that the distribution of $x_0 + B_T$ admits a density with respect to the uniform distribution on $\partial B_1(0)$.

(d) Let $b > 0$, $U = \{x \in \mathbb{R}^d : 0 < \|x\|_2 < b\}$ and $T = \inf\{t \geq 0 : \|x_0 + B_t\|_2 \in \partial U\}$ for $x_0 \in U$. Show that $P(\|x_0 + B_T\|_2 = b) = 1$.

(e) Determine a continuous function g on $\partial\big(B(0, 1) \setminus \{0\}\big)$ such that there is no solution of the Dirichlet problem.

Exercise 15: *(4 points)* Let $U = \{x \in \mathbb{R}^2 : x_2 > 0\}$ and f a continuous function on ∂U. Show that the function

$$h(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u, 0)\frac{x_2}{(u - x_1)^2 + x_2^2}du$$

is harmonic on U, provided that the last expression is finite.
<u>Hint:</u> It is possible to use Exercise 21 from Stochastic Processes and Stochastic Analysis I.

Hand in on: Thursday, 22.11.2018
Revision: Tuesday, 27.11.2018 in Exercise lesson