Mathematisches Institut
der Heinrich-Heine-Universität
Düsseldorf
Jun.-Prof. Dr. C. Rojas-Molina / Dr. E. Sönmez

WS 2018/19
22.11.2018
Sheet 6

Stochastic Processes and Stochastic Analysis II
Exercise sheet

Please gather in groups of 2 people to hand in your answers.

Let $(X_n)_{n\in\mathbb{N}}$ be a Markov chain given by the triple $(S, \vec{p}, P)$ and for a subset $A \subset S$ consider its hitting time

$$T^A := \inf\{n \geq 0 : X_n \in A\}.$$

In this exercise sheet we consider

$$h_i^A := P_i(T^A < \infty), \quad k_i^A := \mathbb{E}_i[T^A]$$

for $i \in S$.

Exercise 16: *(8 points)*

(a) Show that the vector $h^A = (h_i^A : i \in S)$ solves the system of linear equations

$$\begin{aligned} h_i^A &= 1, && i \in A \\ h_i^A &= \sum_{j\in S} p_{ij} h_j^A, && i \notin A. \end{aligned}$$

(b) Assume that $x = (x_i^A : i \in S)$ is another solution in (a). Show that

$$x_i = P_i(X_1 \in A) + P_i(X_1 \notin A, X_2 \in A) + \sum_{j\notin A}\sum_{k\notin A} p_{ij}p_{jk}x_k.$$

(c) Colclude that $h^A = (h_i^A : i \in S)$ is the minimal non-negative solution in (a).

Exercise 17: *(8 points)*

(a) Show that the vector $k^A = (k_i^A : i \in S)$ solves the system of linear equations

$$\begin{aligned} k_i^A &= 0, && i \in A \\ k_i^A &= 1 + \sum_{j\notin A} p_{ij} k_j^A, && i \notin A. \end{aligned}$$

(b) Assume that $y = (y_i^A : i \in S)$ is another solution in (a). Show that

$$y_i = P_i(T^A \geq 1) + P_i(T^A \geq 2) + \sum_{j\notin A}\sum_{k\notin A} p_{ij}p_{jk}y_k.$$

(c) Colclude that $k^A = (k_i^A : i \in S)$ is the minimal non-negative solution in (a).

Exercise 18: *(4 points)* Let $S = \{1, 2, 3, 4\}$ and assume that the stochastic matrix P is given by

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Determine $h_i^{\{4\}}$ and $k_i^{\{1,4\}}$ for every $i \in S$.

Hand in on: Thursday, 29.11.2018
Revision: Tuesday, 04.12.2018 in Exercise lesson