Mathematisches Institut
der Heinrich-Heine-Universität
Düsseldorf
Jun.-Prof. Dr. C. Rojas-Molina / Dr. E. Sönmez

WS 2018/19
13.12.2018
Sheet 9

Stochastic Processes and Stochastic Analysis II
Exercise sheet

Please gather in groups of 2 people to hand in your answers.

Exercise 25: *(6 points)* Let Q be a matrix on a finite set S and define $P(t) = e^{tQ}$ for $t \geq 0$. Show that $\big(P(t)\big)_{t\geq 0}$ satisfies the following:

(a) $P(s+t) = P(s)P(t)$ for every $s, t \geq 0$.

(b) $\big(P(t)\big)_{t\geq 0}$ is the unique solution to the forward equation

$$\frac{d}{dt}P(t) = P(t)Q, \quad P(0) = I_d.$$

(c) $\big(P(t)\big)_{t\geq 0}$ is the unique solution to the backward equation

$$\frac{d}{dt}P(t) = QP(t), \quad P(0) = I_d.$$

(d) For $k = 0, 1, 2, \ldots$ we have

$$\Big(\frac{d^k}{dt^k}P(t)\Big)|_{t=0} = Q^k.$$

Exercise 26: *(6 points)*

(a) Show that a matrix Q on a finite set S is a Q-matrix if and only if $P(t) = e^{tQ}$ is a stochastic matrix for all $t \geq 0$.

(b) Let $X = (X_t)_{t\geq 0}$ be a Markov chain with state space $S = \{1, 2, 3\}$ and Q-matrix

$$Q = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -3 \end{pmatrix}.$$

Compute $p_{11}(t) = P_1(X_t = 1)$ for $t \geq 0$.

Exercise 27: *(8 points)* Let $X = (X_t)_{t\geq 0}$ be a Markov chain with state space $S = \{0, 1\}$ and canonical filtration $(\mathcal{F})_{t\geq 0}$ satisfying

$$\mathbb{E}_0[\mathbf{1}_{\{X_t=1\}}|\mathcal{F}_T] = \mathbb{E}_{X_T}[\mathbf{1}_{\{X_{t-T}=1\}}]$$

on $\{T \leq t\}$ for every stopping time T. Assume that X is generated by the Q-matrix

$$Q = \begin{pmatrix} -\beta & \beta \\ \delta & -\delta \end{pmatrix}$$

for $\beta, \delta > 0$ and consider $p_t(x, y) = P_x(X_t = y)$ for all $x, y \in S$, $t \geq 0$.

(a) Show that for all $t \geq 0$

$$p_t(0,1) = \int_0^t \beta e^{-\beta s} p_{t-s}(1,1) ds.$$

Hint: Consider the stopping time $T = \inf\{t \geq 0 : X_t = 1\}$.

(b) Show that

$$p_t(0,0) = \frac{\delta}{\beta+\delta} + \frac{\beta}{\beta+\delta} e^{-t(\beta+\delta)} \qquad p_t(0,1) = \frac{\beta}{\beta+\delta}(1 - e^{-t(\beta+\delta)})$$

$$p_t(1,1) = \frac{\beta}{\beta+\delta} + \frac{\delta}{\beta+\delta} e^{-t(\beta+\delta)} \qquad p_t(0,0) = \frac{\delta}{\beta+\delta}(1 - e^{-t(\beta+\delta)}).$$

Hint: Observe that $p_t(x,y)$ satisfies an ordinary differential equation.

(c) Verify that Q satisfies

$$q(x,y) = \frac{d}{dt} p_t(x,y)|_{t=0}.$$

Hand in on: Thursday, 20.12.2018

Revision: Tuesday, 08.01.2019 in Exercise lesson