Mathematisches Institut
der Heinrich-Heine-Universität
Düsseldorf
Jun.-Prof. Dr. C. Rojas-Molina / Dr. E. Sönmez

WS 2018/19
06.12.2018

Stochastic Processes and Stochastic Analysis II
Bonus exercise sheet

Please gather in groups of 2 people to hand in your answers.

Exercise 1: Let $(B_t)_{t\geq 0}$ be a one-dimensional Brownian motion and $a, b \in \mathbb{R}$. For $t \in [0.1)$ consider the Gaussian process Z with

$$Z_t = a(1-t) + bt + (1-t)\int_0^t \frac{dB_s}{1-s}.$$

(a) Compute the mean and covariance function of Z. Which process do you obtain if $a = b = 0$?
(b) Show that $Z_t \to b$ in L^2 as $t \to 1$.
(c) Let $f : [0,1) \to \mathbb{R}$ be continuous. Solve the ordinary differential equation

$$z'(t) = \frac{b - z(t)}{1-t} + f(t), \quad 0 \leq t < 1, \quad z(0) = a.$$

(d) Conclude that Z is a solution of the stochastic differential equation

$$dZ_t = \frac{b - Z_t}{1-t}dt + dB_t, \quad 0 \leq t < 1, \quad Z_0 = a.$$

Exercise 2: Let $B = (B_t)_{t\geq 0}$ be a one-dimensional Brownian motion and $\xi \in L^2$ independent of B. Consider the stochastic differential equation given by

$$dX_t = r_t X_t dt + v_t X_t dB_t, \quad X_0 = \xi,$$

where $r, v : \mathbb{R}_+ \to \mathbb{R}$ are bounded functions.
(a) Show that the above SDE admits a unique strong solution.
(b) Determine the solution of the SDE by considering $Y = \log X$.
(c) Assume that $r \equiv \rho \in \mathbb{R}$ and $v \equiv \eta \in \mathbb{R}$ are constants and $\xi = 1$ almost surely. Discuss the asymptotic behaviour of X_t as $t \to \infty$. Which process do you obtain in this case?
(d) Consider another stochastic differential equation given by

$$dZ_t = (r_t Z_t + r'_t)dt + (v_t Z_t + v'_t)dB_t, \quad Z_0 = \xi,$$

where $r, r', v, v' : \mathbb{R}_+ \to \mathbb{R}$ are bounded functions. Show that this SDE admits a unique strong solution.
(e) Determine the solution of the SDE in (d) by considering $C = X^{-1}Z$.

Exercise 3: Let $B = (B_t)_{t\geq 0}$ be a one-dimensional Brownian motion and assume that $X^x = (X_t)_{t\geq 0}$, $x \in \mathbb{R}$, is the strong solution of the stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x,$$

where $b, \sigma : \mathbb{R}_+ \to \mathbb{R}$ are Lipschitz continuous functions with Lipschitz constant $K > 0$. The aim of this exercise is to show that the function $u : \mathbb{R} \to \mathbb{R}$ given by

$$u(x) = \mathbb{E}_x[f(X_t)]$$

is continuous, where $f : \mathbb{R} \to \mathbb{R}$ is bounded and continuous.

(a) Show that for all $t \geq 0, x, y \in \mathbb{R}$ we have

$$\mathbb{E}\big[(X_t^x - X_t^y)^2\big] \leq (x-y)^2 C_t,$$

where $C : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$C_t = 3\exp\big(3K^2 t(1+t)\big).$$

Hint: Use Gronwall's Lemma.

(b) Conclude that u is continuous.

Hint: Recall that a function v is continuous at a point x if and only if for every sequence $x_n \to x$ there exists a subsequence x_{n_k} with $v(x_{n_k}) \to v(x)$.

Hand in on: Thursday, 20.12.2018

Revision: During Exercise lessons