Coalgebra structures in algebras

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Fez, June 18-21, 2014

Overview

- Modules and algebras over commutative rings
- Hopf algebras
- Coalgebras and comodules
- Frobenius algebras
- Entwining algebras and coalgebras
- Bialgebras and Hopf algebras

Module categories over commutative ring R

Basic notions

objects	R-modules
morphisms	R-homomorphisms

Special properties

category $\mathbb{M}_R$ has	products and coproducts
	kernels and cokernels
R	projective generator

$$
\begin{array}{rl}
M \otimes_R N & \text{tensor product} \\
M \otimes_R -,\ \mathrm{Hom}_R(M, -) & \mathbb{M}_R \to \mathbb{M}_R \ \text{ functors} \\
\mathrm{Hom}_R(M \otimes_R N, K) & \simeq\ \mathrm{Hom}_R(N, \mathrm{Hom}_R(M, K)) \\
\mathrm{tw} : M \otimes_R N \to N \otimes_R M & m \otimes n \mapsto n \otimes m
\end{array}
$$

Algebras and modules

R-algebras (A, m, e)

multiplication	$m : A \times A \to A,\ (a, b) \mapsto ab$	R-bilinear
	$m : A \otimes A \to A,\ a \otimes b \mapsto ab$	R-linear
unit element	$\eta : R \to A \quad (\eta(1_R) = 1_A)$	
	$m \circ (I_A \otimes \eta) = I_A = m \circ (\eta \otimes I_A)$	

Associativity and unitality

$$
\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "m \otimes I"] \arrow[d, "I \otimes m"'] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"] & A
\end{tikzcd}
,\qquad
\begin{tikzcd}
A \otimes A \arrow[dr, "m"] & R \otimes A \arrow[l, "\eta \otimes I"'] \arrow[d, "="] \\
A \otimes R \arrow[u, "I \otimes \eta"] \arrow[r, "="] & A
\end{tikzcd}
$$

Product of algebras (A, m, η), (B, m', η')

Multiplication and unit on $A \otimes B$

$$\begin{array}{c} m_{AB} : A \otimes B \otimes A \otimes B \xrightarrow{A \otimes \mathrm{tw} \otimes B} A \otimes A \otimes B \otimes B \xrightarrow{m \otimes m'} A \otimes B \\ a \otimes b \otimes c \otimes d \longmapsto a \otimes c \otimes b \otimes d \longmapsto ab \otimes cd \end{array}$$

$$\eta_{AB} : R \xrightarrow{\eta \otimes \eta'} A \otimes B, \qquad 1_R \mapsto 1_A \otimes 1_B$$

Replace $\mathrm{tw} : B \otimes A \to A \otimes B$

$$\lambda : B \otimes A \to A \otimes B$$

Product of algebras (A, m, η), (B, m', η')

Distributive law $\lambda : B \otimes_R A \to A \otimes_R B$, R-linear

$$\begin{tikzcd} B \otimes A \otimes A \arrow[r, "\lambda\otimes A"] \arrow[d, "B\otimes m"'] & A \otimes B \otimes A \arrow[r, "A\otimes\lambda"] & A \otimes A \otimes B \arrow[d, "m\otimes B"] \\ B \otimes A \arrow[rr, "\lambda"] & & A \otimes B \end{tikzcd}$$

$$\begin{tikzcd} B \otimes A \arrow[rr, "\lambda"] & & A \otimes B \\ B \otimes B \otimes A \arrow[u, "m'\otimes A"] \arrow[r, "B\otimes\lambda"'] & B \otimes A \otimes B \arrow[r, "\lambda\otimes B"'] & A \otimes B \otimes B. \arrow[u, "A\otimes m'"'] \end{tikzcd}$$

$$\begin{tikzcd} B \otimes A \arrow[rr, "\lambda"] & & A \otimes B \\ & B \arrow[ul, "B\otimes\eta"] \arrow[ur, "\eta\otimes B"'] & \end{tikzcd} \quad , \quad \begin{tikzcd} B \otimes A \arrow[rr, "\lambda"] & & A \otimes B \\ & A \arrow[ul, "\eta'\otimes A"] \arrow[ur, "A\otimes\eta'"'] & \end{tikzcd}$$

Modules

Left A-modules, category ${}_A\mathbb{M}$

R-linear map $\quad \rho : A \otimes M \to A,\ a \otimes m \mapsto am$

unitality $\quad M \xrightarrow{\eta \otimes I_M} A \otimes M \xrightarrow{\rho} M = M \xrightarrow{I} M.$

Lifting of functors by λ

$$\begin{tikzcd} {}_B\mathbb{M} \arrow[r, "\widehat{A}"] \arrow[d, "U"'] & {}_B\mathbb{M} \arrow[d, "U"] \\ {}_R\mathbb{M} \arrow[r, "A\otimes -"] & {}_R\mathbb{M} \end{tikzcd}$$

$U : {}_B\mathbb{M} \to {}_R\mathbb{M},\ (M, \rho) \mapsto M$

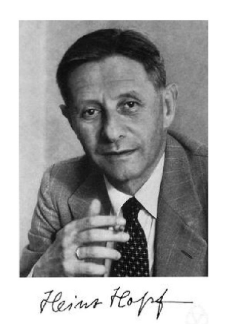
$\rho : B \otimes M \to M,$

$\widehat{A}(M) = A \otimes M,\ B \otimes A \otimes M \xrightarrow{\lambda \otimes I} A \otimes B \otimes M \xrightarrow{I \otimes \rho} A \otimes M,$

$$\widehat{A}(-) = (A \otimes B) \otimes_B - : {}_B\mathbb{M} \to {}_B\mathbb{M}$$

Hopf algebras

Annals of Mathematics 42 (1941)



ÜBER DIE TOPOLOGIE DER GRUPPEN-MANNIGFALTIGKEITEN UND IHRE VERALLGEMEINERUNGEN

By Heinz Hopf

Hopf algebras

Annals of Mathematics 81 (1965)

On the Structure of Hopf Algebras

By JOHN W. MILNOR and JOHN C. MOORE*

The notion of Hopf algebra[1] has been abstracted from the work of Hopf on manifolds which admit a product operation. The homology $H_*(M; K)$ of such a manifold with coefficients in the field K admits not only a diagonal or co-product

$$H_*(M; K) \longrightarrow H_*(M; K) \otimes H_*(M; K)$$

induced by the diagonal $M \to M \times M$, but also a product

$$H_*(M, K) \otimes H_*(M; K) \longrightarrow H_*(M; K)$$

induced by the product $M \times M \to M$. The structure theorem of Hopf concerning such algebras has been generalized by Borel, Leray, and others.

Hopf algebras

Hopf algebras

- algebraic topoloy
- group theory
- group schemes
- homology of Lie algebras
- quantum groups
- non-commutative geometry

Coalgebras

Coalgebra

over a ring R is an R-module C with *linear maps*

$$\Delta : C \to C \otimes_R C, \quad \varepsilon : C \to R,$$

inducing commutative diagrams

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes_R C \arrow[d, "I\otimes\Delta"] \\
C \otimes_R C \arrow[r, "\Delta\otimes I"] & C \otimes_R C \otimes_R C ,
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "="] & C \otimes_R C \arrow[d, "\varepsilon\otimes I"] \\
C \otimes_R C \arrow[r, "I\otimes\varepsilon"'] & C .
\end{tikzcd}$$

Coalgebras

C-comodule

is an R-module M with *R-linear maps*

$$\rho^M : M \longrightarrow C \otimes_R M,$$

with commutative diagram

$$\begin{tikzcd}
M \arrow[r, "\rho^M"] \arrow[d, "\rho^M"'] & C \otimes_R M \arrow[d, "\Delta \otimes I"] \\
C \otimes_R M \arrow[r, "I \otimes \rho^M"] & C \otimes_R C \otimes_R M
\end{tikzcd}$$

Coalgebras

C-comodule homomorphism

R-linear map $f : M \to N$ with commutative diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\rho^M"'] & N \arrow[d, "\rho^M"] \\ C \otimes_R M \arrow[r, "I \otimes f"] & C \otimes_R N \end{tikzcd}$$

Category of left Comodules ${}^C\mathbf{M}$

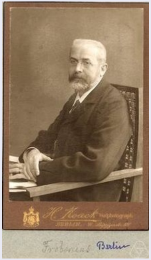
left C-comodules and comodule homomorphisms

Functor

$$C \otimes_R - : {}_R\mathbb{M} \to {}^C\mathbb{M}, \quad X \mapsto (C \otimes_R X, \Delta \otimes I_X).$$

Frobenius algebras

Ferdinand Frobenius, *Theorie der hyperkomplexen Größen, 1903*



Frobenius algebras

A finite dimensional K-algebra
$A^* = \mathrm{Hom}_K(A, K)$ left A-module
$A \simeq A^*$ as left A-modules

$\sigma : A \times A \to K$, nondegenerate, associative $\sigma(ab, c) = \sigma(a, bc)$

Frobenius algebras

Frobenius algebras

- named and studied by Brauer and Nesbitt (1937)
- duality properties (Nakayama 1939)
- representation theory of finite groups
- relation with coalgebras (Lawvere 1967, Abrams 1999)
- topological quantum field theory (Abrams 1996)
- coding theory
- cohomolgy rings of compact oriented manifolds

Frobenius algebras

Coalgebra structure of A^*, A_K finite dimensional

Multiplication and unit on A:

$$m : A \otimes_K A \to A, \quad \eta : K \to A,\ k \mapsto k1_A.$$

Apply $(-)^* = \mathrm{Hom}_K(-, K)$, comultiplication and counit on A^*:

$$A^* \xrightarrow{m^*} (A \otimes_K A)^* \simeq A^* \otimes_K A^*, \quad A^* \xrightarrow{\eta^*} K.$$

Frobenius algebras, L. Abrams (1999)

Coalgebra structure, $\varphi : A \to A^*$, $\varepsilon := \varphi(1_A) : A \to K$

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[d, "\varphi"'] & A \otimes_K A \\ A^* \arrow[r, "m^*"] & A^* \otimes_K A^* \arrow[u, "\varphi^{-1}\otimes\varphi^{-1}"'] \end{tikzcd}, \qquad \begin{tikzcd} A & A \otimes_R A \arrow[l, "A\otimes\varepsilon"'] \arrow[r, "\varepsilon\otimes A"] & A \\ & A \arrow[ul, "="] \arrow[u, "\delta"'] \arrow[ur, "="'] & \end{tikzcd} .$$

Compatibility: Frobenius conditions

$$\begin{tikzcd} A \otimes A \arrow[r, "m"] \arrow[d, "I\otimes\delta"'] & A \arrow[d, "\delta"] \\ A \otimes A \otimes A \arrow[r, "m\otimes I"] & A \otimes A \end{tikzcd}, \qquad \begin{tikzcd} A \otimes A \arrow[r, "m"] \arrow[d, "\delta\otimes I"'] & A \arrow[d, "\delta"] \\ A \otimes A \otimes A \arrow[r, "I\otimes m"] & A \otimes A \end{tikzcd}$$

A-modules $\simeq$ A-comodules: $\quad {}_A\mathbb{M} \simeq {}^A\mathbb{M}$

Frobenius algebras

Characterisations

$$\begin{tikzcd} A \otimes A \arrow[r, "m"] \arrow[d, "I\otimes\delta"'] & A \arrow[d, "\delta"] \\ A \otimes A \otimes A \arrow[r, "m\otimes I"] & A \otimes A; \end{tikzcd}$$

δ is a left A-module morphism;

m is a right A-comodule morphism;

$$\begin{tikzcd} A \otimes A \arrow[r, "m"] \arrow[d, "\delta\otimes I"'] & A \arrow[d, "\delta"] \\ A \otimes A \otimes A \arrow[r, "I\otimes m"] & A \otimes A; \end{tikzcd}$$

δ is a right A-module morphism;

m is a left A-comodule morphism;

$$\delta(1_A) \in A \otimes A \quad \text{with} \quad a\delta(1_A) = \delta(a) = \delta(1_A)a.$$

Frobenius algebras $(A, m, \eta, \delta, \varepsilon)$

Frobenius bimodules $(M,\ \varrho : A \otimes M \to M,\ \omega : M \to A \otimes M)$

$$\begin{tikzcd}
A \otimes M \arrow[r, "\varrho"] \arrow[d, "I\otimes\omega"'] & M \arrow[d, "\omega"] \\
A \otimes A \otimes M \arrow[r, "m\otimes I"] & A \otimes M
\end{tikzcd}, \qquad
\begin{tikzcd}
A \otimes M \arrow[r, "\varrho"] \arrow[d, "\delta\otimes I"'] & M \arrow[d, "\omega"] \\
A \otimes A \otimes M \arrow[r, "I\otimes\omega"] & A \otimes M
\end{tikzcd}$$

Category ${}^A_A\mathbb{M}$

objects:	Frobenius bimodules
morphisms:	A-module and A-comodule morphisms

Functor

$$A \otimes - : {}_R M \to {}^A_A\mathbb{M}, \quad X \mapsto (A \otimes X, m_A \otimes X, \delta_A \otimes X)$$

Frobenius algebras

$A \otimes - : {}_R\mathbb{M} \to {}^A_A\mathbb{M}$

right adjoint $\quad Hom^A_A(A, -) : {}^A_A\mathbb{M} \to {}_R\mathbb{M}$,

$$\mathrm{Hom}^A_A(A \otimes X, M) \simeq \mathrm{Hom}_R(X, \mathrm{Hom}^A_A(A, M))$$

Coinvariants $\mathrm{Hom}^A_A(A, M)$, $\quad A^{coinv} := \mathrm{End}^A_A(A) \simeq A$

Equivalence

$$A \otimes_R - : {}_A\mathbb{M} \to {}^A_A\mathbb{M}$$

Frobenius algebras $(A, m, \eta, \delta, \varepsilon)$

Further equivalences

For $\varrho : A \otimes M \to M$, define

$$\omega : \; M \xrightarrow{\eta \otimes I} A \otimes M \xrightarrow{\delta \otimes I} A \otimes A \otimes M \xrightarrow{I \otimes \varrho} A \otimes M,$$

then (M, ϱ, ω) is a Frobenius bimodule and

$$\Psi : {}_A\mathbb{M} \to {}_A^A\mathbb{M}, \quad (M, \varrho) \mapsto (M\varrho, \omega)$$

is an equivalence of categories.

$${}_A\mathbb{M} \xrightarrow{\Psi} {}_A^A\mathbb{M} \xrightarrow{U_A} {}^A\mathbb{M}, \quad {}^A\mathbb{M} \to {}_A^A\mathbb{M} \xrightarrow{U^A} {}_A\mathbb{M}$$

Separable algebras (A, m, η, δ)

m and δ with Frobenius condition and $m \circ \delta = 1$

The functor

$$A \otimes_R - : {}_R\mathbb{M} \to {}_A\mathbb{M}_A, \quad X \mapsto (A \otimes X, m_A \otimes X)$$

has right adjoint ${}_A Hom_A(A, -) : {}_A\mathbb{M}_A \to {}_R M$,

$${}_A\mathrm{Hom}_A(A \otimes X, M) \simeq \mathrm{Hom}_R(X, {}_A\mathrm{Hom}_A(A, M))$$

Coinvariants ${}_A\mathrm{Hom}_A(A, M)$, $\quad C(A) := {}_A\mathrm{End}_A(A)$ is center of A,

If $R = C(A)$: equivalence (central separable algebra)

$$A \otimes_{C(A)} - : {}_{C(A)}\mathbb{M} \to {}_A\mathbb{M}_A$$

Bialgebras

Bialgebra

An R-module B that is an algebra and a coalgebra,

$$\mu : B \otimes_R B \to B, \qquad \eta : R \to B,$$
$$\Delta : B \to B \otimes_R B, \qquad \varepsilon : B \to R,$$

such that

$$\Delta \text{ and } \varepsilon \text{ are algebra morphisms,}$$

or, equivalently,

$$\mu \text{ and } \eta \text{ are coalgebra morphisms.}$$

Other compatibility conditions ?

Bialgebras $\underline{B} = (B, m, \eta)$, $\overline{B} = (B, \Delta, \varepsilon)$

Δ algebra morphism

$$\begin{tikzcd}
B \otimes B \arrow[r, "m"] \arrow[d, "B\otimes\Delta"'] & B \arrow[r, "\Delta"] & B \otimes B \\
B \otimes B \otimes B \arrow[rr, dashed, "\omega\otimes B"] \arrow[d, "\Delta\otimes B\otimes B"'] & & B \otimes B \otimes B \arrow[u, "B\otimes m"'] \\
B \otimes B \otimes B \otimes B \arrow[rr, "B\otimes \mathrm{tw}\otimes B"] & & B \otimes B \otimes B \otimes B \arrow[u, "m\otimes B\otimes B"']
\end{tikzcd}$$

$$\omega : B \otimes B \xrightarrow{\Delta\otimes B} B \otimes B \otimes B \xrightarrow{B\otimes \mathrm{tw}} B \otimes B \xrightarrow{m\otimes B} B \otimes B$$

$$\overline{\omega} : B \otimes B \xrightarrow{B\otimes\Delta} B \otimes B \otimes B \xrightarrow{\mathrm{tw}\otimes B} B \otimes B \xrightarrow{B\otimes m} B \otimes B$$

Bialgebras $\underline{B}$, $\overline{B}$, $\omega : \underline{B} \otimes \overline{B} \to \overline{B} \otimes \underline{B}$

Hopf modules $\rho : B \otimes M \to M, \quad \nu : M \to B \otimes M$

$$\begin{tikzcd}
& B \otimes M \arrow[r, "\rho"] \arrow[d, "B\otimes\nu"'] & M \arrow[r, "\nu"] & B \otimes M \\
& B \otimes B \otimes M \arrow[rr, "\omega\otimes M"] & & B \otimes B \otimes M \arrow[u, "B\otimes\rho"']
\end{tikzcd}$$

Functor $B \otimes_R - : {}_R\mathbb{M} \to {}^B_B\mathbb{M}, \quad X \mapsto (B \otimes X, m \otimes X, \Delta \otimes X)$

with right adjoint $\mathrm{Hom}^B_B(B, -) : {}^B_B\mathbb{M} \to {}_R\mathbb{M}$,

$$\mathrm{Hom}^B_B(B \otimes X, M) \simeq \mathrm{Hom}_R(X, \mathrm{Hom}^B_B(B, M))$$

Coinvariants $\mathrm{Hom}^B_B(B, M)$, $\quad B^{coinv} = \mathrm{End}^B_B(B) \simeq R$

$(\underline{B}, \overline{B}, \omega)$ Hopf algebra iff $B \otimes_R - : {}_R\mathbb{M} \to {}^B_B\mathbb{M}$ is equivalence

Entwining (A, m, η) and (C, δ, ε)

T. Brzeziński and Sh. Majid

Comodule bundles, Comm. Math. Phys. (1999)

Entwining from A to C, $\quad \omega : A \otimes C \to C \otimes A$

$$
\begin{tikzcd}
A\otimes C\otimes C \arrow[r, "\omega\otimes C"] & C\otimes A\otimes C \arrow[r, "C\otimes\omega"] & C\otimes C\otimes A, \\
A\otimes C \arrow[u, "A\otimes\delta"] \arrow[rr, "\omega"] & & C\otimes A \arrow[u, "\delta\otimes A"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes A\otimes C \arrow[r, "A\otimes\omega"] \arrow[d, "m\otimes C"'] & A\otimes C\otimes A \arrow[r, "\omega\otimes A"] & C\otimes A\otimes A \arrow[d, "C\otimes m"] \\
A\otimes C \arrow[rr, "\omega"] & & C\otimes A
\end{tikzcd}
$$

$$C \otimes \eta = \omega \circ (\eta \otimes C), \qquad A \otimes \varepsilon = (\varepsilon \otimes A) \circ \omega$$

Algebra $A = (B, m, \eta)$, coalgebra $C = (C, \delta, \varepsilon)$

Entwining from C to A, $\overline{\omega} : C \otimes A \to A \otimes C$

$$\begin{tikzcd} C\otimes A\otimes A \arrow[r, "\overline{\omega}\otimes A"] \arrow[d, "C\otimes m"'] & A\otimes C\otimes A \arrow[r, "A\otimes\overline{\omega}"] & A\otimes A\otimes C \arrow[d, "m\otimes C"] \\ C\otimes A \arrow[rr, "\overline{\omega}"] & & A\otimes C \end{tikzcd}, \qquad \begin{tikzcd} C\otimes C\otimes A \arrow[r, "C\otimes\overline{\omega}"] & C\otimes A\otimes C \arrow[r, "\overline{\omega}\otimes C"] & A\otimes C\otimes C \\ C\otimes A \arrow[u, "\delta\otimes A"] \arrow[rr, "\overline{\omega}"] & & A\otimes C \arrow[u, "A\otimes\delta"'] \end{tikzcd}$$

$$\overline{\omega} \circ (C \otimes \eta) = \eta \otimes C, \qquad \varepsilon \otimes A = (A \otimes \varepsilon) \circ \overline{\omega}$$

Algebra $A = (B, m, \eta)$, coalgebra $C = (C, \delta, \varepsilon)$

Entwining from A to C, $\quad \omega : A \otimes C \to C \otimes A$, $\quad$ EM categories

$$
\begin{tikzcd}
{}_A\mathbb{M} \arrow[r, "\widehat{C}"] \arrow[d, "U_A"'] & {}_A\mathbb{M} \arrow[d, "U_A"] \\
{}_R\mathbb{M} \arrow[r, "C\otimes -"] & {}_R\mathbb{M}
\end{tikzcd}
\qquad
\begin{tikzcd}
{}^C\mathbb{M} \arrow[r, "\widehat{A}"] \arrow[d, "U^C"'] & {}^C\mathbb{M} \arrow[d, "U^C"] \\
{}_R\mathbb{M} \arrow[r, "A\otimes -"] & {}_R\mathbb{M}
\end{tikzcd}
$$

Entwining from C to A, $\quad \overline{\omega} : C \otimes A \to A \otimes C$, $\quad$ Kleisli categories

$$
\begin{tikzcd}
{}_R\mathbb{M} \arrow[r, "C\otimes -"] \arrow[d, "\Phi_A"'] & {}_R\mathbb{M} \arrow[d, "\Phi_A"] \\
{}_A\widetilde{\mathbb{M}} \arrow[r, "\widetilde{C}"] & {}_A\widetilde{\mathbb{M}}
\end{tikzcd}
\qquad
\begin{tikzcd}
{}_R\mathbb{M} \arrow[r, "A\otimes -"] \arrow[d, "\Phi^C"'] & {}_R\mathbb{M} \arrow[d, "\Phi^C"] \\
{}^C\widetilde{\mathbb{M}} \arrow[r, "\widetilde{A}"] & {}^C\widetilde{\mathbb{M}}
\end{tikzcd}
$$

Weak Hopf algebras

Journal of Algebra 221 (1999)

Weak Hopf Algebras

I. Integral Theory and C^*-Structure

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Weak Hopf algebras

algebra $\underline{B} = (B, m, \eta)$, coalgebra $\overline{B} = (B, \Delta, \varepsilon)$, $a, b, c \in B$

$$\Delta(ab) = \Delta(a)\Delta(b)$$

$$\varepsilon(abc) = \sum \varepsilon(ab_{\underline{2}})\varepsilon(b_{\underline{1}}c), \qquad [\,\varepsilon(ab) = \varepsilon(a)\varepsilon(b)\,]$$
$$= \sum \varepsilon(ab_{\underline{1}})\varepsilon(b_{\underline{2}}c);$$

$$(I \otimes \Delta) \circ \Delta(1) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1), \qquad [\,\Delta(1) = 1 \otimes 1\,]$$
$$= (\Delta(1) \otimes 1)(1 \otimes \Delta(1)).$$

Weak entwinings

$$\omega : \underline{B} \otimes \overline{B} \to \overline{B} \otimes \underline{B}, \quad \overline{\omega} : \overline{B} \otimes \underline{B} \to \underline{B} \otimes \overline{B}$$

weakened conditions on units and counits

Weak entwinings between (A, m, η) and (C, δ, ε)

Weak entwining

$\omega : A \otimes C \to C \otimes A$ $\qquad$ $\overline{\omega} : C \otimes A \to A \otimes C$

$\xi : \; C \xrightarrow{\eta\otimes C} A\otimes C \xrightarrow{\omega} C\otimes A \xrightarrow{\varepsilon\otimes A} A$ $\qquad$ $\overline{\xi} : \; C \xrightarrow{C\otimes\eta} C\otimes A \xrightarrow{\overline{\omega}} A\otimes C \xrightarrow{A\otimes\varepsilon} A$

$$\begin{tikzcd}
C \arrow[r, "\eta\otimes C"] \arrow[d, "\delta"'] & A \otimes C \arrow[r, "A\otimes\xi"] \arrow[d, "\omega"] & A \otimes A \arrow[d, "m"] \\
C \otimes C \arrow[r, "C\otimes\xi"] & C \otimes A \arrow[r, "\varepsilon\otimes A"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "C\otimes\eta"] \arrow[d, "\delta"'] & C \otimes A \arrow[r, "\overline{\xi}\otimes A"] \arrow[d, "\overline{\omega}"] & A \otimes A \arrow[d, "m"] \\
C \otimes C \arrow[r, "\overline{\xi}\otimes C"] & A \otimes C \arrow[r, "A\otimes\varepsilon"] & A
\end{tikzcd}$$

entwining $\quad \xi = \overline{\xi} : C \xrightarrow{\varepsilon} R \xrightarrow{\eta} A$

General categories $\mathbb{A} = \mathbb{M}_R$, full weak entwining

monad $\quad F = A \otimes_R -$, $\quad$ comonad $\; G = C \otimes_R -$,

weak entwinings $\quad \omega : FG \to GF$, $\quad \overline{\omega} : GF \to FG$

Various algebras

Algebra (A, m), coalgebra (A, δ)

$$\begin{tikzcd} A \otimes A \arrow[r, "m"] \arrow[d, "I\otimes\delta"'] & A \arrow[d, "\delta"] \\ A \otimes A \otimes A \arrow[r, "m\otimes I"] & A \otimes A \end{tikzcd} \qquad \begin{tikzcd} A \otimes A \arrow[r, "m"] \arrow[d, "\delta\otimes I"'] & A \arrow[d, "\delta"] \\ A \otimes A \otimes A \arrow[r, "I\otimes m"] & A \otimes A \end{tikzcd}$$

Frobenius algebra $(A, m, \eta; \delta, \varepsilon)$

equivalence $\quad A \otimes_R - : {}_A\mathbb{M} \to {}_A^A\mathbb{M}$

Separable algebra $(A, m, \eta; \delta)$, $\quad m \circ \delta = I$

Azumaya algebra $(A, m, \eta; \delta, \tau)$, $\quad m \circ \delta = I$

separable and central, equivalence $\quad A \otimes_R - : \mathbb{M}_R \to {}_A\mathbb{M}_A$

Various algebras

Bialgebra: algebra (B, m, η), coalgebra (B, δ, ε), ω

Entwining $\omega : B \otimes B \to B \otimes B$, m, δ-compatibilty

$$\begin{tikzcd}
B \otimes B \arrow[r, "m"] \arrow[d, "B\otimes\delta"'] & B \arrow[r, "\delta"] & B \otimes B \\
B \otimes B \otimes B \arrow[rr, "\omega\otimes B"] & & B \otimes B \otimes B \arrow[u, "B\otimes m"']
\end{tikzcd}$$

ε is an algebra morphism, m is a coalgebra morphism

Hopf algebra

$$\begin{tikzcd}
B \otimes B \arrow[r, "\delta\otimes B"] \arrow[dr, "="'] & B \otimes B \otimes B \arrow[d, "B\otimes m"] \\
 & B \otimes B
\end{tikzcd}$$

equivalence $B \otimes_R - : \mathbb{M}_R \to {}^B_B\mathbb{M}$

antipode $S : B \to B$