Generators in module and comodule categories

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ABSTRACT. The relations between various categories $\mathbb{A}$ and $\mathbb{B}$ are described by functors and important examples for these are the equivalences between categories. They are often of the form $\mathrm{Hom}(P,-) : \mathbb{A} \to \mathbb{B}$ where P is a generator (with additional properties) in $\mathbb{A}$. In this talk we analyse generators in module and comodule categories. While in full module categories any generator is a flat module over its endomorphism ring, this need not be the case for generators in comodule categories, even if they induce equivalences. As special cases Azumaya and Hopf algebras are considered.

1. Preliminaries

We recall some general notions from category theory. Let $\mathbb{A}$ and $\mathbb{B}$ denote two categories. For any functor $F : \mathbb{A} \to \mathbb{B}$ and objects A, A' from $\mathbb{A}$ there is a map

$$F_{A,A'} : \mathrm{Mor}_{\mathbb{A}}(A, A') \to \mathrm{Mor}_{\mathbb{B}}(F(A), F(A')).$$

F is called *faithful (full)* if all $F_{A,A'}$ are injective (surjective), $A, A' \in \mathbb{A}$. An object P in $\mathbb{A}$ is a *generator in* $\mathbb{A}$ if $\mathrm{Mor}_{\mathbb{A}}(P,-) : \mathbb{A} \to \mathbf{Set}$ is a faithful functor.

1.1. **Adjoint functors.** A pair of functors $F : \mathbb{A} \to \mathbb{B}$, $G : \mathbb{B} \to \mathbb{A}$ is said to be *adjoint* if there is a functorial isomorphism

$$\mathrm{Mor}_{\mathbb{B}}(F(A), B) \longrightarrow \mathrm{Mor}_{\mathbb{A}}(A, F(B)), \quad B \in \mathbb{B},\ A \in \mathbb{A}.$$

Associated with such an adjoint pair there are natural transformations

$$\textit{unit } \eta : I_{\mathbb{B}} \to GF \text{ and } \textit{counit } \varepsilon : FG \to I_A.$$

(1) *The following are equivalent:*
   - (a) *G is full and faithful;*
   - (b) *$\varepsilon : FG \to I_{\mathbb{B}}$ is an isomorphism.*

(2) *The following are equivalent:*
   - (a) *F is full and faithful;*
   - (b) *$\eta : I_{\mathbb{A}} \to GF$ is an isomorphism.*

(3) *The following are equivalent:*
   - (a) *F and G are full and faithful;*
   - (b) *$\varepsilon : FG \to I_{\mathbb{B}}$ and $\eta : I_{\mathbb{A}} \to GF$ are isomorphisms;*
   - (c) *F and G are (inverse) equivalences between $\mathbb{A}$ and $\mathbb{B}$.*

In *Grothendieck categories* we have the following properties of generators (see [**Nas**, III, Teoremă 9.1]).

1.2. **Gabriel Popescu Theorem.** *Let $\mathbb{A}$ be a Grothendieck category, P a generator in $\mathbb{A}$, and $S = \mathrm{End}_{\mathbb{A}}(P)$. Then $\mathrm{Hom}_{\mathbb{A}}(P,-) : \mathbb{A} \to {}_S\mathbf{M}$ has a left adjoint $T : {}_S\mathbf{M} \to \mathbb{A}$ and:*

(1) *$\varepsilon : T\mathrm{Hom}_{\mathbb{A}}(P,-) \to I_{\mathbb{A}}$ is a functorial isomorphism;*

(2) *$\mathrm{Hom}_{\mathbb{A}}(P,-)$ is fully faithful;*

(3) *T is left exact.*

2. Generators in the category of A-modules

In this section A will denote an associative ring with unit and ${}_A\mathbf{M}$ denotes the category of left A-modules. Morphisms of left modules should act from the right while those of right modules are acting from the left.

Throughout P will be a left A-module and $S = \mathrm{End}_A(P)$. Thus P is a right S-module and there is a canonical ring morphism

$$\phi : A \to B = \mathrm{End}_S(M), \quad a \mapsto [m \mapsto am].$$

P is called *balanced* provided ϕ is an isomorphism.

2.1. **Generators in ${}_A\mathbf{M}$ - 1.** *For a module P in ${}_A\mathbf{M}$ and $S = \mathrm{End}_A(P)$, the following are equivalent:*

(a) *P is a generator in ${}_A\mathbf{M}$;*

(b) *every A-module N is P-generated, that is, there is an epimorphism $P^{(\Lambda)} \to N$, Λ some index set;*

(c) (i) *$\phi : A \to B$ is an isomorphism and*
(ii) *P_S is finitely generated and projective.*

The characterisation given in (c) is due to Morita [**Mor**] (e.g. [**Fai**, Proposition 3.26], [**WiFo**, 18.8]). The interesting aspect of it is that it describes the *universal* property of P being a generator in ${}_A\mathbf{M}$ by *internal* properties of P as an R- and an S-module. In particular, a generating property on the left side is related to a projectivity property on the right.

2.2. **Adjoint pair $P \otimes_S -$, $\mathrm{Hom}_A(P,-)$.** The functors

$$P \otimes_S - : {}_S\mathbf{M} \to {}_A\mathbf{M}, \quad \mathrm{Hom}_A(P,-) : {}_A\mathbf{M} \to {}_S\mathbf{M},$$

form an adjoint pair with functorial isomorphism for $X \in {}_S\mathbf{M}$, $Y \in {}_A\mathbf{M}$,

$$\mathrm{Hom}_A(P \otimes_S X, Y) \to \mathrm{Hom}_S(X, \mathrm{Hom}_A(P,Y)),$$

$$\text{unit} \quad \eta_X : X \to \mathrm{Hom}_A(P, P \otimes_S X), \quad x \mapsto [p \mapsto p \otimes x];$$

$$\text{counit} \quad \varepsilon_Y : P \otimes_S \mathrm{Hom}_A(P,Y) \to Y, \quad p \otimes f \mapsto (p)f.$$

A special property of the category of A-modules is the fact that for a generator P, the functor $\mathrm{Hom}_A(P,-) : {}_A\mathbf{M} \to {}_S\mathbf{M}$ is not only faithful (required by the definition) but also full. To show this we recall the following (e.g. [**WiFo**, 15.9]).

2.3. **Modules flat over the endomorphism ring.** *For an A-module P and $S = \mathrm{End}_A(P)$, the following are equivalent:*

(a) *P is a flat right S-module;*

(b) *for any $f : P^n \to P$, $n \in \mathbb{N}$, $\mathrm{Ke}\, f$ is P-generated.*

From 1.2 we obtain

2.4. **Generators in ${}_A\mathbf{M}$ - 2.** *For an A-module P the following are equivalent:*

(a) $\mathrm{Hom}_A(P,-) : {}_A\mathbf{M} \to {}_S\mathbf{M}$ *is faithful;*
(b) $\mathrm{Hom}_A(P,-) : {}_A\mathbf{M} \to {}_S\mathbf{M}$ *is fully faithful;*
(c) $\varepsilon : P \otimes_S \mathrm{Hom}_A(P,N) \to N$ *is an isomorphism for any* $N \in {}_A\mathbf{M}$*;*
(d) $\varepsilon : P \otimes_S \mathrm{Hom}_A(P,N) \to N$ *is surjective for any* $N \in {}_A\mathbf{M}$*.*

Proof. (a)⇒(b) follows from the Gabriel Popescu Theorem 1.2.

The other implications are obvious. □

The properties of generators strongly depend on the categories they generate. For example, consider $\mathrm{Gen}(P)$, the full subcategory of ${}_A\mathbf{M}$ whose objects are the P-generated modules. It is the largest subcategory for which P is a generator.

2.5. **Generator in** $\mathrm{Gen}(\mathbf{P})$**.** *For an A-module P the following are trivial.*

(1) P *is a generator in* $\mathrm{Gen}(P)$*;*
(2) $\mathrm{Hom}_A(P,-) : \mathrm{Gen}(P) \to {}_S\mathbf{M}$ *is faithful;*
(3) $\varepsilon_N : P \otimes_S \mathrm{Hom}_A(P,N) \to N$ *is surjective for any* $N \in \mathrm{Gen}(P)$*.*

Here P need not be flat over its endomorphism ring unless $\mathrm{Gen}(P)$ has special properties, e.g., is a Grothendieck category. To study this consider the full subcategory $\sigma[P]$ of ${}_A\mathbf{M}$ whose objects are subgenerated by P, that is, they are submodules of P-generated modules. Since $\sigma[P]$ is a Grothendieck category, the proof for 2.4 applies to show:

2.6. **Generator in $\sigma[P]$ - 1.** *For an A-module P the following are equivalent:*

(a) $\mathrm{Hom}_A(P,-) : \sigma[P] \to {}_S\mathbf{M}$ *is faithful;*
(b) $\mathrm{Hom}_A(P,-) : \sigma[P] \to {}_S\mathbf{M}$ *is fully faithful;*
(c) $\varepsilon_N : P \otimes_S \mathrm{Hom}_A(P,N) \to N$ *is surjective,* $N \in \sigma[P]$*;*
(d) $\varepsilon_N : P \otimes_S \mathrm{Hom}_A(P,N) \to N$ *is an isomorphism,* $N \in \sigma[P]$*;*
(e) P_S *is flat and* ε_V *is an isomorphism for all injectives* $V \in \sigma[P]$*.*

Proof. (e)⇒(d) Let N be any module in $\sigma[P]$ and consider an exact sequence $0 \to N \to I_1 \to I_2$ in $\sigma[P]$ where I_1, I_2 are (P-)injective modules in $\sigma[P]$. Since P_S is flat we obtain a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & P \otimes_S \mathrm{Hom}_A(P,N) \arrow[r] \arrow[d, "\varepsilon_N"] & P \otimes_S \mathrm{Hom}_A(P,I_1) \arrow[r] \arrow[d, "\varepsilon_{I_1}"] & P \otimes_S \mathrm{Hom}_A(P,I_2) \arrow[d, "\varepsilon_{I_2}"] \\
0 \arrow[r] & N \arrow[r] & I_1 \arrow[r] & I_2
\end{tikzcd}$$

By assumption, ε_{I_1} and ε_{I_2} are isomorphisms and hence so is ε_N. Thus P is a generator in $\sigma[P]$ by 2.6.

The remaining assertions are obvious. □

The question arises if one can - similar to Morita's result for ${}_A\mathbf{M}$ (see 2.1) - characterise generators in $\sigma[P]$ by internal properties of P as an R- or S-module. This can be derived from condition (e) in 2.6 by applying ideas from Zimmermann [**Zim**]. For this we recall from [**Zim**] and [**WiSta**]:

2.7. **L-dcc and P-dcc.** For A-modules L, V, P and $S = \mathrm{End}_A(P)$, consider the canonical map

$$\nu_{L,P,V} : L \otimes_S Hom_R(P,V) \to \mathrm{Hom}_R(\mathrm{Hom}_S(L,P),V),\ l \otimes f \mapsto [g \mapsto (g(l))f],$$

which is an isomorphism provided L_S is finitely presented and V is P-injective (e.g., [**WiFo**, 25.5]). Following Zimmermann [**Zim**, 3.2], we say that P_S has *L-dcc* if $\alpha_{L,P}$ is a monomorphism for any injective $V \in \sigma[P]$. The terminology indicates that the condition is related to descending chain conditions on certain matrix subgroups of P. In connection with the density property for P this can be understood as a Mittag-Leffler condition on P_S (e.g. [**WiSta**, 5.4]).

Putting $L = P$ we see that *P-dcc* on P corresponds to the injectivity of

$$\nu_{P,V} : P \otimes_S Hom_A(P,V) \to \mathrm{Hom}_A(\mathrm{Hom}_S(P,P),V).$$

With the canonical map $\phi : A \to B := \mathrm{Hom}_S(P,P)$ we obtain the commutative diagram

$$\begin{tikzcd} P \otimes_S \mathrm{Hom}_A(P,V) \arrow[r, "\nu_{P,V}"] \arrow[dr, "\varepsilon_V"'] & \mathrm{Hom}_A(B,V) \arrow[d, "\phi^*"] \\ & V. \end{tikzcd}$$

As an injective module in $\sigma[P]$, V is P-generated and so ε_V is always surjective. If ϕ^* is an isomorphism, e.g., if $\phi : A \to B$ is dense, then P has P-dcc if and only if ε_V is an isomorphism for all injectives $V \in \sigma[P]$.

Thus Morita's description of generators in ${}_A\mathbf{M}$ can be generalised to generators in $\sigma[P]$ in the following way (see [**WiSta**, Corollary 5.8]):

2.8. **Generators in $\sigma[P]$ - 2.** *For an A-module P the following are equivalent:*

(a) *P is a generator in $\sigma[P]$;*

(b) *P_S is flat and has P-dcc and*

 (i) *$\phi : A \to B$ is dense, or*

 (ii) *$\phi^* : \mathrm{Hom}_A(B,V) \simeq \mathrm{Hom}_A(A,V)$ for all (injective) $V \in \sigma[P]$.*

3. Bimodules and Azumaya algebras

Let A denote an algebra over a commutative associative ring R with unit, with multiplication $\mu : A \otimes_R A \to A$ and unit map $\eta : R \to A$.

3.1. **(A,A)-bimodules.** Denote the category of (A,A)-bimodules by ${}_A\mathbf{M}_A$. This can also be considered as category of left modules over the enveloping algebra $A \otimes_R A^o$, that is we may identify ${}_A\mathbf{M}_A = {}_{A\otimes_R A^o}\mathbf{M}$.

The *center* of an (A,A)-bimodule M is defined as

$$Z(M) = \{m \in M \mid am = ma \text{ for all } a \in A\}$$

and it is easily veryfied that

$${}_A\mathrm{Hom}_A(A,M) \simeq Z(M), \quad {}_A\mathrm{End}_A(A) \simeq Z(A).$$

The multiplication μ factors via $\mu' : A \otimes_{Z(A)} A \to A$

Clearly A itself is an (A, A)-bimodule but in general it is neither projective nor a generator in the category ${}_A\mathbf{M}_A$. To describe this situation we apply the results from the previous section to the adjoint pair of functors

$$A \otimes_{Z(A)} - : {}_{Z(A)}\mathbf{M} \to {}_A\mathbf{M}_A, \quad {}_A\mathrm{Hom}_A(A, -) : {}_A\mathbf{M}_A \to {}_{Z(A)}\mathbf{M}.$$

3.2. **Azumaya algebras.** *For a ring A, the following are equivalent:*

(a) *A is a generator in ${}_A\mathbf{M}_A$;*

(b) *A is a projective generator in ${}_A\mathbf{M}_A$;*

(c) ${}_A\mathrm{Hom}_A(A, -) : {}_A\mathbf{M}_A \to {}_{Z(A)}\mathbf{M}$ *is (fully) faithful;*

(d) $A \otimes_{Z(A)} - : {}_{Z(A)}\mathbf{M} \to {}_A\mathbf{M}_A$ *is an equivalence;*

(e) $A \otimes_{Z(A)} A^o \simeq \mathrm{End}_{Z(A)}(A)$ *and $A_{Z(A)}$ is finitely generated and projective;*

(f) *A is projective in ${}_A\mathbf{M}_A$;*

(g) $\mu : A \otimes_{Z(A)} A \to A$ *splits in ${}_A\mathbf{M}_A$.*

Proof. The equivalence of (a), (c) and (e) is clear from 2.1 and 2.4.

(e)⇒(b) As a finitely generated, projective and faithful module over the commutative ring $Z(A)$, A is a generator in ${}_{Z(A)}\mathbf{M}$ (e.g. [**WiFo**, 18.10]) and thus, by 2.4, A is (finitely generated and) projective over $\mathrm{End}_{Z(A)}(A)$.

(b)⇒(f) and (f)⇔(g) are obvious.

(g)⇒(a) See, e.g., [**WiBim**, 28.7]. □

The theory sketched above can be extended in two different directions. Replace ${}_A\mathbf{M}_A$ by $\sigma[{}_AA_A]$, the full subcategory of bimodules subgenerated by A. The generating property in this category does not imply finiteness over the endomorphism ring. Furthermore, replacing the ring $A \otimes_R A^o$ by the multiplication algebra, we need no longer require the algebra A to be associative.

3.3. **Multiplication algebra.** Let A be a not necessarily associative R-algebra with unit. Then left and right multiplications with any $a \in A$ induce R-linear maps

$$L_a : A \to A, \ x \mapsto ax; \quad R_a : A \to A, \ x \mapsto xa.$$

The *multiplication algebra* of A is the (associative) subalgebra

$$M(A) \subset \mathrm{End}_R(A) \text{ generated by } \{L_a, R_a \mid a \in A\}.$$

We consider A as a left module over $M(A)$ (finitely generated by 1_A); $\mathrm{End}_{M(A)}(A)$ is isomorphic to the center of A. By $\sigma[{}_{M(A)}A]$, or $\sigma[A]$ for short, we denote the full subcategory of ${}_{M(A)}\mathbf{M}$ subgenerated by A.

From 2.1 and 2.4 we derive immediately:

3.4. **A as generator in ${}_{M(A)}\mathbf{M}$.** *For a (non-associative) algebra A with unit the following are equivalent:*

(a) *A is a generator in ${}_{M(A)}\mathbf{M}$;*

(b) *A is a projective generator in ${}_{M(A)}\mathbf{M}$;*

(c) $\mathrm{Hom}_{M(A)}(A, -) : {}_{M(A)}\mathbf{M} \to {}_{Z(A)}\mathbf{M}$ *is (fully) faithful;*

(d) $\varepsilon_N : A \otimes_{Z(A)} \mathrm{Hom}_{M(A)}(A, N) \to N$ *is sur(bi-)jective for any $N \in {}_{M(A)}\mathbf{M}$;*

(e) $\phi : M(A) \simeq \mathrm{End}_{Z(A)}(A)$ *and $A_{Z(A)}$ is finitely generated and projective.*

The properties justify to call the algebras described in 3.4 (non-associative) *Azumaya algebras* (see [**WiBim**, 24.8]). Notice that - unlike in ${}_{A\otimes_{Z(A)}A^o}\mathbf{M}$ - projectivity of A in ${}_{M(A)}\mathbf{M}$ need not imply the generating property.

Applying 2.6 and 2.8 we obtain:

3.5. A **as generator in** $\sigma[A]$**.** *For a (non-associative) algebra A with unit, the following are equivalent:*

(a) $\mathrm{Hom}_{M(A)}(A,-):\sigma[A]\to{}_{Z(A)}\mathbf{M}$ *is (fully) faithful;*

(b) $\varepsilon_N : A\otimes_{Z(A)}\mathrm{Hom}_{M(A)}(A,N)\to N$ *is sur(bi-)jective for any* $N\in\sigma[A]$*;*

(c) $\phi : M(A)\to\mathrm{End}_{Z(A)}(A)$ *is dense and* $A_{Z(A)}$ *is flat and has A-dcc.*

Note that the generating property of A in $\sigma[A]$ need not imply projectivity.

3.6. A **as projective generator in** $\sigma[A]$**.** *For a (non-associative) algebra A with unit, the following are equivalent:*

(a) *A is a projective generator in $\sigma[A]$;*

(b) $\mathrm{Hom}_{M(A)}(A,-):\sigma[A]\to{}_{Z(A)}\mathbf{M}$ *is an equivalence;*

(c) $A\otimes_{Z(A)}-:{}_{Z(A)}\mathbf{M}\to\sigma[A]$ *is an equivalence;*

(d) $\varepsilon_N : A\otimes_{Z(A)}\mathrm{Hom}_{M(A)}(A,N)\to N$ *is surjective (bijective) for any* $N\in\sigma[A]$*;*

(e) $\phi : M(A)\to\mathrm{End}_{Z(A)}(A)$ *is dense and* $A_{Z(A)}$ *is faithfully flat and has A-dcc.*

The rings considered in 3.6 are named *Azumaya rings* (e.g. [**WiBim**, 26.4]). For example, any (non-associative) simple ring with unit is of this type.

Associative Azumaya rings are closely related to the *ideal algebras* studied by M.L. Ranga Rao, a former student of Carl Faith (see [**Rao**]).

The theory sketched above can also be formulated for rings A without unit. In this case the role of the center of A is taken over by the *centroid* of A, $C(A)=\mathrm{End}_{M(A)}(A)$ (see [**WiBim**]).

4. Corings and comodules

Consider any coring $(\mathcal{C},\Delta,\varepsilon)$ over an associative ring A with unit, that is, an (A,A)-bimodule $\mathcal{C}$ with bimodule morphisms $\Delta:\mathcal{C}\to\mathcal{C}\otimes_A\mathcal{C}$ and $\varepsilon:\mathcal{C}\to A$ satisfying the coassociativity and counitality conditions. We recall some elementary facts from [**BrWi**, Chapter 3].

4.1. **Category of comodules.** Right $\mathcal{C}$-comodules ared defined as right A-modules M with an A-linear map (coaction) $\varrho^M : M\to M\otimes_A\mathcal{C}$ satisfying the coassociativity and counitality conditions.

Comodule morphisms $f:M\to N$ between $\mathcal{C}$-comodules are those A-linear maps inducing commutative diagrams

$$\begin{array}{ccc} M & \xrightarrow{\quad f\quad} & N \\ {\scriptstyle \varrho_M}\downarrow & & \downarrow{\scriptstyle \varrho_N} \\ M\otimes_A\mathcal{C} & \xrightarrow{\ f\otimes I\ } & N\otimes_A\mathcal{C}. \end{array}$$

The category of all right $\mathcal{C}$-comodules is denoted by $\mathbf{M}^{\mathcal{C}}$. It is an additive category with direct sums and cokernels, and it is abelian provided $\mathcal{C}$ is flat as a left A-module. There is a bijective natural map

$$\varphi:\mathrm{Hom}^{\mathcal{C}}(M,X\otimes_A\mathcal{C})\to\mathrm{Hom}_A(M,X),\ f\mapsto(I_X\otimes\varepsilon)\circ f,$$

which shows that the functor $-\otimes_A \mathcal{C}$ is right adjoint to the forgetful functor $\mathbf{M}^{\mathcal{C}} \to \mathbf{M}_A$. It also shows that the ring $\mathrm{End}^{\mathcal{C}}(\mathcal{C})$ of comodule endomorphisms of $\mathcal{C}$ is isomorphic to the dual ring $\mathcal{C}^* = \mathrm{Hom}_A(\mathcal{C}, A)$, the latter endowed with the convolution product for $f, g \in \mathcal{C}^*$, $f *^r g(c) = g((f \otimes I)\Delta(c))$.

Similarly, the left dual ${}^*\mathcal{C} = {}_A\mathrm{Hom}(\mathcal{C}, A)$ is a ring with convolution product $f *^l g(c) = f((I \otimes g)\Delta(c))$. The counit ε is the unit for both $\mathcal{C}^*$ and ${}^*\mathcal{C}$.

Every right $\mathcal{C}$-comodule (M, ϱ^M) has a left ${}^*\mathcal{C}$-module structure given by

$$\rightharpoonup : {}^*\mathcal{C} \otimes_R M \to M, \quad f \otimes m \mapsto (I_M \otimes f) \circ \varrho^M(m).$$

Endowed with this structure, any $\mathcal{C}$-comodule morphism $f : M \to N$ between comodules is ${}^*\mathcal{C}$-linear, that is

$$\mathrm{Hom}^{\mathcal{C}}(M, N) \subseteq {}_{^*\mathcal{C}}\mathrm{Hom}(M, N),$$

thus inducing a faithful functor $\mathbf{M}^{\mathcal{C}} \to {}_{^*\mathcal{C}}\mathbf{M}$.

This functor is full if and only if $\mathcal{C}$ is locally projective as a left A-module (α-condition, [**BrWi**, 19.2]) and in this case $\mathbf{M}^{\mathcal{C}}$ is equivalent to $\sigma[{}_{^*\mathcal{C}}C]$, the full subcategory of ${}_{^*\mathcal{C}}\mathbf{M}$ subgenerated by $\mathcal{C}$. The comodule category $\mathbf{M}^{\mathcal{C}}$ is equal to ${}_{^*\mathcal{C}}\mathbf{M}$ if and only if ${}_A\mathcal{C}$ is finitely generated and projective.

In general, $\mathbf{M}^{\mathcal{C}}$ is not an abelian category and hence the characterisations of generators considered in earlier sections need not apply. This kind of defect comes from the observation that monomorphisms in $\mathbf{M}^{\mathcal{C}}$ need not be injective maps. However, if ${}_A\mathcal{C}$ is flat, then $\mathbf{M}^{\mathcal{C}}$ is abelian and results similar to those in section 2 can be obtained. In particular, if ${}_A\mathcal{C}$ is locally projective, $\mathbf{M}^{\mathcal{C}}$ is the same as $\sigma[{}_{^*\mathcal{C}}C]$ and propositions like 2.6 or 2.8 do apply directly.

4.2. **Generators in $\mathbf{M}^{\mathcal{C}}$.** *Let $\mathcal{C}$ be an A-coring with ${}_A\mathcal{C}$ flat, $P \in \mathbf{M}^{\mathcal{C}}$ and $S = \mathrm{End}^{\mathcal{C}}(P)$. The following are equivalent:*

(a) *P is a generator in $\mathbf{M}^{\mathcal{C}}$;*

(b) *$\mathrm{Hom}^{\mathcal{C}}(P, -) : \mathbf{M}^{\mathcal{C}} \to \mathbf{M}_S$ is (fully) faithful;*

(c) *$\varepsilon : \mathrm{Hom}^{\mathcal{C}}(P, -) \otimes_S P \to I_{\mathbf{M}^{\mathcal{C}}}$ is an isomorphism;*

(d) *${}_SP$ is flat and $\varepsilon_V : \mathrm{Hom}^{\mathcal{C}}(P, V) \otimes_S P \to V$ is an isomorphism for all injective comodules $V \in \mathbf{M}^{\mathcal{C}}$.*

The following notion generalises generators (see [**WiGa3**, 4.1]).

4.3. **Galois comodules.** Let $\mathcal{C}$ be an A-coring, $P \in \mathbf{M}^{\mathcal{C}}$ and $S = \mathrm{End}^{\mathcal{C}}(P)$. P is said to be a *Galois comodule* if $\varepsilon_V : \mathrm{Hom}^{\mathcal{C}}(P, V) \otimes_S P \to V$ is an isomorphism for all $(\mathcal{C}, A)$-injective comodules $V \in \mathbf{M}^{\mathcal{C}}$ (i.e., V is injective with respect to A-splitting $\mathcal{C}$-monomorphisms).

In case P_A is finitely generated and projective, P being a Galois comodule can be characterised by the single isomorphism ([**WiGa3**, 5.3])

$$\varepsilon_{\mathcal{C}} : \mathrm{Hom}^{\mathcal{C}}(P, \mathcal{C}) \otimes_S P \to \mathcal{C}.$$

This implies that any coring $\mathcal{C}$ which is finitely generated and projective as a right A-module is a right Galois comodule.

Clearly, if the functor $\mathrm{Hom}^{\mathcal{C}}(P, -)$ is fully faithful, then P is a Galois comodule. On the other hand, if P is a Galois comodule and ${}_SP$ is flat, then $\mathrm{Hom}^{\mathcal{C}}(P, -)$ is fully faithful and ${}_A\mathcal{C}$ is flat, i.e., P is a generator in $\mathbf{M}^{\mathcal{C}}$ (see [**WiGa3**, 4.8]).

For a sketch of the way from Galois field extensions to Galois comodules we refer to [**WiGa1**].

4.4. A **as $\mathcal{C}$-comodule.** Given an A-coring $\mathcal{C}$ one may ask when A itself is a $\mathcal{C}$-comodule. This is the case when $\mathcal{C}$ has a *grouplike* element, that is, some $g \in \mathcal{C}$ with $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1_A$. Then A is a right (and left) $\mathcal{C}$-comodule by the coactions

$$\varrho^A : A \to \mathcal{C},\ a \mapsto ga, \quad ({}^A\varrho : A \to \mathcal{C},\ a \mapsto ag,)$$

and we denote A with this comodule structure by A_g (e.g. [**WiGa2**], [**BrWi**, Section 28]). For $M \in \mathbf{M}^{\mathcal{C}}$, the *$g$-coinvariants* of M are defined as

$$M_g^{co\mathcal{C}} = \{m \in M \mid \varrho^M(m) = m \otimes g\}.$$

(1) $\psi_M : \mathrm{Hom}^{\mathcal{C}}(A_g, M) \to M_g^{co\mathcal{C}},\ f \mapsto f(1_A)$, is an isomorphism.

(2) $\mathrm{End}^{\mathcal{C}}(A_g) \simeq A_g^{co\mathcal{C}} = \{a \in A_g \,|\, ga = ag\}$ (= centraliser of g in A).

(3) For any $X \in \mathbf{M}_A$, $(X \otimes_A \mathcal{C})^{co\mathcal{C}} \simeq \mathrm{Hom}^{\mathcal{C}}(A_g, X \otimes_A \mathcal{C}) \simeq X$, and

$$\mathcal{C}^{co\mathcal{C}} \simeq \mathrm{Hom}^{\mathcal{C}}(A_g, \mathcal{C}) \simeq \mathrm{Hom}_A(A_g, A) \simeq A,$$

which is a left A- and right $\mathrm{End}^{\mathcal{C}}(A_g)$-morphism.

Now one may ask when A_g is a Galois comodule. This leads to

4.5. **Galois corings.** *For an A-coring $\mathcal{C}$ with a grouplike element g and $B = A_g^{co\mathcal{C}}$, the following are equivalent:*

(a) $\varepsilon_{\mathcal{C}} : \mathrm{Hom}^{\mathcal{C}}(A_g, \mathcal{C}) \otimes_B A \to \mathcal{C},\ f \otimes a \mapsto f(a)$, *is an isomorphism;*

(b) *the (A, A)-bimodule map $\chi : A \otimes_B A \to \mathcal{C},\ 1_A \otimes 1_A \mapsto g$, is a (coring) isomorphism;*

(c) $\varepsilon_N : \mathrm{Hom}^{\mathcal{C}}(A_g, N) \otimes_B A \to N$ *is an isomorphism, for every $(\mathcal{C}, A)$-injective $N \in \mathbf{M}^{\mathcal{C}}$.*

$(\mathcal{C}, g)$ is called a *Galois coring* if it satisfies the above conditions.

As seen before Galois comodules are close to generators. This is recalled in the

4.6. **Galois Coring Structure Theorem.** (e.g., [**BrWi**, 28.19]). *Let $\mathcal{C}$ be an A-coring with grouplike element g and $B = A_g^{co\mathcal{C}}$.*

(1) *The following are equivalent:*

  (a) *$(\mathcal{C}, g)$ is a Galois coring and ${}_BA$ is flat;*

  (b) *${}_A\mathcal{C}$ is flat and A_g is a generator in $\mathbf{M}^{\mathcal{C}}$.*

(2) *The following are equivalent:*

  (a) *$(\mathcal{C}, g)$ is a Galois coring and ${}_BA$ is faithfully flat;*

  (b) *${}_A\mathcal{C}$ is flat and A_g is a projective generator in $\mathbf{M}^{\mathcal{C}}$;*

  (c) *${}_A\mathcal{C}$ is flat and* $\mathrm{Hom}^{\mathcal{C}}(A_g, -) : \mathbf{M}^{\mathcal{C}} \to \mathbf{M}_B$ *is an equivalence.*

4.7. **Ring extensions.** To any ring extension $B \to A$, an A-coring can be associated by putting $\mathcal{C} = A \otimes_B A$ with grouplike element $g = 1_A \otimes 1_A$ in $\mathcal{C}$, called the *Sweedler A-coring.*

The category of right comodules over this coring is isomorphic to the category of *descent data* for the given ring extension (e.g. [**BrWi**, 25.4]). In this context the results on Galois corings can be interpreted in the classical *descent theory.*

Recall that if $\mathcal{C}$ satisfies the α-condition, then $\mathrm{End}^{\mathcal{C}}(\mathcal{C}) = {}_{*\mathcal{C}}\mathrm{End}(\mathcal{C}) \simeq \mathcal{C}^*$ and we get (e.g. [**BrWi**, 19.21]):

4.8. **$\mathcal{C}$ as generator in $\mathbf{M}^{\mathcal{C}}$.** *Let $\mathcal{C}$ be an A-coring with ${}_A\mathcal{C}$ locally projective. Then $\mathbf{M}^{\mathcal{C}} = \sigma[{}_{*\mathcal{C}}C]$ and the following are equivalent:*

(a) *$\mathcal{C}$ is a generator in $\mathbf{M}^{\mathcal{C}}$;*

(b) *${}_{*\mathcal{C}}\mathrm{Hom}(\mathcal{C}, -) : \mathbf{M}^{\mathcal{C}} \to \mathbf{M}_{\mathcal{C}^*}$ is (fully) faithful;*

(c) *$\mathcal{C}$ is flat as a $\mathcal{C}^*$-module and $\varepsilon_N : \mathcal{C} \otimes_{\mathcal{C}^*} \mathrm{Hom}^{\mathcal{C}}(\mathcal{C}, N) \to N$ is an isomorphism for $N \in \mathbf{M}^{\mathcal{C}}$.*

5. Bimodules and Hopf algebras

In this section R denotes again a commutative associative ring with unit.

5.1. **Bialgebras and bimodules.** An R-module B is called a *bialgebra* if it has an algebra structure (B, μ, η) and a coalgebra structure (B, Δ, ε),

$$\mu : B \otimes_R B \to B,\ \eta : R \to B, \quad \Delta : B \to B \otimes_R B,\ \varepsilon : B \to R,$$

such that Δ and ε are algebra morphisms, or, equivalently, μ and η are coalgebra morphisms where $B \otimes_R B$ is endowed with the canonical algebra and coalgebra structure (induced by the twist map $B \otimes_R B \to B \otimes_R B$).

A *(mixed) B-bimodule* is an R-module M with structural maps

$$\rho_M : M \otimes_R B \to M, \quad \rho^M : M \to M \otimes_R B.$$

satisfying the compatibility condition

$$\rho^M(mb) = \rho^M(m) \cdot \Delta(b), \quad \text{for } b \in B,\ m \in M.$$

Category of B-bimodules. The category of right B-bimodules is denoted by $\mathbf{M}_B^B$. Its objects are B-bimodules and the morphisms are R-linear maps which are both B-module and B-comodule morphisms.

$\mathbf{M}_B^B$ is an additive category with coproducts and cokernels. Similar to the category of comodules over a coalgebra (and for the same reasons), $\mathbf{M}_B^B$ has coproducts and cokernels but not necessarily kernels. It is abelian (and in fact Grothendieck) provided B is flat as an R-module.

Naturally, B is a B-bimodule and $\mathrm{End}_B^B(B) = R$. However, B need not be a subgenerator in $\mathbf{M}_B^B$ while $B \otimes_R B$ always is.

$B \otimes_R B$ can be considered as a B-coring and the category $\mathbf{M}_B^B$ is equivalent to the category $\mathbf{M}^{B \otimes_R B}$ of right $B \otimes_R B$-comodules. In particular, B is in $\mathbf{M}^{B \otimes_R B}$ and $1_B \otimes 1_B$ is a group like element (see 4.5).

B induces the (free) functor

$$\phi_B^B : \mathbf{M}_R \to \mathbf{M}_B^B, \quad M \mapsto (M \otimes_R B, I_M \otimes \mu, I_M \otimes \Delta),$$

which is full and faithful by the natural isomorphisms for $X, Y \in \mathbf{M}_R$,

$$\mathrm{Hom}_B^B(B \otimes_R X, B \otimes_R Y) \simeq \mathrm{Hom}_R^B(X, B \otimes_R Y) \simeq \mathrm{Hom}_R(X, Y).$$

We have the adjoint pair of functors

$$\phi_B^B : \mathbf{M}_R \to \mathbf{M}_B^B, \quad \mathrm{Hom}_B^B(B, -) : \mathbf{M}_B^B \to \mathbf{M}_R,$$

and since ϕ_B^B is full and faithful we know (from 1.1) that $\eta : I \to \mathrm{Hom}_B^B(B, B \otimes_R -)$ is an isomorphism.

Handling bialgebras there is a notion which does not show up in the settings considered so far.

5.2. **Antipodes.** Given a bialgebra B, besides composition, $\mathrm{End}_R(B)$ has another associative product, the *convolution product*, given by

$$f * g(b) = (f \otimes g)\Delta(b) \text{ for } f, g \in \mathrm{End}_R(B) \text{ and } b \in B.$$

An element $S \in \mathrm{End}_R(B)$ is called an *antipode* of B if it is inverse to I_B with respect to the convolution product $*$. So, by definition,

$$\mu \circ (S \otimes I_B) \circ \Delta = \mu \circ (I_B \otimes S) \circ \Delta = \eta \circ \varepsilon \,.$$

Notice that as an endomorphism of B, S need neither be injective nor surjective.

In the next theorem no assumptions are made on the R-module structure of B (see [**BrWi**, 15.5 and 36.16]).

5.3. **Hopf algebras - 1.** *For an R-bialgebra B the following are equivalent:*

(a) $\mathrm{Hom}^B_B(B, -) : \mathbf{M}^B_B \to \mathbf{M}_R$ *is full and faithful;*

(b) $\varepsilon_N : \mathrm{Hom}^B_B(B, N) \otimes_R B \to N$ *is an isomorphism for any* $N \in \mathbf{M}^B_B$*;*

(c) $\mathrm{Hom}^B_B(B, -) : \mathbf{M}^B_B \to \mathbf{M}_R$ *is an equivalence (with inverse* $- \otimes_R B$*);*

(d) $B \otimes_R B$ *is a Galois* B*-coring;*

(e) $\gamma_B : B \otimes_R B \to B \otimes_R B,\ a \otimes b \mapsto (a \otimes 1_B)\Delta(b),$ *is an isomorphism;*

(f) B *has an antipode.*

Bialgebras B with these properties are called *Hopf algebras*.

The equivalence of (c) and (f) is known as the *Fundamental Theorem* of Hopf algebras. Notice that here $\mathrm{Hom}^B_B(B, -)$ is fully faithful but nevertheless B need not be flat over its endomorphism ring R. This comes from the fact that monomorphisms in $\mathbf{M}^B_B$ need not be injective maps.

If B is assumed to be flat as an R-module, B is a Hopf algebra if and only if B is a generator in the (Grothendieck) category $\mathbf{M}^B_B$. Assuming B to be a locally projective R-module we can get more characterisations for such bialgebras. For this we consider the

5.4. **Smash product.** If B_R is locally projective, then the comodules in $\mathbf{M}^B$ can be identified with the B^*-modules subgenerated by the comodule B.

Thus, if B is a bialgebra, the objects in $\mathbf{M}^B_B$ can be considered as B-modules with a B^*-module structure satisfying some compatibility conditions. Thus the bimodules allow the structure of left modules over the *smash product* $B\#B^*$, which is defined as the R-module $B \otimes_R B^*$ with multiplication

$$(a \otimes f)(b \otimes g) := ((\Delta b)(a \otimes f))\,(1_B \otimes g).$$

Similar to the case of comodules we may identify

$$\mathbf{M}^B_B = \sigma_{B\#B^*}\,[B \otimes_R B] \subset {}_{B\#B^*}\mathbf{M},$$

where $\mathbf{M}^B_B = {}_{B\#B^*}\mathbf{M}$ if and only if B_R is finitely generated (and projective).

In the situation described above we can refer to 2.6 and 2.8 to describe Hopf algebras (see also [**BrWi**, 15.5]). The results are similar to the characterisations of Azumaya rings (see 3.6).

5.5. **Hopf algebras - 2.** *For an R-bialgebra B with B_R is locally projective, the following are equivalent:*

(a) *B is a generator in $\mathbf{M}_B^B$;*

(b) *$\mathrm{Hom}_B^B(B,-) : \mathbf{M}_B^B \to \mathbf{M}_R$ is (full and) faithful;*

(c) *B is a subgenerator in $\mathbf{M}_B^B$ and $\phi : B\#B^* \to \mathrm{End}_R(B)$ is dense;*

(d) *$\varepsilon_V : \mathrm{Hom}_B^B(B,V) \otimes_R B \to V$ is an isomorphism for injectives $V \in \mathbf{M}_B^B$.*

Special cases of the preceding results can be found in [**WiMod**] where further applications of these techniques to module algebras and group actions on algebras are outlined.

In [**WiAlg**] and [**MeWi**] the theory of bialgebras and Hopf algebras is generalised to endofunctors of arbitrary categories.

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