A categorical view on rings and modules

Robert Wisbauer

University of Düsseldorf, Germany

Balikesir, August 12-15, 2013

Overview

- Historic notes

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- Historic notes
- Categories

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- Frobenius algebras and monads

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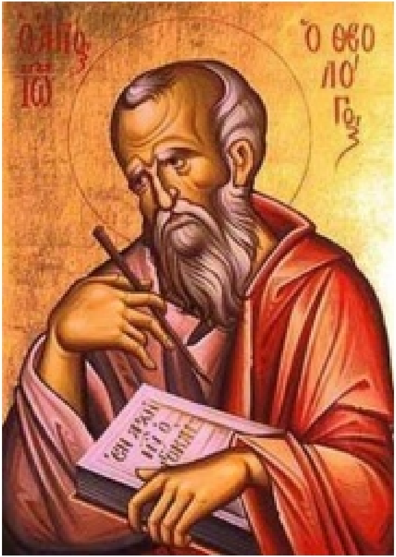
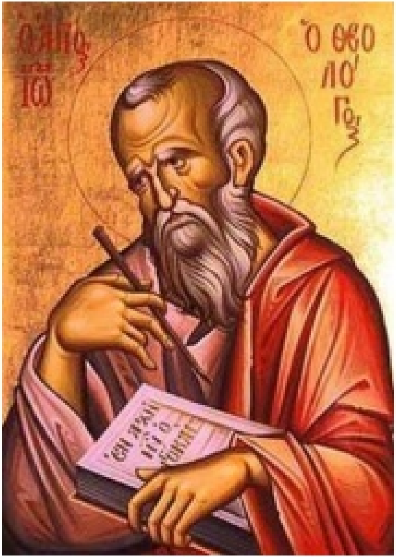
- Historic notes
- Categories
- Frobenius algebras and monads
- Categorical aspects of rings and modules

St. John 20 - 101 St. Patrick 387 - 460(?)



St. John 20 - 101



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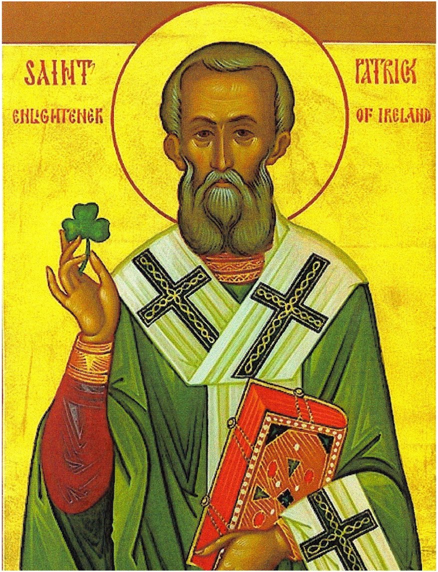

John and Patrick *1943

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"All the News That's Fit to Print"

The New York Times.

LATE CITY EDITION

NEW YORK, TUESDAY, MAY 8, 1945.

THREE CENTS

THE WAR IN EUROPE IS ENDED! SURRENDER IS UNCONDITIONAL; V-E WILL BE PROCLAIMED TODAY; OUR TROOPS ON OKINAWA GAIN

ISLAND-WIDE DRIVE

Marines Reach Village a Mile From Naha and Army Lines Advance

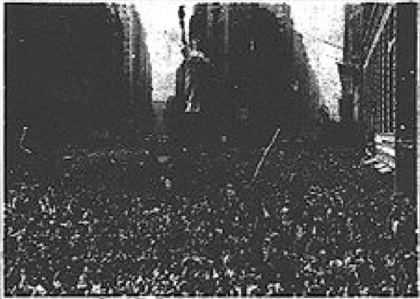
7 MORE SHIPS SUNK

Search Planes Again Hit Japan's Life Line—Kyushu Bombed

The Pulitzer Awards For 1944 Announced

MOLOTOFF HAILS BASIC 'UNANIMITY'

GERMANY SURRENDERS: NEW YORKERS MASSED UNDER SYMBOL OF LIBERTY



GERMANS CAPITULATE ON ALL FRONTS

American, Russian and French Generals Accept Surrender in Eisenhower Headquarters, a Reims School

REICH CHIEF OF STAFF ASKS FOR MERCY

Doenitz Orders All Military Forces of Germany To Drop Arms—Troops in Norway Give Up —Churchill and Truman on Radio Today

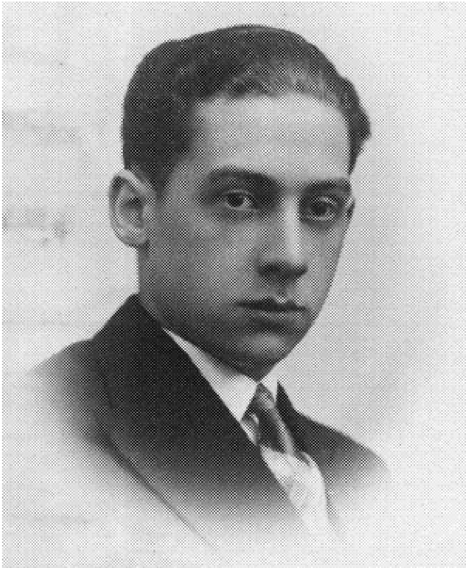
By EDWARD KENNEDY

PRAGUE SAYS FOES

Wild Crowds Greet News In City While Others Pray

SHAEF BAN ON AP

Samuel Eilenberg 1913-98 Saunders MacLane 1909-2005



Alfred Goldie 1920-2005 Christian U. Jensen *1936

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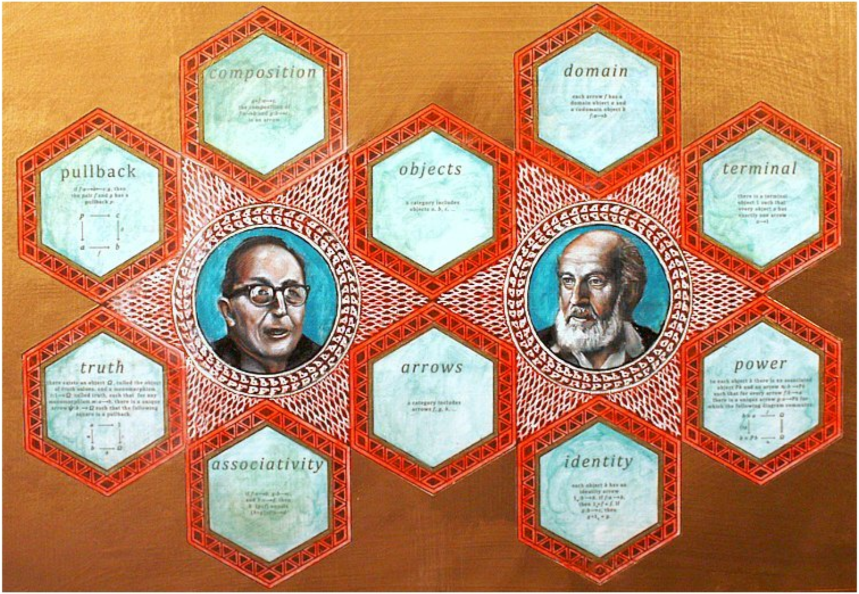
Robert 1945

little John 1945

Eilenberg - MacLane



Transactions Amer. Math. Soc., May 1945

GENERAL THEORY OF NATURAL EQUIVALENCES

BY

SAMUEL EILENBERG AND SAUNDERS MacLANE

CONTENTS

GENERAL THEORY OF NATURAL EQUIVALENCES

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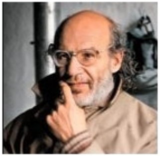
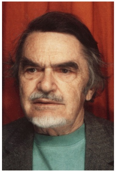
SAMUEL EILENBERG AND SAUNDERS MacLANE

CONTENTS

In a metamathematical sense our theory provides general concepts applicable to all branches of abstract mathematics, and so contributes to the current trend towards uniform treatment of different mathematical disciplines. In particular, it provides opportunities for the comparison of constructions and of the isomorphisms occurring in different branches of mathematics; in this way it may occasionally suggest new results by analogy.

Alexander Grothendieck *1928, Berlin
Bill Lawvere *1937, Indiana
Joachim Lambek *1922, Leipzig
Eugenio Moggi *1960 (?), Italy



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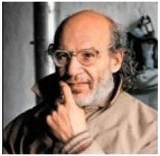  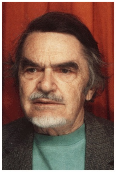 

1957 major results in algebraic geometry

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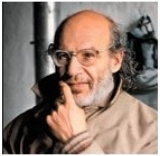

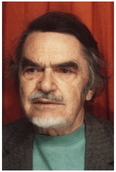


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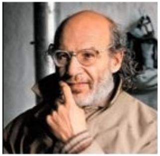  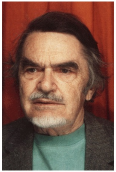 

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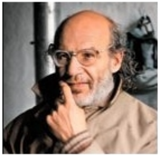

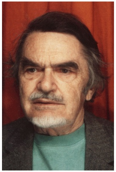


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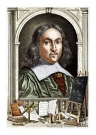
1966	logic beautifully captured in category theory
1980	types / programs used in computer
1989	use of monads to structure programs

Pierre de Fermat $\sim$1640



Satz von Fermat $a^n + b^n \neq c^n, \quad a, b, c \in \mathbb{N}, 3 \leq n$

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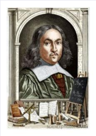

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Georg Cantor, Crelle Journal 1874

Über eine Eigenschaft des Inbegriffs aller reellen Zahlen

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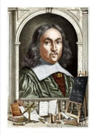

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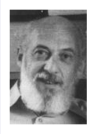
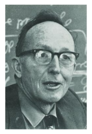
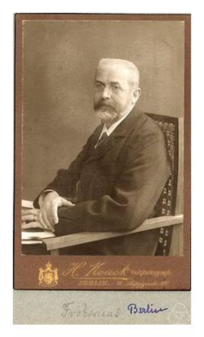
Eilenberg - Mac Lane, Trans. AMS 1945

General Theory of natural equivalences

Ferdinand Frobenius, *Theorie der hyperkomplexen Größen, 1903*



Frobenius algebras

A finite dimensional K-algebra
$A^* = \mathrm{Hom}_K(A, K)$ left A-module
$A \simeq A^*$ as left A-modules

Categorical aspects of rings and modules

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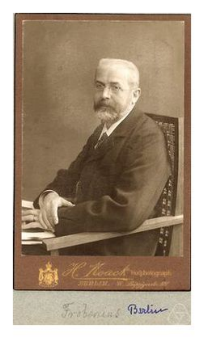

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$\sigma : A \times A \to K$, nondegenerate, associative $\sigma(ab, c) = \sigma(a, bc)$

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named and studied by Brauer and Nesbitt (1937)

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topological quantum field theory (Abrams 1996)
Frobenius monads in categories (Street 2004)

Frobenius algebras

Coalgebra structure of A^*, A_K finite dimensional

Multiplication and unit on A:

$$m : A \otimes_K A \to A, \quad \eta : K \to A,\ k \mapsto k1_A.$$

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Coalgebra structure of A^*, A_K finite dimensional

Multiplication and unit on A:

$$m : A \otimes_K A \to A, \quad \eta : K \to A,\ k \mapsto k1_A.$$

Apply $(\ )^* = \mathrm{Hom}_K(-, K)$, comultiplication and counit on A^*:

$$A^* \xrightarrow{m^*} (A \otimes_K A)^* \simeq A^* \otimes_K A^*, \quad A^* \xrightarrow{\eta^*} K.$$

Frobenius algebras

Coalgebra structure (L. Abrams, 1999), $\lambda : A \to A^*$, $\varepsilon := \lambda(1_A) : A \to K$

$$
\begin{array}{ccc}
A & \xrightarrow{\delta} & A \otimes_K A \\
{\scriptstyle \lambda}\downarrow & & \uparrow{\scriptstyle \lambda^{-1}\otimes\lambda^{-1}} \\
A^* & \xrightarrow{m^*} & A^* \otimes_K A^*
\end{array},
\qquad
\begin{array}{ccccc}
A & \xleftarrow{A\otimes\varepsilon} & A \otimes_R A & \xrightarrow{\varepsilon\otimes A} & A \\
 & {\scriptstyle =}\nwarrow & \uparrow{\scriptstyle \delta} & \nearrow{\scriptstyle =} & \\
 & & A & &
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satisfies Frobenius conditions

$$
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A \otimes A & \xrightarrow{m} & A \\
{\scriptstyle I\otimes\delta}\downarrow & & \downarrow{\scriptstyle \delta} \\
A \otimes A \otimes A & \xrightarrow{m\otimes I} & A \otimes A
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\end{array}
$$

Theorem (Abrams): A-modules $\simeq$ A-comodules: $\mathbb{M}_A \simeq \mathbb{M}^A$

Categories

Category $\mathbb{A}$

class of **objects**, **morphism sets** $\mathrm{Mor}_{\mathbb{A}}(A, B)$,

composition $\quad \mathrm{Mor}_{\mathbb{A}}(A, B) \times \mathrm{Mor}_{\mathbb{A}}(B, C) \to \mathrm{Mor}_{\mathbb{A}}(A, C)$,

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composition $\quad \mathrm{Mor}_{\mathbb{A}}(A, B) \times \mathrm{Mor}_{\mathbb{A}}(B, C) \to \mathrm{Mor}_{\mathbb{A}}(A, C)$,

Functors $F : \mathbb{A} \to \mathbb{B}$

morphism $f : A \to A'$	sent to $F(f) : F(A) \to F(A')$ of $\mathbb{B}$,
composition $f \circ g$ in $\mathbb{A}$	sent to $F(f) \circ F(g)$ in $\mathbb{B}$,
identity $A \to A$	sent to identity $F(A) \to F(A)$.

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Natural transformations $\psi : F \to G, \quad F, G : \mathbb{A} \to \mathbb{B}$

$$
\begin{array}{ccc}
A & & F(A) & \xrightarrow{\psi_A} & G(A) \\
\Big\downarrow h & & F(h)\Big\downarrow & & \Big\downarrow G(h) \\
A', & & F(A') & \xrightarrow{\psi_{A'}} & G(A')
\end{array}
$$

Categories

Adjoint functors $L : \mathbb{A} \to \mathbb{B}$ **and** $R : \mathbb{B} \to \mathbb{A}$

Natural isomorphism $\varphi : \mathrm{Mor}_{\mathbb{B}}(L(A), B) \to \mathrm{Mor}_{\mathbb{A}}(A, R(B))$

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unit and **counit** (natural transformations)

$$\eta : I \to RL, \quad \varepsilon : LR \to I,$$

with *triangular identities*

$$\begin{array}{ccc} L & \xrightarrow{L\eta} & LRL \\ & \underset{=}{\searrow} & \downarrow{\scriptstyle \varepsilon L} \\ & & L \end{array} , \qquad \begin{array}{ccc} R & \xrightarrow{\eta R} & RLR \\ & \underset{=}{\searrow} & \downarrow{\scriptstyle R\varepsilon} \\ & & R. \end{array}$$

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$$\varphi : \quad L(A) \xrightarrow{f} B \quad \longmapsto \quad A \xrightarrow{\eta_A} RL(A) \xrightarrow{R(f)} R(B)$$

$$\varphi^{-1} : \quad A \xrightarrow{h} R(B) \quad \longmapsto \quad L(A) \xrightarrow{L(h)} LR(B) \xrightarrow{\varepsilon_B} B.$$

General categories

Monads on $\mathbb{A}$

$\mathcal{F} = (F, m, \eta)$, where $F : \mathbb{A} \to \mathbb{A}$ is a functor with natural transf.

$$m : FF \to F, \quad \eta : I_{\mathbb{A}} \to F,$$

satisfying certain commutative diagrams (as for algebras).

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F-modules - $\mathbb{A}_F$

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and certain commutative diagrams (as for the usual modules).

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Free functor $\phi_F : \mathbb{A} \to \mathbb{A}_F, \quad A \mapsto (F(A), FF(A) \xrightarrow{m_A} F(A)),$

forgetful functor $U_F : \mathbb{A}_F \to \mathbb{A}$ (right adjoint).

General categories

Comonad on $\mathbb{A}$

$\mathbf{G} = (G, \delta, \varepsilon)$, where $G : \mathbb{A} \to \mathbb{A}$ is a functor with natural transf.

$$\delta : G \to GG, \quad \varepsilon : G \to I_{\mathbb{A}},$$

satisfying certain commuting diagrams (reversed to module case).

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objects $A \in \mathrm{Obj}(\mathbb{A})$ with morphisms $\psi : A \to G(A)$ in $\mathbb{A}$
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and certain commutative diagrams.

Cofree functor	$\phi^G : \mathbb{A} \to \mathbb{A}^G,\ A \mapsto (G(A), G(A) \xrightarrow{\delta_A} GG(A)),$
forgetful functor	$U^G : \mathbb{A}^G \to \mathbb{A}$ (left adjoint).

General categories

Adjoint endofunctors $F : \mathbb{A} \to \mathbb{A}, \quad G : \mathbb{A} \to \mathbb{A}$

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$$\begin{array}{ccc}
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{\scriptstyle \mathrm{Mor}(m^*, Y)} \downarrow & & \vdots \; ? \\
\mathrm{Mor}_{\mathbb{A}}(FF(X), Y) & & \vdots \\
{\scriptstyle \varphi_{F(X),Y}} \downarrow & & \downarrow \\
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\end{array}$$

implies G is comonad

$$\delta : G \to GG, \qquad \varepsilon : G \to I_{\mathbb{A}}$$

General categories

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(F, G) adjoint with unit $u : I \to GF$ and counit $\tilde{\varepsilon} : FG \to I$. Then

every F-module $\quad F(A) \xrightarrow{\rho} A$

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Theorem (Eilenberg-Moore 1965)

The functor

$$\mathbb{A}_F \to \mathbb{A}^G, \quad F(A) \xrightarrow{\rho} A \quad \longmapsto \quad A \xrightarrow{u_A} GF(A) \xrightarrow{G\rho} G(A)$$

induces an isomorphism of categories.

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(1) $F : \mathbb{A} \to \mathbb{A}$ is a monad;

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(a) F is a Frobenius monad;

(b) $\overline{F} = (F, \delta, \varepsilon)$ comonad with isomorphism $K : \mathbb{A}_F \to \mathbb{A}^F$ and commutative diagram

$$\begin{array}{ccccc} \mathbb{A} & \xrightarrow{\phi_F} & \mathbb{A}_F & \xrightarrow{U_F} & \mathbb{A} \\ = \big\downarrow & & \big\downarrow K & & \big\downarrow = \\ \mathbb{A} & \xrightarrow[\phi^F]{} & \mathbb{A}^F & \xrightarrow[U^F]{} & \mathbb{A}\,. \end{array}$$

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Monad (F, m, η) on category $\mathbb{A}$. Equivalent:

(a) F is a separable monad;

(b) $m : FF \to F$ has a natural section $\delta : F \to FF$ with commutative diagrams

$$
\begin{array}{ccc}
FF & \xrightarrow{\delta F} & FFF \\
{\scriptstyle m}\downarrow & & \downarrow{\scriptstyle Fm} \\
F & \xrightarrow{\delta} & FF
\end{array}
\qquad
\begin{array}{ccc}
FF & \xrightarrow{F\delta} & FFF \\
{\scriptstyle m}\downarrow & & \downarrow{\scriptstyle mF} \\
F & \xrightarrow{\delta} & FF
\end{array}
\qquad
\begin{array}{ccc}
F & \xrightarrow{\delta} & FF \\
& {\scriptstyle =}\searrow & \downarrow{\scriptstyle m} \\
& & F.
\end{array}
$$

Azumaya monads

Azumaya monad $(F, m, e; \lambda)$ on category $\mathbb{A}$

distributive law $\lambda : FF \to FF$ satisfying *Yang-Baxter equation*

$$\begin{array}{ccccc} FFF & \xrightarrow{F\lambda} & FFF & \xrightarrow{\lambda F} & FFF \\ {\scriptstyle \lambda F}\downarrow & & & & \downarrow{\scriptstyle F\lambda} \\ FFF & \xrightarrow{F\lambda} & FFF & \xrightarrow{\lambda F} & FFF. \end{array}$$

(1) $\mathcal{F}^\lambda = (F, m \cdot \lambda, e)$ is a monad on $\mathbb{A}$;

(2) $\mathcal{F}\mathcal{F}^\lambda = (FF, mm \cdot FF\lambda \cdot F\lambda F, ee)$ is a monad:

(3) comparison functor $K : \mathbb{A} \to \mathbb{A}_{\mathcal{F}\mathcal{F}^\lambda}$ sending $A \in \mathbb{A}$ to

$$(F(A),\ FFF(A) \xrightarrow{F(\lambda_A)} FFF(A) \xrightarrow{F(m_A)} FF(A) \xrightarrow{m_A} F(A)).$$

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Azumaya monad – if K is an equivalence of categories.

Bimonads

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$F : \mathbb{A} \to \mathbb{A}$ a monad (F, m, e) and a comonad (F, δ, ε), double entwining $\tau : FF \to FF$, $\varepsilon \cdot e = 1$, commutative diagrams

$$
\begin{array}{ccccc}
FF & \xrightarrow{m} & F & \xrightarrow{\delta} & FF \\
{\scriptstyle \delta\delta}\downarrow & & & & \uparrow{\scriptstyle mm} \\
FFFF & & \xrightarrow{F\tau F} & & FFFF,
\end{array}
\qquad
\begin{array}{ccc}
FF & \xrightarrow{F\varepsilon} & F \\
{\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\
F & \xrightarrow{\varepsilon} & 1,
\end{array}
\qquad
\begin{array}{ccc}
1 & \xrightarrow{e} & F \\
{\scriptstyle e}\downarrow & & \downarrow{\scriptstyle \delta} \\
F & \xrightarrow{eF} & FF.
\end{array}
$$

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$$\begin{array}{ccccc} FF & \xrightarrow{m} & F & \xrightarrow{\delta} & FF \\ {\scriptstyle \delta\delta}\downarrow & & & & \uparrow{\scriptstyle mm} \\ FFFF & & \xrightarrow{F\tau F} & & FFFF, \end{array} \qquad \begin{array}{ccc} FF & \xrightarrow{F\varepsilon} & F \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ F & \xrightarrow{\varepsilon} & 1, \end{array} \qquad \begin{array}{ccc} 1 & \xrightarrow{e} & F \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle \delta} \\ F & \xrightarrow{eF} & FF. \end{array}$$

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$$FF \xrightarrow{F\delta} FFF \xrightarrow{mF} FF \quad \text{is an isomorphism.}$$

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$$FF \xrightarrow{F\delta} FFF \xrightarrow{mF} FF \quad \text{is an isomorphism.}$$

$$\overline{K} : \mathbb{A} \to \mathbb{A}_F^F(\tau),\ A \mapsto (F(A), \delta_A, m_A) \qquad \text{is an equivalence.}$$

Various algebras

Algebras and coalgebras, A R-module

$$A \otimes_R A \xrightarrow{m} A,\ R \xrightarrow{\eta} A; \quad A \xrightarrow{\delta} A \otimes_R A,\ A \xrightarrow{\varepsilon} R$$

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Frobenius algebra $(A, m, \eta; \delta, \varepsilon)$

m is right A-module and right A-comodule morphism

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Separable algebra $(A, m, \eta; \delta)$

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equivalence $\mathbb{M}_R \to {}_A\mathbb{M}^A$ (separable and central)

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Azumaya algebra $(A, m, \eta; \tau)$

equivalence $\mathbb{M}_R \to {}_A\mathbb{M}^A$ (separable and central)

Hopf algebra $(A, m, \eta, \delta, \varepsilon)$

Bialgebra: m is a coalgebra morphism (δ is an algebra morphism)
Hopf algebra: $(m \otimes I) \cdot (I \otimes \delta) = I$, equivalence $\mathbb{M}_R \to \mathbb{M}_A^A$

Fiona 1989 Felicity 1990

Fiona 1989 Felicity 1990

Patrick and Felicity 1990

References

- Brauer, R. and Nesbitt, C., *On the regular representations of algebras*, Proc. Nat. Acad. Sci. USA 23(4), 236-240 (1937),
- Böhm, G., Brzeziński, T. and Wisbauer, R., *Monads and comonads on module categories*, J. Algebra 322 (2009).
- Cantor, G., *Über eine Eigenschaft des Inbegriffs aller reellen algebraischen Zahlen*, Journal Reine Angew. Mathematik 77, 258-262 (1874).
- Eilenberg, S. and MacLane, S., *General theory of natural equivalences*, Trans. Amer. Math. Soc. 58, 231-294 (1945).
- Eilenberg, S. and Moore, J.C., *Adjoint functors and triples*, Illinois J. Math. 9, 381-398 (1965).
- Eilenberg, S. and Nakayama, T., *On the dimension of modules and algebras II, Frobenius algebras and quasi-Frobenius rings*, Nagoya Math. J. 9, 1-16 (1955).

References

- Frobenius, G. *Theorie der hyperkomplexen Grössen I, II*, Berl. Ber. 1903, 504-537, 634-645 (1903).
- Greferath, M., Nechaev, A., and Wisbauer, R., *Finite quasi-Frobenius modules and linear codes*, J. Algebra Appl. 3(3), 247-272 (2004).
- Grothendieck, A., *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. (2) 9, 119-221 (1957).
- Lambek, J., *From types to sets*, Adv. in Math. 36(2), 113-164 (1980).
- Lawvere, F.W., *The category of categories as a foundation for mathematics*, Proc. Conf. Categorical Algebra (La Jolla), Springer, New York, 1-20 (1966).
- Mesablishvili, B. and Wisbauer, R., *Bimonads and Hopf monads on categories*, J. K-Theory 7(2), 349-388 (2011).

References

- Mesablishvili, B. and Wisbauer, R., *Galois functors and entwining structures*, J. Algebra 324, 464-506 (2010).
- Mesablishvili, B. and Wisbauer, R., *Notes on bimonads and Hopf monads*, Theory Appl. Categ. **26**, 281-303 (2012).
- Mesablishvili, B. and Wisbauer, R., *QF functors and (co)monads*, J. Algebra 376, 101-122 (2013).
- Moggi, E., *A category-theoretic account of program modules*, Category theory and computer science (Manchester), Lecture Notes in Comput. Sci., 389, Springer, Berlin, 101-117, (1989).
- Nakayama, T., *On Frobeniusean algebras I*, Ann. of Math. (2) 40, 611-633 (1939).
- Street, R., *Frobenius monads and pseudomonoids*, J. Math. Phys. 45(10), 3930-3948 (2004).