

A categorical view on rings and modules

Robert Wisbauer

University of Düsseldorf, Germany

Balikesir, August 12-15, 2013

Overview

- Historic notes

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- Categories

Overview

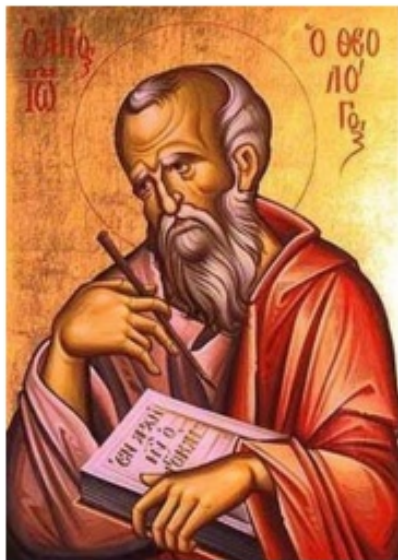
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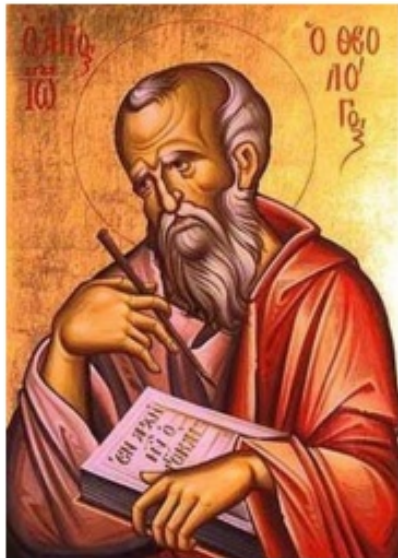
- Historic notes
- Categories
- Frobenius algebras and monads
- Categorical aspects of rings and modules

St. John 20 - 101

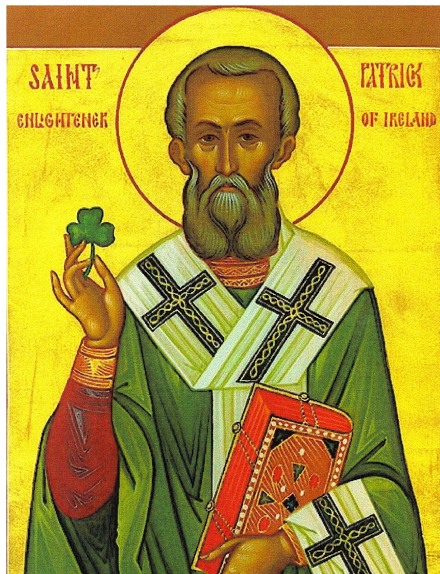
St. Patrick 387 - 460(?)



St. John 20 - 101



St. Patrick 387 - 460(?)



John and Patrick *1943



John and Patrick *1943



"All the News
That's Fit to Print"

The New York Times

LATE CITY EDITION

Ready with advance today. Printed
continuously and without interruption.
Published by The New York Times Co., Inc.
500 N. 4th St., New York, N. Y.

VOL. XXIV No. 12,812

NEW YORK, TUESDAY, MAY 8, 1945

THREE CENTS

THE WAR IN EUROPE IS ENDED! SURRENDER IS UNCONDITIONAL; V-E WILL BE PROCLAIMED TODAY; OUR TROOPS ON OKINAWA GAIN

ISLAND-WIDE DRIVE

Marines Reach Village a
Mile From Naha and
Army Lines Advance

7 MORE SHIPS SUNK

Search Planes Again Hit
Japan's Life Line—
Kyushu Bombed

By WARREN BROWN
The Marine Corps today announced that it had captured the island of Naha, Okinawa, and that it had advanced its lines to a village a mile from Naha, the main Japanese base on the island. The capture of Naha, which is a strategic point, is a major step in the drive to take the island of Okinawa. The Marines also reported that they had sunk seven Japanese ships in the waters around the island. The Japanese navy has been heavily defeated in the Pacific, and the capture of Naha is a major blow to its morale. The Marines are now preparing to take the island of Okinawa, which is a strategic point in the Pacific. The Japanese navy has been heavily defeated in the Pacific, and the capture of Naha is a major blow to its morale. The Marines are now preparing to take the island of Okinawa, which is a strategic point in the Pacific.

The Pulitzer Awards For 1944 Announced

The Pulitzer Prize awards announced yesterday by the board of trustees, University of Columbia, included the following: Best newspaper article to "A Bill for Adam" by John Hersey, for an original article that told of the nuclear attack on "Hiroshima" by most of them. Among the newspaper articles was that in the Boston Globe, which was written by the late John H. Garvin, for the reporting of the "Hiroshima" article. The award for best newspaper article for the coverage of the war was given to the "Hiroshima" article by the Pulitzer Prize Board. The award for best newspaper article for the coverage of the war was given to the "Hiroshima" article by the Pulitzer Prize Board. The award for best newspaper article for the coverage of the war was given to the "Hiroshima" article by the Pulitzer Prize Board.

MOLOTOFF HAILS BASIC 'UNANIMITY'

He Stresses The Points in
World Charter, but His View
on One Is Questioned

By EDGAR S. SNODGRASS
MOSCOW, May 7.—The Soviet premier, Joseph Stalin, today hailed the basic unanimity of the world charter for peace, but his view on one point was questioned.

GERMANY SURRENDERS: NEW YORKERS MASSES UNDER SYMBOL OF LIBERTY



Thousands of New Yorkers gathered in Times Square to celebrate the German surrender.

GERMANS CAPITULATE ON ALL FRONTS

American, Russian and French Generals
Accept Surrender in Eisenhower
Headquarters, a Reims School

REICH CHIEF OF STAFF ASKS FOR MERCY

Doerflinger Orders All Military Forces of Germany
To Drop Arms—Troops in Norway Give Up
—Churchill and Truman on Radio Today

By EDWARD KENNEDY
Special Staff Correspondent

REIMS, France, May 7.—Germany surrendered unconditionally to the Western Allies and the Soviet Union at 8:41 A. M. French time today. [This was at 6:41 P. M. Eastern War-time Sunday.] The surrender took place at a 500-bed schoolhouse that is the headquarters of Gen. Dwight D. Eisenhower.

PRAGUE SAYS FIRES LACERATE TERRITORIES

Wild Crowds Greet News In City While Others Prevail

SHARF BAN ON AP LITERS IN A BARRAGE

Prague, May 7.—The city of Prague today was a scene of wild celebration as news of the German surrender spread. Crowds of people gathered in the streets, and many people were seen dancing and singing. The news was greeted with joy and relief in the city. The news was greeted with joy and relief in the city. The news was greeted with joy and relief in the city.

Samuel Eilenberg 1913-98 Saunders MacLane 1909-2005



Alfred Goldie 1920-2005

Christian U. Jensen *1936



Alfred Goldie 1920-2005



Christian U. Jensen *1936



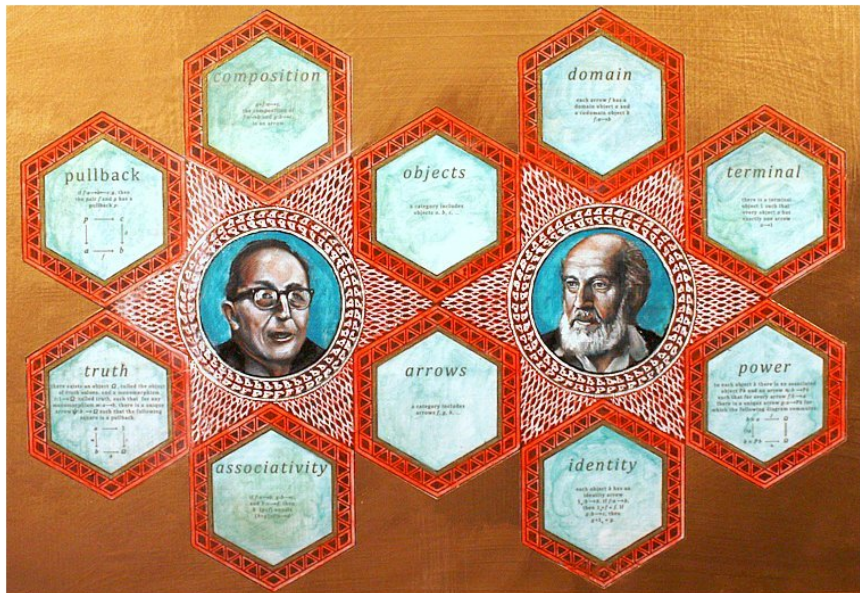
Robert 1945



little John 1945



Eilenberg - MacLane



GENERAL THEORY OF NATURAL EQUIVALENCES

BY

SAMUEL EILENBERG AND SAUNDERS MACLANE

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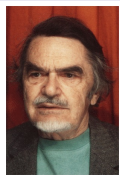
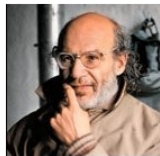
In a metamathematical sense our theory provides general concepts applicable to all branches of abstract mathematics, and so contributes to the current trend towards uniform treatment of different mathematical disciplines. In particular, it provides opportunities for the comparison of constructions and of the isomorphisms occurring in different branches of mathematics; in this way it may occasionally suggest new results by analogy.

Alexander Grothendieck *1928, Berlin

Bill Lawvere *1937, Indiana

Joachim Lambek *1922, Leipzig

Eugenio Moggi *1960 (?), Italy

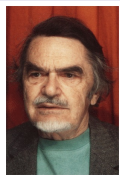
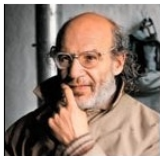


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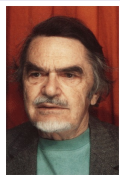
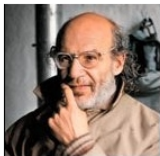
1957 major results in algebraic geometry

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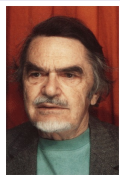
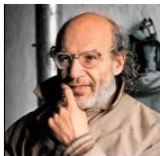
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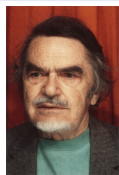
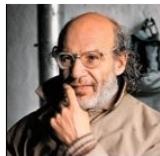
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- 1980 types / programs used in computer

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- 1957 major results in algebraic geometry
- 1966 logic beautifully captured in category theory
- 1980 types / programs used in computer
- 1989 use of monads to structure programs

Pierre de Fermat ~1640



Satz von Fermat $a^n + b^n \neq c^n, \quad a, b, c \in \mathbb{N}, 3 \leq n$

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Georg Cantor, Crelle Journal 1874



Über eine Eigenschaft des Inbegriffs aller reellen Zahlen

Pierre de Fermat ~1640



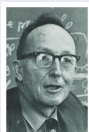
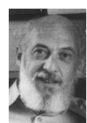
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Eilenberg - Mac Lane, Trans. AMS 1945



General Theory of natural equivalences

Categorical aspects of rings and modules

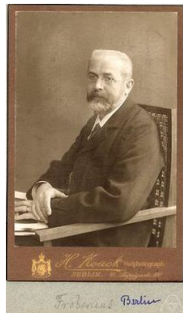
Ferdinand Frobenius, *Theorie der hyperkomplexen Größen*, 1903

Frobenius algebras

A finite dimensional K -algebra

$A^* = \text{Hom}_K(A, K)$ left A -module

$A \simeq A^*$ as left A -modules



Categorical aspects of rings and modules

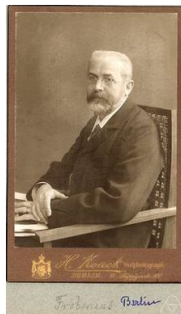
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$\sigma : A \times A \rightarrow K$, nondegenerate, associative $\sigma(ab, c) = \sigma(a, bc)$

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named and studied by Brauer and Nesbitt (1937)

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topological quantum field theory (Abrams 1996)
Frobenius monads in categories (Street 2004)

Frobenius algebras

Coalgebra structure of A^* , A_K finite dimensional

Multiplication and unit on A :

$$m : A \otimes_K A \rightarrow A, \quad \eta : K \rightarrow A, \quad k \mapsto k1_A.$$

Frobenius algebras

Coalgebra structure of A^* , A_K finite dimensional

Multiplication and unit on A :

$$m : A \otimes_K A \rightarrow A, \quad \eta : K \rightarrow A, \quad k \mapsto k1_A.$$

Apply $()^* = \text{Hom}_K(-, K)$, comultiplication and counit on A^* :

$$A^* \xrightarrow{m^*} (A \otimes_K A)^* \simeq A^* \otimes_K A^*, \quad A^* \xrightarrow{\eta^*} K.$$

Frobenius algebras

Coalgebra structure (L. Abrams, 1999), $\lambda : A \rightarrow A^*$,
 $\varepsilon := \lambda(1_A) : A \rightarrow K$

$$\begin{array}{ccc}
 A & \xrightarrow{\delta} & A \otimes_K A \\
 \lambda \downarrow & & \uparrow \lambda^{-1} \otimes \lambda^{-1} \\
 A^* & \xrightarrow{m^*} & A^* \otimes_K A^*
 \end{array}$$

$$\begin{array}{ccccc}
 & A \otimes \varepsilon & & \varepsilon \otimes A & \\
 & \longleftarrow & A \otimes_R A & \longrightarrow & A \\
 & \searrow = & \uparrow \delta & \nearrow = & \\
 & A & & &
 \end{array}$$

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 \end{array}$$

satisfies Frobenius conditions

$$\begin{array}{ccc}
 A \otimes A & \xrightarrow{m} & A \\
 I \otimes \delta \downarrow & & \downarrow \delta \\
 A \otimes A \otimes A & \xrightarrow{m \otimes I} & A \otimes A
 \end{array}$$

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 A \otimes A \otimes A & \xrightarrow{I \otimes m} & A \otimes A
 \end{array}$$

Theorem (Abrams): A -modules \simeq A -comodules: $M_A \simeq M^A$

Categories

Category \mathbb{A}

class of **objects**, **morphism sets** $\text{Mor}_{\mathbb{A}}(A, B)$,

composition $\text{Mor}_{\mathbb{A}}(A, B) \times \text{Mor}_{\mathbb{A}}(B, C) \rightarrow \text{Mor}_{\mathbb{A}}(A, C)$,

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composition $\text{Mor}_{\mathbb{A}}(A, B) \times \text{Mor}_{\mathbb{A}}(B, C) \rightarrow \text{Mor}_{\mathbb{A}}(A, C)$,

Functors $F : \mathbb{A} \rightarrow \mathbb{B}$

morphism $f : A \rightarrow A'$ sent to $F(f) : F(A) \rightarrow F(A')$ of \mathbb{B} ,

composition $f \circ g$ in \mathbb{A} sent to $F(f) \circ F(g)$ in \mathbb{B} ,

identity $A \rightarrow A$ sent to identity $F(A) \rightarrow F(A)$.

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identity $A \rightarrow A$ sent to identity $F(A) \rightarrow F(A)$.

Natural transformations $\psi : F \rightarrow G, \quad F, G : \mathbb{A} \rightarrow \mathbb{B}$

$$\begin{array}{ccccc} A & & F(A) & \xrightarrow{\psi_A} & G(A) \\ \downarrow h & & \downarrow F(h) & & \downarrow G(h) \\ A' & & F(A') & \xrightarrow{\psi_{A'}} & G(A') \end{array}$$

Categories

Adjoint functors $L : \mathbb{A} \rightarrow \mathbb{B}$ and $R : \mathbb{B} \rightarrow \mathbb{A}$

Natural isomorphism $\varphi : \text{Mor}_{\mathbb{B}}(L(A), B) \rightarrow \text{Mor}_{\mathbb{A}}(A, R(B))$

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unit and counit (natural transformations)

$$\eta : I \rightarrow RL, \quad \varepsilon : LR \rightarrow I,$$

with *triangular identities*

$$\begin{array}{ccc} L & \xrightarrow{L\eta} & LRL \\ & \searrow = & \downarrow \varepsilon L \\ & & L \end{array} \quad , \quad \begin{array}{ccc} R & \xrightarrow{\eta R} & RLR \\ & \searrow = & \downarrow R\varepsilon \\ & & R. \end{array}$$

Categories

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$$\varphi : L(A) \xrightarrow{f} B \mapsto A \xrightarrow{\eta_A} RL(A) \xrightarrow{R(f)} R(B)$$

$$\varphi^{-1} : A \xrightarrow{h} R(B) \mapsto L(A) \xrightarrow{L(h)} LR(B) \xrightarrow{\varepsilon_B} B.$$

General categories

Monads on \mathbb{A}

$\mathcal{F} = (F, m, \eta)$, where $F : \mathbb{A} \rightarrow \mathbb{A}$ is a functor with natural transf.

$$m : FF \rightarrow F, \quad \eta : I_{\mathbb{A}} \rightarrow F,$$

satisfying certain commutative diagrams (as for algebras).

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F -modules - \mathbb{A}_F

objects $A \in \text{Obj}(\mathbb{A})$ with morphisms $\varrho : F(A) \rightarrow A$
and certain commutative diagrams (as for the usual modules).

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F -modules - \mathbb{A}_F

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and certain commutative diagrams (as for the usual modules).

Free functor $\phi_F : \mathbb{A} \rightarrow \mathbb{A}_F, \quad A \mapsto (F(A), FF(A) \xrightarrow{m_A} F(A)),$
forgetful functor $U_F : \mathbb{A}_F \rightarrow \mathbb{A}$ (right adjoint).

General categories

Comonad on \mathbb{A}

$\mathbf{G} = (G, \delta, \varepsilon)$, where $G : \mathbb{A} \rightarrow \mathbb{A}$ is a functor with natural transf.

$$\delta : G \rightarrow GG, \quad \varepsilon : G \rightarrow I_{\mathbb{A}},$$

satisfying certain commuting diagrams (reversed to module case).

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Comonad on \mathbb{A}

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\mathbf{G} -comodules - \mathbb{A}^G

objects $A \in \text{Obj}(\mathbb{A})$ with morphisms $\psi : A \rightarrow G(A)$ in \mathbb{A}
and certain commutative diagrams.

General categories

Comonad on \mathbb{A}

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and certain commutative diagrams.

Cofree functor $\phi^G : \mathbb{A} \rightarrow \mathbb{A}^G, A \mapsto (G(A), G(A) \xrightarrow{\delta_A} GG(A))$,
forgetful functor $U^G : \mathbb{A}^G \rightarrow \mathbb{A}$ (left adjoint).

General categories

Adjoint endofunctors $F : \mathbb{A} \rightarrow \mathbb{A}, G : \mathbb{A} \rightarrow \mathbb{A}$

$$\text{Mor}_{\mathbb{A}}(F(X), Y) \xrightarrow{\varphi_{X,Y}} \text{Mor}_{\mathbb{A}}(X, G(Y))$$

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Adjoint endofunctors $F : \mathbb{A} \rightarrow \mathbb{A}$, $G : \mathbb{A} \rightarrow \mathbb{A}$

$$\text{Mor}_{\mathbb{A}}(F(X), Y) \xrightarrow{\varphi_{X,Y}} \text{Mor}_{\mathbb{A}}(X, G(Y))$$

F monad, $m : FF \rightarrow F$, $\eta : I_{\mathbb{A}} \rightarrow F$

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implies G is comonad

$$\delta : G \rightarrow GG, \quad \varepsilon : G \rightarrow I_{\mathbb{A}}$$

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Theorem (Eilenberg-Moore 1965)

The functor

$$\mathbb{A}_F \rightarrow \mathbb{A}^G, \quad F(A) \xrightarrow{\rho} A \quad \mapsto \quad A \xrightarrow{u_A} GF(A) \xrightarrow{G\rho} G(A)$$

induces an isomorphism of categories.

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(1) $F : \mathbb{A} \rightarrow \mathbb{A}$ is a monad;

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- (a) F is a Frobenius monad;
- (b) $\bar{F} = (F, \delta, \varepsilon)$ comonad with isomorphism $K : \mathbb{A}_F \rightarrow \mathbb{A}^F$ and commutative diagram

$$\begin{array}{ccccc} \mathbb{A} & \xrightarrow{\phi_F} & \mathbb{A}_F & \xrightarrow{U_F} & \mathbb{A} \\ \downarrow = & & \downarrow K & & \downarrow = \\ \mathbb{A} & \xrightarrow{\phi^F} & \mathbb{A}^F & \xrightarrow{U^F} & \mathbb{A} \end{array}$$

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$$\begin{array}{ccc} FF & \xrightarrow{\delta F} & FFF \\ m \downarrow & & \downarrow Fm \\ F & \xrightarrow{\delta} & FF \end{array} \quad \begin{array}{ccc} FF & \xrightarrow{F\delta} & FFF \\ m \downarrow & & \downarrow mF \\ F & \xrightarrow{\delta} & FF \end{array} \quad \begin{array}{ccc} F & \xrightarrow{\delta} & FF \\ & \searrow = & \downarrow m \\ & & F. \end{array}$$

Azumaya monads

Azumaya monad $(F, m, e; \lambda)$ on category \mathbb{A}

distributive law $\lambda : FF \rightarrow FF$ satisfying *Yang-Baxter equation*

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- (3) comparison functor $K : \mathbb{A} \rightarrow \mathbb{A}_{\mathcal{F}\mathcal{F}^\lambda}$ sending $A \in \mathbb{A}$ to

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Azumaya monad – if K is an equivalence of categories.

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$F : \mathbb{A} \rightarrow \mathbb{A}$ a monad (F, m, e) and a comonad (F, δ, ε) ,
double entwining $\tau : FF \rightarrow FF$, $\varepsilon \cdot e = 1$, commutative diagrams

$$\begin{array}{ccc} FF & \xrightarrow{m} & F & \xrightarrow{\delta} & FF \\ \delta\delta \downarrow & & & & \uparrow mm \\ FFFF & \xrightarrow{F\tau F} & FFFF & & \end{array}, \quad \begin{array}{ccc} FF & \xrightarrow{F\varepsilon} & F \\ m \downarrow & & \downarrow \varepsilon \\ F & \xrightarrow{\varepsilon} & 1, \end{array} \quad \begin{array}{ccc} 1 & \xrightarrow{e} & F \\ e \downarrow & & \downarrow \delta \\ F & \xrightarrow{eF} & FF. \end{array}$$

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 FFFF & \xrightarrow{F\tau F} & FFFF & & & & F & \xrightarrow{\varepsilon} & 1 & & F & \xrightarrow{eF} & FF
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$FF \xrightarrow{F\delta} FFF \xrightarrow{mF} FF$ is an isomorphism.

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$FF \xrightarrow{F\delta} FFF \xrightarrow{mF} FF$ is an isomorphism.

$\bar{K} : \mathbb{A} \rightarrow \mathbb{A}_F^F(\tau)$, $A \mapsto (F(A), \delta_A, m_A)$ is an equivalence.

Various algebras

Algebras and coalgebras, A R -module

$$A \otimes_R A \xrightarrow{m} A, R \xrightarrow{\eta} A; \quad A \xrightarrow{\delta} A \otimes_R A, A \xrightarrow{\varepsilon} R$$

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m is right A -module and right A -comodule morphism

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Separable algebra $(A, m, \eta; \delta)$

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equivalence $\mathbb{M}_R \rightarrow {}_A\mathbb{M}^A$ (separable and central)

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Hopf algebra $(A, m, \eta, \delta, \varepsilon)$

Bialgebra: m is a coalgebra morphism (δ is an algebra morphism)

Hopf algebra: $(m \otimes I) \cdot (I \otimes \delta) = I$, equivalence $\mathbb{M}_R \rightarrow \mathbb{M}_A^A$

Fiona 1989

Felicity 1990



Fiona 1989









Felicity 1990









Patrick and Felicity 1990









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