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On modules with the Kulikov property and pure semisimple modules and rings

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Abstract

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For an R-module M let $\sigma[M]$ denote the category of submodules of M-generated modules. M has the Kulikov property if submodules of pure projective modules in $\sigma[M]$ are pure projective. The following is proved: Assume M is a locally noetherian module with the Kulikov property and there are only finitely many simple modules in $\sigma[M]$. Then, for every $n \in \mathbb{N}$, there are only finitely many indecomposable modules of length $\leq n$ in $\sigma[M]$.

With our techniques we provide simple proofs for some results on left pure semisimple rings obtained by Prest and Zimmermann-Huisgen and Zimmermann with different methods.

For an R-module M the category of submodules of M-generated modules is denoted by $\sigma[M]$. We say that M has the *Kulikov property* if submodules of pure projective modules in $\sigma[M]$ are again pure projective. Making use of our knowledge of the functor ring we prove that for a locally noetherian module M with the Kulikov property and only finitely many simple modules in $\sigma[M]$ the following is true:

For every $n \in \mathbb{N}$ there are only finitely many indecomposable modules of length $\leq n$ in $\sigma[M]$.

Examples for this situation are the $\mathbb{Z}$-modules $\mathbb{Z}_{p^\infty}$ for prime numbers $p \in \mathbb{N}$ (Prüfer groups). The theorem also applies to pure semisimple modules M (every module in $\sigma[M]$ is pure projective) of finite length and yields a corresponding result for left pure semisimple rings proved in [14] and [9].

Finally we get new proofs for various characterizations of *right* pure semisimple rings. From these we deduce another result of [9] and [14]:

For a right pure semisimple ring R the number of finitely presented indecomposable left R-modules of length $\leq n$ is finite for every $n \in \mathbb{N}$.

We want to give an entirely module theoretic proof of our results. To assure this, results from the literature are cited not only where they occured first, but also where module theoretic proofs are given. No use is made of the 'tool kit' of [14] which includes Auslander–Bridger duality and matrix subgroups, nor is model theory referred to as in [9].

1. Preliminaries

Throughout this note R will be an associative ring with identity and R-**Mod** stands for the category of unitary left R-modules. Morphisms of left modules are written on the right-hand side. For an R-module M we denote by $\sigma[M]$ the full subcategory of R-**Mod** whose objects are submodules of M-generated modules (see [11, 13]).

Consider an R-module $V = \bigoplus_\Lambda V_\lambda$ with finitely generated modules V_λ and $N \in R$-**Mod**. We use the notation (e.g. [4, 11])

$$\hat{\mathrm{H}}\mathrm{om}(V,N) = \{f \in \mathrm{Hom}(V,N) \mid (V_\lambda)f = 0 \text{ for almost all } \lambda \in \Lambda\},$$

and $\hat{\mathrm{E}}\mathrm{nd}(V) = \hat{\mathrm{H}}\mathrm{om}(V,V)$. The ring $\hat{\mathrm{E}}\mathrm{nd}(V)$ has no identity but enough idempotents. We apply the theory of modules over such rings as presented in [6] or [13, §49].

Every direct summand W of V is of the form $W = Ve$ with an idempotent $e \in \mathrm{End}(V)$ and $\mathrm{End}(W) = e\mathrm{End}(V)e$. If $W = \bigoplus_{\Lambda_1} V_\lambda$ for a subset $\Lambda_1 \subset \Lambda$, then $\hat{\mathrm{E}}\mathrm{nd}(W) = e\hat{\mathrm{E}}\mathrm{nd}(V)e$.

Taking V as right module over $\hat{\mathrm{E}}\mathrm{nd}(V)$ and $\mathrm{End}(V)$ the following statements are easily established:

Proposition 1.1. *With the notations above we have:*

(1) *The* $\mathrm{End}(V)$*-submodules of V are exactly the* $\hat{\mathrm{E}}\mathrm{nd}(V)$*-submodules of V.*

(2) *If V is a noetherian* $\mathrm{End}(V)$*-module, then every direct summand W of V is a noetherian* $\mathrm{End}(W)$*-module.*

(3) *If* $\hat{\mathrm{E}}\mathrm{nd}(V)$ *is locally noetherian on the left (right) and $W = \bigoplus_{\Lambda_1} V_\lambda$, $\Lambda_1 \subset \Lambda$, then* $\hat{\mathrm{E}}\mathrm{nd}(W)$ *is also locally noetherian on the left (right).* □

The proof of the first part of Theorem 9 in [14] yields the next result. For the sake of completeness we repeat the short argument:

Lemma 1.2. *With the notations above assume Λ to be infinite and consider non-zero elements $v_\lambda \in V_\lambda$. Suppose that V is a noetherian* End(V)*-module. Then there exist infinitely many distinct indices $\lambda_0, \lambda_1, \ldots$ in Λ and homomorphisms $f_k : V_{\lambda_{k-1}} \to V_{\lambda_k}$ such that for all $n \in \mathbb{N}$,*

$$(v_{\lambda_0}) f_1 f_2 \cdots f_n \neq 0.$$

Proof. For $S = \mathrm{End}(V)$ consider the S-submodule of V generated by $\{v_\lambda\}_\Lambda$. Since V_S is noetherian this is a finitely generated module, i.e. for suitable indices we have $v_{\lambda_0}S + \cdots + v_{\lambda_k}S = \sum_\Lambda v_\lambda S$.

Hence, without loss of generality we may assume that $v_{\lambda_0}\mathrm{Hom}(V_{\lambda_0}, V_\lambda) \neq 0$ for all λ in an infinite subset $\Lambda_1 \subset \Lambda \setminus \{\lambda_0\}$. We choose a family $\{f_\lambda^{(1)}\}$ with $(v_{\lambda_0}) f_\lambda^{(1)} \neq 0$ for all $\lambda \in \Lambda_1$.

As observed above, $W = \bigoplus_{\Lambda_1} V_\lambda$ is again noetherian over its endomorphism ring. Therefore we may repeat the above construction with the v_λ replaced by $(v_{\lambda_0}) f_\lambda^{(1)}$ to find an index $\lambda_1 \in \Lambda_1$, an infinite subset $\Lambda_2 \subset \Lambda_1 \setminus \{\lambda_1\}$, and homomorphisms $f_\lambda^{(2)} \in \mathrm{Hom}(V_{\lambda_1}, V_\lambda)$ for which $(v_{\lambda_0}) f_{\lambda_1}^{(1)} f_\lambda^{(2)} \neq 0$ for all $\lambda \in \Lambda_2$.

Continuing this process we get the desired sequence of homomorphisms. □

2. Modules with the Kulikov property

For the R-module M let $\{U_\alpha\}_A$ be a representing set of the finitely presented modules in $\sigma[M]$ and $U = \bigoplus_A U_\alpha$. $T = \widehat{\mathrm{End}}(U)$ is called the *functor ring* of (the finitely presented modules in) $\sigma[M]$.

The study of categories with the *Kulikov property* was initiated by Brune. Let us recall the following results ([2, 2.1;7, 5.1], see also [12;13, 53.3]):

2.1. *Assume M to be a locally noetherian R-module with functor ring T. Then:*

(1) *${}_TT$ is locally noetherian if and only if submodules of pure projective modules in $\sigma[M]$ are again pure projective (Kulikov property).*

(2) *${}_TT$ is perfect if and only if M is pure semisimple.*

Of special interest for our considerations are modules M with only finitely many non-isomorphic simple modules in $\sigma[M]$, e.g. modules over semilocal rings, modules of finite length or $\mathbb{Z}_{p^\infty}$. For these we show the following:

Theorem 2.2. *Let M be a locally noetherian R-module and assume that there are only finitely many simple modules in $\sigma[M]$.*

If M has the Kulikov property, then for every $n \in \mathbb{N}$ there are only finitely many indecomposable modules of length $\leq n$ in $\sigma[M]$.

Proof. Let T be the functor ring of $\sigma[M]$. Choose $\{V_\lambda\}_\Lambda$ to be a representing set of the indecomposable modules of length $\leq n$ in $\sigma[M]$ and put $V = \bigoplus_\Lambda V_\lambda$. By 2.1,

${}_T T$ is locally noetherian. Since V is a direct summand of U, the ring $\hat{\mathrm{E}}\mathrm{nd}(V)$ is also locally left noetherian by Proposition 1.1.

For every $\lambda \in \Lambda$, there is an idempotent $e_\lambda \in \hat{\mathrm{E}}\mathrm{nd}(V)$ with $\hat{\mathrm{H}}\mathrm{om}(V, V_\lambda) = \hat{\mathrm{E}}\mathrm{nd}(V)e_\lambda$ and $e_\lambda \hat{\mathrm{E}}\mathrm{nd}(V)e_\lambda \simeq \mathrm{End}(V_\lambda)$ is a local ring. Hence the factor of $\hat{\mathrm{E}}\mathrm{nd}(V)$ by its radical is left semisimple. Moreover, by the Harada–Sai Lemma (e.g. [13, 54.1]), the radical of $\hat{\mathrm{E}}\mathrm{nd}(V)$ is nilpotent. Therefore $\hat{\mathrm{E}}\mathrm{nd}(V)$ is a (left and) right perfect ring and enjoys the descending chain condition for finitely generated left ideals ([6, Theorem 5], see also [13, 49.9]).

Combining the two properties we see that $\hat{\mathrm{E}}\mathrm{nd}(V)$ has locally finite length on the left.

For the rest of the proof we apply ideas from [1]: Passage to the projective cover provides a bijection between the (isomorphism types of) simple modules and the indecomposable projective modules in $\hat{\mathrm{E}}\mathrm{nd}(V)$-**Mod**. With the functor $\hat{\mathrm{H}}\mathrm{om}(V, -)$ we obtain a bijection between the isomorphism classes of indecomposable summands of V and the projective covers of simple modules in $\hat{\mathrm{E}}\mathrm{nd}(V)$-**Mod.**

Let $E_1, \dots, E_k$ denote the simple modules in $\sigma[M]$. Then for every indecomposable summand X of V there is an epimorphism $g : X \to E_i$ for some $i \le k$ and $0 \ne \hat{\mathrm{H}}\mathrm{om}(V, g) : \hat{\mathrm{H}}\mathrm{om}(V, X) \to \hat{\mathrm{H}}\mathrm{om}(V, E_i)$. Hence the simple factor of $\hat{\mathrm{H}}\mathrm{om}(V, X)$ is a composition factor of $\hat{\mathrm{H}}\mathrm{om}(V, E_i)$.

Since the E_i occur among the V_λ, the left $\hat{\mathrm{E}}\mathrm{nd}(V)$-modules $\hat{\mathrm{H}}\mathrm{om}(V, E_i)$ are of the form $\hat{\mathrm{E}}\mathrm{nd}(V)e_i$ for a suitable idempotent e_i. Thus the fact that $\hat{\mathrm{E}}\mathrm{nd}(V)$ has locally finite length on the left guarantees that these modules have finite length. Hence there are only finitely many simple modules in $\hat{\mathrm{E}}\mathrm{nd}(V)$-**Mod** and only finitely many indecomposable direct summands of V. □

An R-module M is called *pure semisimple* if every module in $\sigma[M]$ is pure projective in $\sigma[M]$ (e.g. [7; 13, 53.4]). If such a module has finite length, then there are only finitely many simple modules in $\sigma[M]$ (see [13, 32.4]) and the conditions for the above theorem are satisfied. In particular, any left pure semisimple ring R is left artinian (e.g. [3, Theorem 4.4]) and hence we get the first part of Corollary 10 in [14].

Corollary 2.3. *If R is a left pure semisimple ring, then for every $n \in \mathbb{N}$ there are only finitely many indecomposable left R-modules of of length $\le n$.* □

It was shown by Kulikov in [8] that subgroups of direct sums of finitely generated abelian groups are again direct sums of finitely generated groups, i.e. $\mathbb{Z}$ has the Kulikov property. However, $\mathbb{Z}$ does not satisfy the conditions of the above theorem.

It is easily seen from Kulikov's result that also the $\mathbb{Z}$-modules $\mathbb{Z}_{p^\infty}$, for prime numbers $p \in \mathbb{N}$, have the Kulikov property. Since there is only one simple module in $\sigma[\mathbb{Z}_{p^\infty}]$ (= the category of abelian p-groups, [13, 15.10]) our theorem applies. Evidently, $\mathbb{Z}_{p^\infty}$ is not a pure semisimple $\mathbb{Z}$-module. Hence by 2.1, the functor ring of $\sigma[\mathbb{Z}_{p^\infty}]$ is left noetherian but not left perfect.

3. Right pure semisimple rings

Let $\{U_\alpha\}_A$ be a representing set for all finitely presented left R-modules and $U = \bigoplus_A U_\alpha$. Then $T = \widehat{\mathrm{End}}(U)$ is called the *left functor ring* of (the finitely presented left modules over) R. It is well known that U is finitely generated and projective as T-module.

Theorem 3.1. *For the ring R with left functor ring T the following statements are equivalent:*

(a) *R is right pure semisimple;*

(b) *T is locally right noetherian;*

(c) *U_T is noetherian ($= U$ is noetherian over* $\mathrm{End}(U)$);

(d) *every pure projective left R-module P is noetherian over* $\mathrm{End}(P)$;

(e) *every direct sum V of finitely presented left R-modules is noetherian over* $\widehat{\mathrm{End}}(V)$ (*or* $\mathrm{End}(V)$);

(f) $\mathrm{Hom}_R(K, P)$ *is noetherian over* $\mathrm{End}(P)$ *for all finitely generated K and all pure projective P in R-***Mod**.

Proof. The equivalence of (a), (b) and (c) can be seen from [5, Proposition 10.7]. For a module theoretic version see [11, Satz 2.4] or [13, 53.7].

(c) $\Rightarrow$ (e) Since every direct sum of copies of U is also a direct sum of a representing set of the finitely presented left R-modules we may assume that V is a direct summand of U. Then the assertion follows from Proposition 1.1.

(e) $\Rightarrow$ (d) The pure projective modules are direct summands of direct sums of finitely presented modules. Hence the statement again is derived from Proposition 1.1.

(d) $\Leftrightarrow$ (f) is easily seen and (d) $\Rightarrow$ (c) is trivial. $\square$

Remarks. (1) The characterization (d) in the above theorem corresponds to one obtained in Theorem 6(I) in connection with Observation 8 in [14].

(2) Using [11, Satz 3.1] we obtain with the above proof that a ring R is of finite representation type if and only if all (pure projective) left (right) R-modules have finite length over their endomorphism rings (compare Théorème 10.10 in [5], Theorem 6(II) in [14] and Theorem 1.5 in [9]).

(3) It was suggested by Simson to include the characterization (f) in the theorem which extends assertions in Proposition 2.4 of [10].

We finally use our result to draw the following conclusion which is also proved in Corollary 10 of [14] and Theorem 3.6 of [9]:

Corollary 3.2. *If R is a right pure semisimple ring, then for every $n \in \mathbb{N}$ there are only finitely many indecomposable finitely presented left R-modules of length $\leq n$.*

Proof. Let $\{V_\alpha\}_A$ be a minimal representing set of the indecomposable finitely presented left R-modules of length $\leq n$. We know from Theorem 3.1 that $\bigoplus_A V_\alpha$ is noetherian over its endomorphism ring.

If A is infinite then, by Lemma 1.2, there is an infinite subset $\{\alpha_i \mid i \in \mathbb{N}\} \subset a$ and a sequence of homomorphisms $f_k : V_{\alpha_{k-1}} \to V_{\alpha_k}$ with $f_1 \cdots f_n \neq 0$ for every $n \in \mathbb{N}$. Since the f_k are non-isomorphisms this contradicts the Harada-Sai Lemma and hence A has to be finite. $\square$

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