QF FUNCTORS AND (CO)MONADS

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Abstract. One reason for the universal interest in Frobenius algebras is that their characterisation can be formulated in arbitrary categories: a functor $K : \mathbb{A} \to \mathbb{B}$ between categories is *Frobenius* if there exists a functor $G : \mathbb{B} \to \mathbb{A}$ which is at the same time a right and left adjoint of K; a monad F on $\mathbb{A}$ is a *Frobenius monad* provided the forgetful functor $\mathbb{A}_F \to \mathbb{A}$ is a Frobenius functor, where $\mathbb{A}_F$ denotes the category of F-modules. With these notions, an algebra A over a field k is a Frobenius algebra if and only if $A \otimes_k -$ is a Frobenius monad on the category of k-vector spaces.

The purpose of this paper is to find characterisations of *quasi-Frobenius algebras* by just referring to constructions available in any categories. To achieve this we define *QF functors* between two categories by requiring conditions on pairings of functors which weaken the axioms for adjoint pairs of functors. *QF monads* on a category $\mathbb{A}$ are those monads F for which the forgetful functor $U_F : \mathbb{A}_F \to \mathbb{A}$ is a QF functor. Applied to module categories (or Grothendieck categories), our notions coincide with definitions first given K. Morita (and others). Further applications show the relations of QF functors and QF monads with Frobenius (exact) categories.

Key Words: (quasi-)Frobenius functors, (quasi-)Frobenius monads, (quasi-)Frobenius algebras, Frobenius categories.

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Introduction

Investigating the Frobenius and quasi-Frobenius ring extensions studied by F. Kasch, T. Nakayama, T. Tsuzuku, B. Pareigis, B. Müller and others, K. Morita defines (in [24]) two objects X, Y of any module category to be *similar* provided there are natural numbers n, k such that X is a direct summand of (the (co)product) Y^n and Y is a direct summand of X^k. Given rings A and B, he calls two functors $S, S' : {}_A\mathbb{M} \to {}_B\mathbb{M}$ between the categories of A-modules and B-modules *similar* provided $S(M)$ and $S'(M)$ are similar for any object $M \in {}_A\mathbb{M}$. Notice that similarity defines an equivalence relation on the class of objects and the class of functors, respectively. K. Morita uses these notions to characterise quasi-Frobenius ring extensions $B \to A$ for which both ${}_BA$ and A_B have to be finitely generated and projective.

Let $F : \mathbb{A} \to \mathbb{B}$ and $G : \mathbb{B} \to \mathbb{A}$ be a pair of (covariant) functors between additive categories. In [14], G. Guo calls G *left quasi-adjoint* to F, provided there are a natural number n and natural transformations $\eta : I_{\mathbb{B}} \to (FG)^n$ and $\zeta : (GF)^n \to I_{\mathbb{A}}$ such that $\zeta_G \circ G\eta = I_G$. He defines (G, F) to be a *left quasi-Frobenius pair* in case (F, G) is an adjoint pair and (G, F) is a quasi-adjoint pair of functors. He shows that a ring extension $\iota : B \to A$ is *left quasi-Frobenius* in the sense of Müller [25, 26], provided (F_1, G_1) is a left quasi-Frobenius pair where $F_1 : {}_A\mathbb{M} \to {}_B\mathbb{M}$ is the restriction of scalars functor and $G_1 = A \otimes_B - : {}_B\mathbb{M} \to {}_A\mathbb{M}$ is the induction functor.

The notions mentioned above are formulated for Grothendieck categories by Castaño Iglesias, Năstăsescu and Vercruysse in [7]. They call a functor $F : \mathbb{A} \to \mathbb{B}$ with left and right adjoints $L, R : \mathbb{B} \to \mathbb{A}$ a *quasi-Frobenius triple* provided L and R are similar functors. This similarity enforces an a priori symmetry for the definitions.

Here we will modify the ideas sketched above to define quasi-Frobenius functors on any categories without requiring finiteness conditions. In particular, we will consider *QF monads* and show that in their module categories the relative injectives coincide with the relative projective objects. A special case of all these functors are *Frobenius functors* between any categories.

In Section 1 we collect elementary properties of pairings of functors weakening the conditions for adjoint pairs of functors. The notion of *right QF functors* handled in Section 2 generalises the notions of Frobenius functors. The latter are functors F with a right adjoint R which is also a left adjoint. Here we require F to have a right adjoint R for which a retract of some product R^{Λ} is left adjoint to F. In these investigations, adjoint triples (L, F, R) of functors (that is $L \dashv F \dashv R$) are of special interest. The main properties of QF triples of functors are listed in Proposition 2.6 and their interplay with functor categories is sketched at the end of this section. Hereby also the relation with *separable functors of the second kind* as defined by Caenepeel and Militaru in [6] is described.

Section 3 begins with recalling some categorical constructions which are of use in studying QF monads and comonads, that is, monads and comonads for which the forgetful functors from the (co)module category to the base category are QF functors. Hereby features known for QF rings and QF corings are shown in a more general context.

In Section 4 the results are considered for module and comodule categories. It turns out that the restriction of our notions coincide with the notions defined for these special cases elsewhere. Finally we outline the relevance of QF functors for *Frobenius categories*, which are defined as exact categories with enough projectives and enough injectives such that projectives and injectives coincide. Recall that rings whose module categories have these properties are precisely the (noetherian) QF rings (e.g. [32, 48.15]).

1. Preliminaries

One of our main tools will be a generalised form of adjoint pairs of functors and in this section we present the basic facts of this setting.

Throughout $\mathbb{A}$ and $\mathbb{B}$ will denote arbitrary categories. By I_A, A or just by I, we denote the identity morphism of an object $A \in \mathbb{A}$, I_F or F stand for the identity on the functor F, and $I_{\mathbb{A}}$ or I mean the identity functor on a category $\mathbb{A}$.

1.1. **Pairing of functors.** Let $L : \mathbb{A} \to \mathbb{B}$ and $R : \mathbb{B} \to \mathbb{A}$ be covariant functors. Assume there are maps, natural in $A \in \mathbb{A}$ and $B \in \mathbb{B}$,

$$\alpha_{A,B} : \mathrm{Mor}_{\mathbb{B}}(L(A), B) \to \mathrm{Mor}_{\mathbb{A}}(A, R(B)),$$
$$\beta_{A,B} : \mathrm{Mor}_{\mathbb{A}}(A, R(B)) \to \mathrm{Mor}_{\mathbb{B}}(L(A), B).$$

These maps correspond to natural transformations α and β between obvious functors $\mathbb{A}^{op} \times \mathbb{B} \to \mathsf{Set}$. The quadruple (L, R, α, β) is called a *(full) pairing of functors.*

1.2. **Quasi-unit and quasi-counit.** Given a pairing (L, R, α, β), the morphisms, for $A \in \mathbb{A}$, $B \in \mathbb{B}$,

$$\eta_A := \alpha_{A,L(A)}(I) : A \to RL(A) \quad \text{and} \quad \varepsilon_B := \beta_{R(B),B}(I) : LR(B) \to B$$

yield natural transformations

$$\eta : I_{\mathbb{A}} \to RL, \quad \varepsilon : LR \to I_{\mathbb{B}},$$

called *quasi-unit* and *quasi-counit* of (L, R, α, β), respectively. They, in turn, determine the transformations α and β by

$$\alpha_{A,B} : \quad L(A) \xrightarrow{f} B \quad \longmapsto \quad A \xrightarrow{\eta_A} RL(A) \xrightarrow{R(f)} R(B),$$

$$\beta_{A,B} : \quad A \xrightarrow{g} R(B) \quad \longmapsto \quad L(A) \xrightarrow{L(g)} LR(B) \xrightarrow{\varepsilon_B} B.$$

1.3. **Definition.** A pairing (L, R, α, β) with quasi-unit η and quasi-counit ε (see 1.2) is called

left semi-adjoint if $\beta \cdot \alpha = I$, that is, $L \xrightarrow{L\eta} LRL \xrightarrow{\varepsilon L} L = L \xrightarrow{I} L$,
right semi-adjoint if $\alpha \cdot \beta = I$, that is, $R \xrightarrow{\eta R} RLR \xrightarrow{R\varepsilon} R = R \xrightarrow{I} R$,
an adjunction if it is left and right semi-adjoint.

The following observations are essentially made in [20, Section IV.1, Exercise 4].

1.4. **Lemma.** *Let (L, R, α, β) be a pairing.*

(1) *If (L, R, α, β) is left semi-adjoint, then*
   (i) *the natural transformation $R\varepsilon \cdot \eta R : R \to R$ is an idempotent;*
   (ii) *L has a right adjoint if and only if this idempotent splits.*

(2) *If (L, R, α, β) is right semi-adjoint, then*
   (i) *the natural transformation $\varepsilon L \cdot L\eta : L \to L$ is an idempotent;*
   (ii) *R has a left adjoint if and only if this idempotent splits.*

Proof. (1) If the idempotent $R\varepsilon \cdot \eta R$ is split by $R \xrightarrow{p} R' \xrightarrow{i} R$, then R' is a right adjoint of L with unit $pL \cdot \eta$ and counit $\varepsilon \cdot Li$.

(2) is shown by a similar argument. □

Recall that a category $\mathbb{A}$ is said to be *Cauchy complete* provided all idempotent morphisms split in $\mathbb{A}$.

1.5. **Corollary.** *Let (L, R, α, β) be a pairing.*

(1) *If the category $\mathbb{A}$ is Cauchy complete, then (L, R, α, β) is left semi-adjoint if and only if the functor R has a retract R' (i.e. there are natural transformations $\tau : R' \to R$ and $\tau' : R \to R'$ with $\tau' \cdot \tau = I_R$) which is right adjoint to L.*

(2) *If the category $\mathbb{B}$ is Cauchy complete, then (L, R, α, β) is right semi-adjoint if and only if the functor L has a retract L' which is left adjoint to R.*

1.6. **Proposition.** *Let $\eta, \varepsilon : L \dashv R : \mathbb{B} \to \mathbb{A}$ be an adjunction.*

(i) *Assume there are a functor $\overline{R} : \mathbb{B} \to \mathbb{A}$ and natural transformations $\tau : R \to \overline{R}$ and $\overline{\tau} : \overline{R} \to R$ with $\overline{\tau} \cdot \tau = I_R$. Then $(L, \overline{R})$ is left semi-adjoint with*

$$\text{quasi-unit} \quad \overline{\eta} = \tau L \cdot \eta \quad \text{and} \quad \text{quasi-counit} \quad \overline{\varepsilon} = \varepsilon \cdot L\overline{\tau}.$$

(ii) *Assume there are a functor $\overline{L} : \mathbb{A} \to \mathbb{B}$ and natural transformations $\overline{\kappa} : \overline{L} \to L$ and $\kappa : L \to \overline{L}$ with $\overline{\kappa} \cdot \kappa = I_L$. Then $(\overline{L}, R)$ is right semi-adjoint with*

$$\textit{quasi-unit } \overline{\eta} = R\kappa \cdot \eta \quad \textit{and} \quad \textit{quasi-counit } \overline{\varepsilon} = \varepsilon \cdot \overline{\kappa} R.$$

Proof. All these assertions are easy to verify. □

The following result can be obtained by adapting the proof of [21, Lemma 3.13]:

1.7. **Lemma.** *Let $H, H' : \mathbb{A} \to \mathbb{B}$ be functors such that H is a retract of H'. Then any (co)limit that is preserved by the functor H' is also preserved by the functor H.*

Proof. Since H is a retract of H', there are natural transformations $\tau : H \to H'$ and $\tau' : H' \to H$ with $\tau' \cdot \tau = I_H$. Now, let $F : \mathcal{C} \to \mathbb{A}$ be an arbitrary functor with $\mathcal{C}$ a small category such that the functor H' preserves its limits. Since $\tau' \cdot \tau = I_H$, the diagram

$$\begin{tikzcd} H \arrow[r, "\tau"] & H' \arrow[r, shift left, "\tau\cdot\tau'"] \arrow[r, shift right, "I"'] & H' \end{tikzcd}$$

is a split equaliser diagram, and thus it is preserved by any functor. Then, in the commutative diagram

$$\begin{tikzcd} H(\varprojlim F) \arrow[r, "\tau_{(\varprojlim F)}"] \arrow[d, "k_1"'] & H'(\varprojlim F) \arrow[r, shift left, "(\tau\cdot\tau')_{(\varprojlim F)}"] \arrow[r, shift right, "I"'] \arrow[d, "k_2"'] & H'(\varprojlim F) \arrow[d, "k_3"] \\ \varprojlim(HF) \arrow[r, "\varprojlim(\tau F)"'] & \varprojlim(H'F) \arrow[r, shift left, "\varprojlim((\tau\cdot\tau')F)"] \arrow[r, shift right, "I"'] & \varprojlim(H'F)\,, \end{tikzcd}$$

where the vertical morphisms are the comparison ones, the rows are (split) equaliser diagrams. Since the functor H' preserves the limit of F, the morphisms k_2 and k_3 are both isomorphisms, implying that k_1 is also an isomorphism.

The dual statement can be shown in the same way by using the split coequaliser diagram

$$\begin{tikzcd} H' \arrow[r, shift left, "\tau\cdot\tau'"] \arrow[r, shift right, "I"'] & H' \arrow[r, "\tau'"] & H. \end{tikzcd}$$

□

Note that to say that (L, R, α, β) is left semi-adjoint is to say that, for any $B \in \mathbb{B}$, the functor $\mathrm{Mor}_{\mathbb{B}}(L(-), B)$ is a retract of the functor $\mathrm{Mor}_{\mathbb{A}}(-, R(B))$, natural in B. It then follows from Lemma 1.7:

1.8. **Proposition.** *Let (L, R, α, β) be a pairing.*

(i) *If (L, R, α, β) is left semi-adjoint, then L preserves any colimits existing in $\mathbb{A}$.*

(ii) *If (L, R, α, β) is right semi-adjoint, then R preserves any limits existing in $\mathbb{B}$.*

1.9. **Relative injectives and projectives.** Let $\mathbb{E}$ be a class of morphisms in a category $\mathbb{A}$. An object $A \in \mathbb{A}$ is said to be $\mathbb{E}$-*projective* if for any diagram

$$\begin{tikzcd} & A \arrow[d, "g"] \arrow[dl, dotted, "h"'] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

with $f \in \mathbb{E}$, there is a morphism $h : A \to X$ such that $fh = g$. Dually, the notion of $\mathbb{E}$-*injective* objects is defined. Note that the class of $\mathbb{E}$-projectives is closed under small coproducts, while the class of $\mathbb{E}$-injectives is closed under small products.

Given a functor $F : \mathbb{A} \to \mathbb{B}$, classes $\mathbb{E}$ of morphisms may be defined by collecting those morphisms $f : A \to A'$ in $\mathbb{A}$, for which $F(f) : F(A) \to F(A')$ is a split monomorphism or a split epimorphism in $\mathbb{B}$. This leads to the notions of *F-injective* or *F-projective* objects, respectively.

1.10**. Proposition.** *Let (L, R, α, β) be a pairing and $\mathbb{E}$ and $\mathbb{E}'$ classes of morphisms in $\mathbb{A}$ and $\mathbb{B}$, respectively.*

(i) *If (L, R, α, β) is left semi-adjoint and $R(\mathbb{E}') \subseteq \mathbb{E}$, then L takes $\mathbb{E}$-projectives into $\mathbb{E}'$-projectives.*

(ii) *If (L, R, α, β) is right semi-adjoint and $L(\mathbb{E}) \subseteq \mathbb{E}'$, then R takes $\mathbb{E}'$-injectives into $\mathbb{E}$-injectives.*

Proof. (i) Let $A \in \mathbb{A}$ be an $\mathbb{E}$-projective object and

$$\begin{tikzcd} & L(A) \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

a diagram in $\mathbb{B}$ with $f \in \mathbb{E}'$. Consider the transform

$$\begin{tikzcd} A \arrow[r, "\eta_A"] & RL(A) \arrow[d, "R(g)"] \\ R(X) \arrow[r, "R(f)"'] & R(Y) \end{tikzcd}$$

of this diagram under the adjunction. Since $R(\mathbb{E}') \subseteq \mathbb{E}$ and $f \in \mathbb{E}'$, $R(f) \in \mathbb{E}$ by assumption and $\mathbb{E}$-projectivity of A implies that there is a morphism $h : A \to R(X)$ making the diagram commute. This leads to the commutative diagram

$$\begin{tikzcd} L(A) \arrow[r, "L\eta_A"] \arrow[d, "L(h)"'] & LRL(A) \arrow[d, "LR(g)"] \\ LR(X) \arrow[r, "LR(f)"'] \arrow[d, "\varepsilon_X"'] & LR(Y) \arrow[d, "\varepsilon_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

and we get

$$f \cdot \varepsilon_X \cdot L(h) = \varepsilon_Y \cdot LR(g) \cdot L\eta_A = g \cdot \varepsilon_{L(A)} \cdot L\eta_A = g,$$

where the last equality follows from left semi-adjointness of the pairing. This shows that $L(A)$ is $\mathbb{E}'$-projective. (ii) is shown dually. □

The following setting will encounter us repeatedly in what follows.

1.11**. Definition.** A triple (L, F, R) of functors $F : \mathbb{A} \to \mathbb{B}$, $L, R : \mathbb{B} \to \mathbb{A}$ is called an *adjoint triple* provided (L, F) and (F, R) are adjoint pairs of functors.

In any category, consider the classes

$\mathbb{E}_1$ of all epimorphisms,	$\mathbb{M}_1$ of all monomorphisms,
$\mathbb{E}_2$ of all strong epimorphisms,	$\mathbb{M}_2$ of all strong monomorphisms,
$\mathbb{E}_3$ of all regular epimorphisms,	$\mathbb{M}_3$ of all regular monomorphisms.

1.12**. Proposition.** *Let (L, F, R) be an adjoint triple of functors.*

(1) *The functors F, L preserve all colimits while F, R preserve all limits existing in $\mathbb{A}$ or $\mathbb{B}$, respectively. Moreover, L preserves small objects.*

(2) *L preserves $\mathbb{E}_i$-projectives, while R preserves $\mathbb{M}_i$-injectives ($i = 1, 2, 3$).*

Proof. (1) The first properties are well-known for adjoint functors. Let B be a small object in $\mathbb{B}$. We have to show that $\mathrm{Mor}_{\mathbb{A}}(L(B), -)$ preserves coproducts. For any family $\{X_i\}_I$ of objects in $\mathbb{A}$ with coproduct $\coprod_I X_i$,

$$\begin{aligned} \mathrm{Mor}_{\mathbb{A}}(L(B), \textstyle\coprod_I X_i) &\simeq \mathrm{Mor}_{\mathbb{B}}(B, F(\textstyle\coprod_I X_i)) \\ &\simeq \mathrm{Mor}_{\mathbb{B}}(B, \textstyle\coprod_I F(X_i)) \\ &\simeq \textstyle\coprod_I \mathrm{Mor}_{\mathbb{B}}(B, F(X_i)) \simeq \textstyle\coprod_I \mathrm{Mor}_{\mathbb{A}}(L(B), X_i)). \end{aligned}$$

(2) Right adjoint functors preserve epimorphisms, strong epimorphisms and regular epimorphisms, while left adjoint functors preserve monomorphisms, strong monomorphisms and regular epimorphisms. Now apply Proposition 1.10. □

2. QF functors

The following definitions generalise the corresponding notions in [24], [14] and [7] to arbitrary categories. Again $\mathbb{A}$ and $\mathbb{B}$ denote any categories. As customary in ring theory we will write *QF* for *quasi-Frobenius*.

2.1. **Definitions.** A functor $F : \mathbb{A} \to \mathbb{B}$ is called

right QF	if it allows for a right adjoint functor $R : \mathbb{B} \to \mathbb{A}$ such that the pair (R^{Λ}, F) is right semi-adjoint for some index set Λ;
left QF	if it allows for a left adjoint functor $L : \mathbb{B} \to \mathbb{A}$ such that the pair $(F, L^{(\Lambda)})$ is left semi-adjoint for some index set Λ;
QF	if it is left and right QF;
Frobenius	if it has a right adjoint functor which is also left adjoint (see [24]).

Clearly a Frobenius functor is QF with $|\Lambda| = 1 = |\Lambda'|$. However, a QF functor with this property need not be Frobenius. The condition only means that R is a retract of L and L is a retract of R. In general this need not imply that $R \simeq L$ (but see Proposition 2.3).

2.2. **Proposition.** *Let (L, F, R) be an adjoint triple of functors (see 1.11).*

(1) *The following are equivalent:*
   (a) *F is left QF;*
   (b) *R is a retract of $L^{(\Lambda)}$ for some index set Λ.*

   If this holds, R preserves colimits, F preserves small objects in $\mathbb{B}$, and every F-injective object in $\mathbb{A}$ is F-projective.

(2) *The following are equivalent:*
   (a) *F is right QF;*
   (b) *L is a retract of R^{Λ} for some index set Λ.*

   In this case L preserves all limits which exist in $\mathbb{B}$ and every F-projective object in $\mathbb{A}$ is F-injective.

Proof. (1) By Proposition 1.12, the functor L preserves colimits and since colimits are preserved by coproducts, it follows from Lemma 1.7 that R also preserves all colimits existing in $\mathbb{B}$. Then the proof of Proposition 1.12 shows that F preserves small objects.

Since R is right adjoint to F, an object $a \in \mathbb{A}$ is F-injective if and only if a is a retract of $R(b)$, with $b \in \mathbb{B}$ (e.g., [28], [23, Proposition 1.5]). Since F is left QF, R is a retract of $L^{(\Lambda)}$, Λ some index set. Then $R(b)$ is a retract of $L^{(\Lambda)}(b) = L(b)^{(\Lambda)}$. But, by the dual of [23, Proposition 1.5], $L(b)$ is F-projective, and since small coproducts of F-projectives are F-projective, it follows that a is a retract of an F-projective object $L^{(\Lambda)}(b)$. Thus a is also F-projective.

(2) By Proposition 1.12, the functor R preserves all limits and since limits are preserved by products, it follows from Lemma 1.7 that L also preserves all limits existing in $\mathbb{B}$. Dual to (1) one proves that any F-projective object is F-injective. □

2.3. **Proposition.** *Let $(\mathbb{A}, \otimes, \mathbb{I}, [-,-])$ be a symmetric monoidal closed category with small complete and cocomplete $\mathbb{A}$. For any $\mathbb{A}$-object A, the following are equivalent:*

(a) *the endofunctor $A \otimes - : \mathbb{A} \to \mathbb{A}$ is Frobenius;*
(b) *the endofunctor $A \otimes - : \mathbb{A} \to \mathbb{A}$ is a QF functor;*
(c) *the object $A \in \mathbb{A}$ is* nuclear*, i.e., the canonical morphism $[A, \mathbb{I}] \otimes A \to [A, A]$ is an isomorphism (e.g.* [15]*).*

Proof. Clearly (a)⇒(b).

(b)⇒(c) Suppose the endofunctor $A \otimes -$ to be QF. Then its right adjoint $[A,-]$ preserves all colimits by Proposition 2.2. Applying [11, Corollary (5.8)] gives that there is a natural isomorphism $[A,-] \simeq [A,\mathbb{I}] \otimes -$, implying in particular that the canonical morphism $[A,\mathbb{I}] \otimes A \to [A,A]$ is an isomorphism. Thus, A is nuclear.

(c)⇒(a) If A is nuclear, then by [15, Theorem 2.5], there is a natural isomorphism $[A,-] \simeq [A,\mathbb{I}] \otimes -$. It then follows that the functor $[A,-]$ is left adjoint to $A \otimes -$. Indeed, to say that the functor $[A,-] \simeq [A,\mathbb{I}]\otimes-$ is right adjoint to the functor $A\otimes-$ is to say that the object $[A,\mathbb{I}]$ is right adjoint to the object A in the monoidal category $\mathbb{A}$. Hence, by symmetry of $\mathbb{A}$, there is an adjunction $[A,\mathbb{I}] \dashv A$ in $\mathbb{A}$, inducing an adjunction $[A,-] \dashv A\otimes-$ of functors. This proves that the endofunctor $A\otimes- : \mathbb{A} \to \mathbb{A}$ is Frobenius. □

2.4. **Proposition.** *(Composition of QF functors) Let (L, F, R) be as in 1.11 and let (L_1, G, R_1) be an adjoint triple with functors $G : \mathbb{B} \to \mathbb{C}$ and $L_1, R_1 : \mathbb{C} \to \mathbb{B}$.*

If F and G are left (right) QF functors, then $GF : \mathbb{A} \to \mathbb{C}$ is again a left (right) QF functor.

Proof. Assume F and G to be left QF functors. Then there are index sets Λ, Λ_1 such that R is a retract of $L^{(\Lambda)}$ and R_1 is a retract of $L_1^{(\Lambda_1)}$. That is, there are natural transformations

$$k : R \to L^{(\Lambda)}, \quad l : L^{(\Lambda)} \to R, \quad k_1 : R_1 \to L_1^{(\Lambda_1)}, \quad l_1 : L_1^{(\Lambda_1)} \to R_1,$$

such that $l \cdot k = I_R$ and $l_1 \cdot k_1 = I_{R_1}$. By Proposition 2.2(1), the functor R preserves small colimits in $\mathbb{B}$ and thus there is an isomorphism

$$\omega : RL_1^{(\Lambda_1)} \simeq (RL_1)^{(\Lambda_1)}.$$

It is now easy to see that the composites

$$RR_1 \xrightarrow{Rk_1} RL_1^{(\Lambda_1)} \xrightarrow{\omega} (RL_1)^{(\Lambda_1)} \xrightarrow{(kL_1)^{(\Lambda_1)}} (L^{(\Lambda)}L_1)^{(\Lambda_1)} = ((LL_1)^{(\Lambda)})^{(\Lambda_1)} \simeq (LL_1)^{(\Lambda\times\Lambda_1)},$$

$$RR_1 \xleftarrow{Rl_1} RL_1^{(\Lambda_1)} \xleftarrow{\omega^{-1}} (RL_1)^{(\Lambda_1)} \xleftarrow{(lL_1)^{(\Lambda_1)}} (L^{(\Lambda)}L_1)^{(\Lambda_1)} = ((LL_1)^{(\Lambda)})^{(\Lambda_1)} \simeq (LL_1)^{(\Lambda\times\Lambda_1)}$$

make RR_1 a retract of $(LL_1)^{(\Lambda\times\Lambda_1)}$. This shows that the functor GF with left and right adjoints LL_1 and RR_1 is left QF.

A similar proof shows the claim for right QF functors. □

2.5. **Definition.** An adjoint triple (L, F, R) (as in 1.11) is said to be a *(left, right) QF triple* provided F is a (left, right) QF functor as in Definition 2.1.

Summarising the above observations yields generalisations of [7, Lemma 2.4(a)]:

2.6. **Proposition.** *Let (L, F, R) be a QF triple. Then:*

(i) *The functors L, F and R preserve all limits and colimits in $\mathbb{A}$ or $\mathbb{B}$, respectively.*
(ii) *The functors L and F preserve small objects.*
(iii) *L and R preserve both $\mathbb{E}_i$-projectives and $\mathbb{M}_i$-injectives ($i = 1, 2, 3$).*
(iv) *F preserves $\mathbb{E}_1$-projectives, $\mathbb{E}_3$-projectives, $\mathbb{M}_1$-injectives and $\mathbb{M}_3$-injectives.*
(v) *Every F-injective object in $\mathbb{A}$ is F-projective and vice versa.*
(vi) *If $\mathbb{B}$ is small complete, well-powered, and with a small cogenerating set, then the functor L admits a left adjoint.*
(vii) *If $\mathbb{B}$ is small cocomplete, well-copowered, and with a small generating set, then the functor R admits a right adjoint.*

Proof. (i), (ii) and (v) follow by Proposition 2.2.

(iii) It follows from Proposition 1.12(ii) that L preserves $\mathbb{E}_i$-projectives and R preserves $\mathbb{M}_i$-injectives ($i = 1, 2, 3$). By Proposition 2.2(1), R (resp. L) is a retract of $L^{(\Lambda)}$ (resp. R^{Λ}). But since $\mathbb{E}_i$-projectives (resp. $\mathbb{M}_i$-injectives) are closed under coproducts (resp. products) and retracts, it follows that R (resp.L) also preserves $\mathbb{E}_i$-projectives (resp. $\mathbb{M}_i$-injectives) ($i = 1, 2, 3$).

(iv) Since R preserves all small colimits, it in particular preserves epimorphisms (by the dual of [3, Proposition 2.9.2]) and regular epimorphisms. It now follows from Proposition 1.10 that F preserves $\mathbb{E}_1$-projectives and $\mathbb{E}_3$-projectives.

Dually, F preserves $\mathbb{M}_1$-injectives and $\mathbb{M}_3$-injectives.

(vi) and (vii) follow from (i) and the Special Adjoint Theorem (e.g. [20]) and its dual, respectively. □

2.7. **The functors Π and Σ.** Given a category $\mathbb{A}$ with small products and coproducts and an index set Λ, we write $\Pi^{\mathbb{A}}_{\Lambda}$ (resp. $\Sigma^{\mathbb{A}}_{\Lambda}$) for the functor $\mathbb{A} \to \mathbb{A}$ that takes an object A from $\mathbb{A}$ to A^{Λ} (resp. $A^{(\Lambda)}$). For any functor $H : \mathbb{X} \to \mathbb{A}$, we write H^{Λ} (resp. $H^{(\Lambda)}$) for the composite $\Pi^{\mathbb{A}}_{\Lambda} H$ (resp. $\Sigma^{\mathbb{A}}_{\Lambda} H$). Note that, if a functor $H : \mathbb{X} \to \mathbb{A}$ preserves products (coproducts), then $H^{\Lambda} = \Pi^{\mathbb{A}}_{\Lambda} H \simeq H\Pi^{\mathbb{X}}_{\Lambda}$ ($H^{(\Lambda)} = \Sigma^{\mathbb{A}}_{\Lambda} H \simeq H\Sigma^{\mathbb{X}}_{\Lambda}$). When no confusion can occur, we shall write Π_{Λ} (Σ_{Λ}) instead of $\Pi^{\mathbb{A}}_{\Lambda}$ ($\Sigma^{\mathbb{A}}_{\Lambda}$).

Given two categories $\mathbb{X}$ and $\mathbb{Y}$, we write $[\mathbb{X}, \mathbb{Y}]$ for the functor category.

2.8. **Functor categories and adjoint triples.** Let (L, F, R) be an adjoint triple (as in 1.11). For unit and counit of the adjunction $F \dashv R$ ($L \dashv F$) write $\eta^R : I_{\mathbb{A}} \to RF$ and $\varepsilon^R : FR \to I_{\mathbb{B}}$ ($\eta^L : I_{\mathbb{B}} \to FL$ and $\varepsilon^L : LF \to I_{\mathbb{A}}$). Then, for any category $\mathbb{X}$, one has adjunctions

$$\eta^{\mathbb{X}}, \varepsilon^{\mathbb{X}} : [\mathbb{X}, F] \dashv [\mathbb{X}, R] : [\mathbb{X}, \mathbb{B}] \to [\mathbb{X}, \mathbb{A}],$$
$$\eta_{\mathbb{X}}, \varepsilon_{\mathbb{X}} : [\mathbb{X}, L] \dashv [\mathbb{X}, F] : [\mathbb{X}, \mathbb{A}] \to [\mathbb{X}, \mathbb{B}],$$

where

$$\eta^{\mathbb{X}} = [\mathbb{X}, \eta^R], \quad \varepsilon^{\mathbb{X}} = [\mathbb{X}, \varepsilon^R], \quad \eta_{\mathbb{X}} = [\mathbb{X}, \eta^L], \quad \varepsilon_{\mathbb{X}} = [\mathbb{X}, \varepsilon^L].$$

Now assume (L, F, R) to be a QF triple. Then there are index sets Λ and Λ' such that R is a retract of $L^{(\Lambda)}$ and L is a retract of $R^{\Lambda'}$ (see Proposition 2.2). Since

$$[\mathbb{B}, L](\Sigma^{\mathbb{B}}_{\Lambda}) = L\Sigma^{\mathbb{B}}_{\Lambda} \simeq \Sigma^{\mathbb{A}}_{\Lambda} L = L^{(\Lambda)}, \quad [\mathbb{B}, R](\Pi^{\mathbb{B}}_{\Lambda'}) = R\Pi^{\mathbb{B}}_{\Lambda'} \simeq \Pi^{\mathbb{A}}_{\Lambda'} R = R^{\Lambda'},$$

it follows that R is a retract of $[\mathbb{B}, L](\Sigma^{\mathbb{B}}_{\Lambda})$, while L is a retract of $[\mathbb{B}, R](\Pi^{\mathbb{B}}_{\Lambda'})$. Using now that $L, R \in [\mathbb{B}, \mathbb{A}]$ and that $[\mathbb{B}, L]$ (resp. $[\mathbb{B}, R]$) is a left (resp. right) adjoint to $[\mathbb{B}, F]$, it follows (e.g. from [23, Proposition 1.5] and its dual) that the R-component $(\varepsilon_{\mathbb{X}})_R = \varepsilon^L R$ of $\varepsilon_{\mathbb{X}}$ is a split epimorphism, while the L-component $(\eta^{\mathbb{X}})_L = \eta^R L$ of $\eta^{\mathbb{X}}$ is a split monomorphism.

Similarly, considering the adjunctions

$$[R, \mathbb{A}] \dashv [F, \mathbb{A}] \dashv [L, \mathbb{A}],$$

one gets that $R\eta^L$ is a split monomorphism, while $L\varepsilon^R$ is a split epimorphism.

Summarising we have proved:

2.9. **Theorem.** *Let (L, F, R) be a QF triple (as in 2.5). Then - with the notation from 2.8 - $\eta^R L$ and $R\eta^L$ are split monomorphisms, while $\varepsilon^L R$ and $L\varepsilon^R$ are split epimorphisms.*

2.10. **Separability and adjoint triples.** In [6], Caenepeel and Militaru introduced the notion of *separable functors of the second kind*. Applied to an adjoint triple (L, F, R), they prove in [6, Theorem 2.7]:

(i) *the functor L is R-separable if and only if $R\eta^L$ is a split monomorphism;*

(ii) *the functor R is L-separable if and only if $L\varepsilon^R$ is a split epimorphism.*

As shown in Theorem 2.9, for a QF triple (L, F, R), the conditions in (i) and (ii) are satisfied and hence [6, Proposition 2.4] applies and yields:

2.11. **Proposition.** *Let (L, F, R) be a QF triple and consider a morphism $f : B \to B'$ in $\mathbb{B}$. Then $R(f)$ has a left (right) inverse in $\mathbb{A}$ if and only if $L(f)$ has a left (right) inverse in $\mathbb{A}$.*

3. QF monads and comonads

Before coming to the main topics of this section we recall some constructions from category theory. For a monad $\mathbf{F} = (F, \mu, \eta)$ on a category $\mathbb{A}$, we write

- $\mathbb{A}_F$ for the Eilenberg-Moore category of $\mathbf{F}$-modules and $\phi_F \dashv U_F : \mathbb{A}_F \to \mathbb{A}$ for the corresponding forgetful-free adjunction;
- $\widetilde{\mathbb{A}}_F$ for the Kleisli category of the monad $\mathbf{F}$ (as a full subcategory of $\mathbb{A}_F$, e.g. [4]) and $\phi_F \dashv u_F : \widetilde{\mathbb{A}}_F \to \mathbb{A}$ for the corresponding Kleisli adjunction.

Dually, if $\mathbf{G} = (G, \delta, \varepsilon)$ is a comonad on $\mathbb{A}$, we write

- $\mathbb{A}^G$ for the Eilenberg–Moore category of $\mathbf{G}$-comodules and $U^G \dashv \phi^G : \mathbb{A} \to \mathbb{A}^G$ for the corresponding forgetful-cofree adjunction;
- $\widetilde{\mathbb{A}}^G$ for the Kleisli category of the comonad $\mathbf{G}$ and $u^G \dashv \phi^G : \mathbb{A} \to \widetilde{\mathbb{A}}^G$ for the corresponding Kleisli adjunction.

3.1. **Monads on functor categories.** Let $\mathbf{F} = (F, \mu, \eta)$ be a monad on a category $\mathbb{A}$. Then the precomposition with F induces a monad $\mathbf{F}_{\mathbb{X}}$ on $[\mathbb{X}, \mathbb{A}]$,

$$F_{\mathbb{X}} : [\mathbb{X}, \mathbb{A}] \to [\mathbb{X}, \mathbb{A}], \qquad \begin{array}{rcl} f & \longmapsto & Ff, \\ f \to f' & \longmapsto & Ff \to Ff'. \end{array}$$

It is easy to see that the corresponding Eilenberg-Moore category $[\mathbb{X}, \mathbb{A}]_{F_{\mathbb{X}}}$ of $\mathbf{F}_{\mathbb{X}}$-modules are just left $\mathbf{F}$-modules (see [22]), that is, functors $f : \mathbb{X} \to \mathbb{A}$ together with a natural transformation $\alpha : Ff \to f$ inducing commutativity of the diagrams

$$\begin{tikzcd} f \arrow[r, "\eta f"] \arrow[dr, equal] & Ff \arrow[d, "\alpha"] \\ & f, \end{tikzcd} \qquad \begin{tikzcd} FFf \arrow[r, "\mu f"] \arrow[d, "F\alpha"'] & Ff \arrow[d, "\alpha"] \\ Ff \arrow[r, "\alpha"'] & f . \end{tikzcd}$$

Dually, $\mathbf{F}$ induces the monad $\mathbf{F}^{\mathbb{X}}$ on the category $[\mathbb{A}, \mathbb{X}]$,

$$F^{\mathbb{X}} : [\mathbb{A}, \mathbb{X}] \to [\mathbb{A}, \mathbb{X}], \qquad \begin{array}{rcl} f & \longmapsto & fF, \\ f \to f' & \longmapsto & fF \to f'F. \end{array}$$

The corresponding Eilenberg-Moore category $[\mathbb{A}, \mathbb{X}]_{F^{\mathbb{X}}}$ of $\mathbf{F}^{\mathbb{X}}$-modules are just right $\mathbf{F}$-modules.

3.2. **Theorem.** ([29, Theorem 8]) *Let* $\mathbf{F} = (F, \mu, \eta)$ *be a monad on* $\mathbb{A}$.

(1) *The assignments* $\begin{array}{rcl} f : \mathbb{X} \to \mathbb{A}_F & \longmapsto & U_F f : \mathbb{X} \to \mathbb{A}, \\ f \to f' : \mathbb{X} \to \mathbb{A}_F & \longmapsto & U_F f \to U_F f' : \mathbb{X} \to \mathbb{A}, \end{array}$

yield an isomorphism of categories

$$[\mathbb{X}, \mathbb{A}_F] \simeq [\mathbb{X}, \mathbb{A}]_{F_{\mathbb{X}}}.$$

(2) *The assignments* $\begin{array}{rcl} f : \widetilde{\mathbb{A}}_F \to \mathbb{X} & \longmapsto & f\phi_F : \mathbb{A} \to \mathbb{X}, \\ f \to f' : \widetilde{\mathbb{A}}_F \to \mathbb{X} & \longmapsto & f\phi_F \to f'\phi_F : \mathbb{A} \to \mathbb{X}, \end{array}$

yield an isomorphism of categories

$$[\widetilde{\mathbb{A}}_F, \mathbb{X}] \simeq [\mathbb{A}, \mathbb{X}]_{F^{\mathbb{X}}}.$$

3.3. **Density presentation.** For a monad $\mathbf{F} = (F, \mu, \eta)$ on $\mathbb{A}$, consider the family

$$\mathcal{P} = \{(FF(A), \mu_{F(A)}) \underset{Fh}{\overset{\mu_A}{\rightrightarrows}} (F(A), \mu_A))\}_{(A,h)} \in \mathbb{A}_F$$

of parallel morphisms. We know from [8] that $\mathcal{P}$ is a *density presentation* (in the sense of Kelly [18]) of the fully-faithful and dense canonical embedding $i : \widetilde{\mathbb{A}}_F \to \mathbb{A}_F$. For any category $\mathbb{B}$ with coequalisers, we write $[\mathbb{A}_F, \mathbb{B}]^{\mathcal{P}}$ for the full subcategory of $[\mathbb{A}_F, \mathbb{B}]$ given by those functors $H : \mathbb{A}_F \to \mathbb{B}$ that preserve the coequaliser of each member of $\mathcal{P}$, that is, for all $(A, h) \in \mathbb{A}_F$, H preserves the coequaliser diagram

$$(FF(A), \mu_{F(A)}) \underset{Fh}{\overset{\mu_A}{\rightrightarrows}} (F(A), \mu_A)) \xrightarrow{h} (A, h) \ .$$

Then, according to [18, Theorem 5.31], the functor

$$[i, \mathbb{B}] : [\mathbb{A}_F, \mathbb{B}]^{\mathcal{P}} \to [\widetilde{\mathbb{A}}_F, \mathbb{B}]$$

is an equivalence of categories. Now, by Theorem 3.2(2), the composite

$$[\mathbb{A}_F, \mathbb{B}]^{\mathcal{P}} \xrightarrow{[i,\mathbb{B}]} [\widetilde{\mathbb{A}}_F, \mathbb{B}] \xrightarrow{\simeq} [\mathbb{A}, \mathbb{B}]_{F^{\mathbb{B}}}, \quad \mathbb{A}_F \xrightarrow{H} \mathbb{B} \longmapsto \mathbb{A} \xrightarrow{\phi_F} \widetilde{\mathbb{A}}_F \xrightarrow{i} \mathbb{A}_F \xrightarrow{H} \mathbb{B},$$

is an equivalence of categories. But since $i \cdot \phi_F = \phi_F$, this equivalence is just the functor $[\phi_F, \mathbb{B}]$. Thus, we have proved:

3.4. **Theorem.** *For any monad* (F, μ, η) *on* $\mathbb{A}$ *and any category* $\mathbb{B}$ *with coequalisers, the functor*

$$[\phi_F, \mathbb{B}] : [\mathbb{A}_F, \mathbb{B}]^{\mathcal{P}} \to [\mathbb{A}, \mathbb{B}]_{F^{\mathbb{B}}}$$

is an equivalence of categories.

We now come back to QF functors.

3.5. **Right adjoints of monads.** Let $\mathbf{F} = (F, \mu, \eta)$ be monad on $\mathbb{A}$. For an adjunction $\bar\eta, \bar\varepsilon : F \dashv G$, the monad structure on F induces canonically a comonad $\mathbf{G} = (G, \delta, \varepsilon)$, called a *right adjoint of the monad* $\mathbf{F}$ (e.g. [10]). The categories $\mathbb{A}_F$ and $\mathbb{A}^G$ are isomorphic by

$$T : \mathbb{A}_F \to \mathbb{A}^G, \quad F(A) \xrightarrow{\varrho} A \;\mapsto\; A \xrightarrow{\bar\eta_A} GF(A) \xrightarrow{G(\varrho)} G(A),$$
$$T^{-1} : \mathbb{A}^G \to \mathbb{A}_F, \quad A \xrightarrow{\omega} G(A) \;\mapsto\; F(A) \xrightarrow{F(\omega)} FG(A) \xrightarrow{\bar\varepsilon_A} A.$$

The forgetful functor $U_F : \mathbb{A}_F \to \mathbb{A}$ is right adjoint to the free functor $\phi_F : \mathbb{A} \to \mathbb{A}_F$ and the forgetful functor $U^G : \mathbb{A}^G \to \mathbb{A}$ is left adjoint to the free functor $\phi^G : \mathbb{A} \to \mathbb{A}^G$.

With these notions we have the diagram with commutative triangle

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "\phi_F"] & \mathbb{A}_F \arrow[r, "U_F"] \arrow[d, "T"'] & \mathbb{A} \\
\mathbb{A} \arrow[r, "\phi^G"] & \mathbb{A}^G . \arrow[ur, "U^G"'] &
\end{tikzcd}$$

This shows that U_F can be written as composition of functors with right adjoints and hence also allows for a right adjoint. More precisely,

$$(\phi_F, U_F, T^{-1}\phi^G) \text{ is an adjoint triple of functors.}$$

Similar arguments show, given a comonad (G, δ, ε) with left adjoint F,

$$(T\phi_F, U^G, \phi^G) \text{ is an adjoint triple of functors.}$$

3.6. **Definitions.** A monad (F, μ, η) is called a *(left, right) QF monad* if F has a right adjoint G such that $(\phi_F, U_F, T^{-1}\phi^G)$ is a (left, right) QF triple.

A comonad (G, δ, ε) is called a *(left, right) QF comonad* if G has a left adjoint F such that $(T\phi_F, U^G, \phi^G)$ is a (left, right) QF triple.

3.7. **Proposition.** (*Properties of QF monads*) *Let* $\mathbf{F} = (F, \mu, \eta)$ *be a monad on* $\mathbb{A}$ *with right adjoint comonad* $\mathbf{G} = (G, \delta, \varepsilon)$.

(1) $\mathbf{F}$ *is a QF monad if and only if* $\mathbf{G}$ *is a QF comonad.*

(2) *If this is the case, then*

- (i) ϕ_F *and* ϕ^G *preserve all limits and colimits;*
- (ii) ϕ_F *preserves small objects;*
- (iii) *the* U_F*-injective objects in* $\mathbb{A}_F$ *are the same as the* U_F*-projectives;*
- (iv) *the* U^G*-injective objects in* $\mathbb{A}^G$ *are the same as the* U^G*-projectives;*
- (v) F *and* G *preserve all limits and colimits;*
- (vi) *if* $\mathbb{A}$ *is small complete, well-powered, and with a small cogenerating set, then the functor* F *admits a left adjoint;*
- (vii) *if* $\mathbb{A}$ *is small cocomplete, well-copowered, and with a small generating set, then the functor* G *admits a right adjoint.*

Proof. (1) The comonad $\mathbf{G}$ being right adjoint to the monad $\mathbf{F}$, the diagram

$$\begin{tikzcd}
\mathbb{A}_F \arrow[rr, "T"] \arrow[dr, "U_F"'] & & \mathbb{A}^G \arrow[dl, "U^G"] \\
& \mathbb{A} &
\end{tikzcd}$$

commutes. Since T is an isomorphism of categories, it follows that $(\phi_F, U_F, T^{-1}\phi^G)$ is a QF triple if and only if $(T\phi_F, U^G, \phi^G)$ is so.

(2) (i), (ii), (iii), (iv), (vi) and (vii) follow from Proposition 2.6.

(v) Since the forgetful functor $U_F : \mathbb{A}_F \to \mathbb{A}$ admits both left and right adjoints, it preserves all limits and colimits. Then the functor $F = U_F \phi_F$ also preserves all limits and colimits by (i). Similarly, G preserves all limits and colimits. $\square$

3.8. **Module structures on right adjoints of monads.** Let (F, μ, η) be a monad on $\mathbb{A}$ with right adjoint comonad (G, δ, ε). Then one has the commutative diagram

$$\begin{tikzcd} \mathbb{A} \arrow[r, "\phi^G"] \arrow[dr, "G"'] & \mathbb{A}^G \arrow[r, "T^{-1}"] \arrow[d, "U^G"] & \mathbb{A}_F \arrow[dl, "U_F"] \\ & \mathbb{A}. & \end{tikzcd}$$

Hence there is a left $\mathbf{F}$-module structure $\alpha_G : FG \to G$ on the functor G (e.g. [22]), thus $(G, \alpha_G) \in [\mathbb{A}, \mathbb{A}]_{F_{\mathbb{A}}}$. Hereby α_G corresponds to $\delta : G \to GG$ under the bijection

$$\mathrm{Mor}_{\mathbb{A}}(F(-), ?) \simeq \mathrm{Mor}_{\mathbb{A}}(-, G(?)),$$

and hence $\alpha_G = \overline{\varepsilon} G \cdot F\delta$, where $\overline{\varepsilon} : FG \to I_{\mathbb{A}}$ is the counit of $F \dashv G$.

Now let (F, μ, η) be a QF monad with a right adjoint comonad (G, δ, ε). Since the forgetful functor $U_F : \mathbb{A}_F \to \mathbb{A}$ creates limits, for any index set Λ, we have commutativity of the diagram

$$\begin{tikzcd} \mathbb{A} \arrow[r, "T^{-1}\phi^G"] \arrow[dr, "G"'] & \mathbb{A}_F \arrow[r, "\Pi_\Lambda"] \arrow[d, "U_F"] & \mathbb{A}_F \arrow[d, "U_F"] \\ & \mathbb{A} \arrow[r, "\Pi_\Lambda"'] & \mathbb{A}. \end{tikzcd}$$

It follows that the functor $G^\Lambda = \Pi_\Lambda G$ is an object of the category $[\mathbb{A}, \mathbb{A}]_{F_{\mathbb{A}}}$. Write $\alpha_{G^\Lambda} : FG^\Lambda \to G^\Lambda$ for the corresponding left action of the monad $\mathbf{F}$ on G^Λ. It is easy to see that $(G, \alpha_G)^\Lambda$ in $[\mathbb{A}, \mathbb{A}]_{F_{\mathbb{A}}}$ is just the pair $(G^\Lambda, \alpha_{G^\Lambda})$.

Similarly, consider the diagram

$$\begin{tikzcd} \mathbb{A} \arrow[r, "\phi_F"] \arrow[dr, "F"'] & \mathbb{A}_F \arrow[r, "\Sigma_\Lambda"] \arrow[d, "U_F"] & \mathbb{A}_F \arrow[d, "U_F"] \\ & \mathbb{A} \arrow[r, "\Sigma_\Lambda"'] & \mathbb{A}, \end{tikzcd}$$

in which the square commutes since F preserves colimits by Proposition 3.7, and thus the forgetful functor U_F creates them. Then commutativity of the outer diagram implies – since the functor $F^{(\Lambda)}$ equals $\Sigma_\Lambda \phi_F$ – that $F^{(\Lambda)}$ is also an object of the category $[\mathbb{A}, \mathbb{A}]_{F_{\mathbb{A}}}$. Write $\alpha_{F^{(\Lambda)}} : FF^{(\Lambda)} \to F^{(\Lambda)}$ for the corresponding left action of the monad $\mathbf{F}$ on $F^{(\Lambda)}$. Then $(F, \mu)^{(\Lambda)}$ in $[\mathbb{A}, \mathbb{A}]_{F_{\mathbb{A}}}$ is just the pair $(F^{(\Lambda)}, \alpha_{F^{(\Lambda)}})$.

3.9. **Proposition.** *Let $\mathbf{F} = (F, \mu, \eta)$ be a monad on $\mathbb{A}$ and G a right adjoint to F.*

(1) $\mathbf{F}$ *is a left QF monad if and only if, for some index set Λ, there is a natural coretraction $G \to F^{(\Lambda)}$ of left* $\mathbf{F}$*-modules.*

(2) $\mathbf{F}$ *is a right QF monad if and only if, for some index set Λ', there is a natural coretraction $F \to G^{\Lambda'}$ of left* $\mathbf{F}$*-modules.*

Proof. (1) Since $U_F T^{-1}\phi^G = U^G\phi^G = G$ and $U_F(\phi_F)^{(\Lambda)} = (U_F\phi_F)^{(\Lambda)} = F^{(\Lambda)}$, it follows from Theorem 3.2 that $T^{-1}\phi^G \in [\mathbb{A}, \mathbb{A}_F]$ is a coretraction of $(\phi_F)^{(\Lambda)} \in [\mathbb{A}, \mathbb{A}_F]$ if and only if (G, δ) is a coretraction of $(F^{(\Lambda)}, \alpha_{F^{(\Lambda)}})$ in $[\mathbb{A}, \mathbb{A}]_{F_{\mathbb{A}}}$.

(2) can be proved in a similar manner. □

3.10. **Definition.** A monad (F, μ, η) on $\mathbb{A}$ is said to be a *Frobenius monad* provided the forgetful functor $U_F : \mathbb{A}_F \to \mathbb{A}$ is Frobenius.

A comonad (G, δ, ε) is said to be a *Frobenius comonad* provided the forgetful functor $U^G : \mathbb{A}^G \to \mathbb{A}$ is Frobenius (Definition 2.1).

By an argument similar to the proof of Proposition 3.9 we get:

3.11. **Proposition.** *A monad* $\mathbf{F}$ *on* $\mathbb{A}$ *with a right adjoint comonad* $\mathbf{G}$ *is Frobenius if and only if the functors* F *and* G *are isomorphic as left* $\mathbf{F}$*-modules.*

As pointed out in 3.5, for any monad (F, μ, η), any right adjoint functor G of F has the structure of a comonad; in particular, for a Frobenius monad F the functors ϕ_F and ϕ^G have to be isomorphic and hence the functor F allows for a comonad structure. This leads to the following characterisation of Frobenius monads given in [30].

3.12. **Proposition.** *A monad* (F, μ, η) *on* $\mathbb{A}$ *is Frobenius provided there exist natural transformations* $\varepsilon : F \to I_{\mathbb{A}}$ *and* $\varrho : I_{\mathbb{A}} \to FF$ *satisfying the equations*

$$F\mu \cdot \varrho F = \mu F \cdot F\varrho \quad \text{and} \quad F\varepsilon \cdot \varrho = \eta = \varepsilon F \cdot \varrho.$$

Putting $\delta = F\mu \cdot \varrho F = \mu F \cdot F\varrho$, *we have*

(i) $F\mu \cdot \delta F = \delta \cdot \mu = \mu F \cdot F\delta$;
(ii) $F\varepsilon \cdot \delta = I_F = \varepsilon F \cdot \delta$;
(iii) $\varrho = \delta \cdot \eta$;
(iv) (F, F) *is an adjoint pair with counit* $\sigma = \varepsilon \cdot \mu : FF \to I_{\mathbb{A}}$ *and unit* $\varrho : I_{\mathbb{A}} \to FF$;
(v) (F, δ, ε) *is a comonad on* $\mathbb{A}$.

It was observed by L. Abrams in [1, Theorem 3.3] that over a Frobenius algebra A, the category of right modules over A is isomorphic to the category of right comodules over A. The following theorem shows that this holds more generally for Frobenius functors and such an isomorphism is characteristic for these functors.

3.13. **Theorem.** *Let* $\mathbf{F} = (F, \mu, \eta)$ *be a monad on* $\mathbb{A}$*. The following are equivalent:*

(a) F *is a Frobenius monad;*
(b) F *allows for a comonad structure* $\overline{\mathbf{F}} = (F, \delta, \varepsilon)$ *and an isomorphism*

$$\kappa : \mathbb{A}_F \to \mathbb{A}^F$$

that is compatible with the forgetful functors (i.e. $U^F\kappa = U_F$*) and restricts to an isomorphism of the Kleisli (sub-)categories* $\widetilde{\mathbb{A}}_F$ *and* $\widetilde{\mathbb{A}}^F$.

Proof. (a)⇒(b) If (F, μ, η) is a Frobenius monad it has a right adjoint (comonad) G which is isomorphic to F. This defines a comonad $\overline{\mathbf{F}} = (F, \delta, \varepsilon)$ and, in view of 3.5, we get an isomorphism

$$\kappa : \mathbb{A}_F \xrightarrow{T} \mathbb{A}^G \xrightarrow{\simeq} \mathbb{A}^F$$

which satisfies the compatibility condition required in (b). Moreover, $\kappa\phi_F = \phi^F$: For any $A \in \mathbb{A}$, κ takes $\phi_F(A) = (F(A), \mu_A)$ to

$$(F(A), F(A) \xrightarrow{\overline{\eta}_{F(A)}} FFF(A) \xrightarrow{F\mu_A} FF(A)),$$

where $\overline{\eta} : I_{\mathbb{A}} \to FF$ is the unit of the adjunction $F \dashv F$. By Proposition 3.12 this means $F\mu_A \cdot \overline{\eta}_{F(A)} = \delta_A$ and $\kappa(\phi_F(A)) = \phi^F(A)$, that is, κ restricts to an isomorphism between the corresponding Kleisli categories.

(b)⇒(a) We claim that, under the conditions given in (b), the comonad $\overline{\mathbf{F}} = (F, \delta, \varepsilon)$ is right adjoint to the monad $\mathbf{F}$. Indeed, if $\kappa : \mathbb{A}_F \to \mathbb{A}^F$ is an isomorphism compatible with the forgetful functors, then the composite $\kappa^{-1}\phi^F$ is right adjoint to the functor U_F. It then follows that the composite $F = U_F\phi_F$ admits as a right adjoint the composite $U_F\kappa^{-1}\phi^F$, which – since $U_F\kappa^{-1} = U^F$ – is just $U^F\phi^F = F$. Thus F is right adjoint to itself implying that it is a Frobenius functor. Then F allows for another comonad structure $\overline{\mathbf{F}}' = (F, \delta', \varepsilon')$ and an isomorphism $\kappa' : \mathbb{A}_F \to \mathbb{A}^{F'}$ that is compatible with the forgetful functors. It follows that the composite isomorphism $\kappa(\kappa')^{-1} : \mathbb{A}^{F'} \to \mathbb{A}^F$ is also compatible with the forgetful functors. Hence the comonads $\overline{\mathbf{F}}$ and $\overline{\mathbf{F}}'$ are isomorphic, and thus the comonad $\overline{\mathbf{F}}$ is also right adjoint to the monad $\mathbf{F}$. Now, since to say that κ restricts to an isomorphism $\widetilde{\mathbb{A}}_{\mathbf{F}} \simeq \widetilde{\mathbb{A}}_{\overline{\mathbf{F}}}$ is to say that $\kappa\phi_F \simeq \phi^F$, it follows that $\phi_F \simeq \kappa^{-1}\phi^F$ is right adjoint to U_F. Thus, $\mathbf{F}$ is a Frobenius monad. □

To answer the question when the free-module and free-comodule functors are QF we need the following observations.

3.14. Lemma. *Let (F, μ, η) be a monad and (G, δ, ε) a comonad on $\mathbb{A}$.*

(1) *The functor $F : \mathbb{A} \to \mathbb{A}$ has a left adjoint L if and only if the free-module functor $\phi_F : \mathbb{A} \to \mathbb{A}_F$ does so.*

In this case L has a right $\mathbf{F}$-module structure which we denote by $\alpha : LF \to L$.

(2) *The functor $G : \mathbb{A} \to \mathbb{A}$ has right adjoint R if and only if the free-comodule functor $\phi^G : \mathbb{A} \to \mathbb{A}^G$ does so.*

In this case R has a left $\mathbf{G}$-comodule structure denoted by $\beta : R \to GR$.

Proof. (1) Indeed, if ϕ_F has a left adjoint, then the functor F, being the composite $U_F\phi_F$, also has a left adjoint. Conversely, suppose that F has a left adjoint. Then since the functor U_F is clearly monadic, one can apply Dubuc's Adjoint Triangle Theorem [9] to the diagram

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "\phi_F"] \arrow[dr, "F"'] & \mathbb{A}_F \arrow[d, "U_F"] \\
& \mathbb{A}
\end{tikzcd}$$

to deduce that the functor ϕ_F also admits a left adjoint. In case F has a left adjoint functor $L : \mathbb{A} \to \mathbb{A}$, the above commutative triangle implies a right $\mathbf{F}$-module structure on L.

(2) is proved in a similar way. □

3.15. Theorem. *Let $\mathbf{F} = (F, \mu, \eta)$ be a monad and $\mathbf{G} = (G, \delta, \varepsilon)$ a comonad on $\mathbb{A}$.*

(1) *If F admits a left adjoint $L : \mathbb{A} \to \mathbb{A}$, the following are equivalent:*
   - (a) *the functor $\phi_F : \mathbb{A} \to \mathbb{A}_F$ is QF;*
   - (b) *there are index sets Λ, Λ' such that*
     - (i) *the right $\mathbf{F}$-module (L, α) is a retract of the right $\mathbf{F}$-module $(F, \mu)^{\Lambda}$,*
     - (ii) *the right $\mathbf{F}$-module (F, μ) is a retract of the right $\mathbf{F}$-module $(L, \alpha)^{(\Lambda')}$.*

(2) *If G admits a right adjoint $R : \mathbb{A} \to \mathbb{A}$, the following are equivalent:*

(a) *the functor* $\phi^G : \mathbb{A} \to \mathbb{A}^G$ *is QF;*

(b) *there are index sets* Λ, Λ' *such that*

(i) *the left* $\mathbf{G}$*-comodule* (R, β) *is a retract of the left* $\mathbf{G}$*-comodule* $(G, \delta)^{(\Lambda)}$,

(ii) *the left* $\mathbf{G}$*-comodule* (G, δ) *is a retract of the left* $\mathbf{G}$*-comodule* $(L, \alpha)^{\Lambda'}$.

Proof. (1) Since F admits a left adjoint $L : \mathbb{A} \to \mathbb{A}$ by our assumption on $\mathbf{F}$, the functor ϕ_F also admits a left adjoint $\overline{L} : \mathbb{A}_F \to \mathbb{A}$. Since left adjoints are unique up to natural isomorphism, $\overline{L}$ may be chosen in such a way that the composite $\overline{L}\phi_F$ is just L. Since $\overline{L}$ is a left adjoint, it preserves all colimits. Since the functor $\sum_{\Lambda'}$ also preserves colimits, it follows that the functor $\overline{L}^{(\Lambda')} = \sum_{\Lambda'} \overline{L}$ preserves colimits, too. Thus, in particular, $\overline{L}^{(\Lambda')} \in [\mathbb{A}_F, \mathbb{A}]^{\mathcal{P}}$. Next, since the functor U_F takes – for any F-module (A, h) – each coequaliser

$$(FF(A), \mu_{F(A)}) \underset{Fh}{\overset{\mu_A}{\rightrightarrows}} (F(A), \mu_A)) \xrightarrow{h} (A, h)$$

to a (split) coequaliser

$$\begin{tikzcd} FF(A) \arrow[r, shift left, "\mu_A"] \arrow[r, shift right, "Fh"'] & F(A) \arrow[l, bend right=60, "\eta_{F(A)}"'] \arrow[r, "h"'] & A \arrow[l, bend right=60, "\eta_A"'] \end{tikzcd}$$

and since any product of split coequalisers is split, it follows that the functor $U_F^\Lambda = \Pi_\Lambda U_F$ also lies in $[\mathbb{A}_F, \mathbb{A}]^{\mathcal{P}}$. Now, since

$$\begin{array}{ll} \overline{L}^{(\Lambda')}\phi_F = (\overline{L}\phi_F)^{(\Lambda')} = L^{(\Lambda')}, & U_F^\Lambda \phi_F = (U_F\phi_F)^\Lambda = F^\Lambda, \\ (L, \alpha)^{(\Lambda')} = (L^{(\Lambda')}, \alpha^{(\Lambda')}), & (F, \mu)^\Lambda = (F^\Lambda, \mu^\Lambda), \end{array}$$

the result follows from Theorem 3.4.

(2) is shown by a similar proof. □

3.16. **Proposition.** *Let* (F, μ, η) *be a monad and* (G, δ, ε) *a comonad on* $\mathbb{A}$.

(1) *If* F *admits a left adjoint, then the functor* $\phi_F : \mathbb{A} \to \mathbb{A}_F$ *is QF if and only if the functor* $\phi_F : \mathbb{A} \to \widetilde{\mathbb{A}}_F$ *is so.*

(2) *If* G *admits a right adjoint, then the functor* $\phi^G : \mathbb{A} \to \mathbb{A}^G$ *is QF if and only if the functor* $\phi^G : \mathbb{A} \to \widetilde{\mathbb{A}}^G$ *is so.*

Proof. (1) Write $L : \mathbb{A} \to \mathbb{A}$ for the left adjoint to F and $\alpha : LF \to L$ for the corresponding right F-module structure on L (see Lemma 3.14). Since L is left adjoint to F, L allows for a canonical comonad structure (L, δ, ε) (e.g. [10]). Moreover, there is an isomorphism between the Kleisli categories, $K : \widetilde{\mathbb{A}}_F \to \widetilde{\mathbb{A}}^L$, given by the natural bijections (e.g. [19], [4, 2.6])

$$\begin{aligned} \mathrm{Mor}_{\mathbb{A}_F}(\phi_F(A), \phi_F(A')) &\simeq \mathrm{Mor}_{\mathbb{A}}(A, F(A)') \\ &\simeq \mathrm{Mor}_{\mathbb{A}}(L(A), A') \simeq \mathrm{Mor}_{\mathbb{A}^L}(\phi^L(A), \phi^L(A')), \end{aligned}$$

leading to the diagram with commutative triangle

$$\begin{tikzcd} \mathbb{A} \arrow[r, "\phi_F"] \arrow[dr, "\phi^L"'] & \widetilde{\mathbb{A}}_F \arrow[r, "u_F"] \arrow[d, "K"] & \mathbb{A} \\ & \widetilde{\mathbb{A}}^L \arrow[r, "u^L"'] & \mathbb{A}. \end{tikzcd}$$

Now the functor ϕ_F - which has a right adjoint u_F - is composed by functors which have left adjoints and hence also allows for a left adjoint, that is,

$$(u^L K, \phi_F, u_F) \text{ is an adjoint triple of functors.}$$

Then, since for any index set Λ,

- $(u_F)^\Lambda \phi_F = \Pi_\Lambda u_F \phi_F = \Pi_\Lambda F = F^\Lambda$, and
- $(u^L K)^{(\Lambda)} \phi_F = \Sigma_\Lambda u^L K \phi_F = \Sigma_\Lambda u^L \phi^L = \Sigma_\Lambda L = L^{(\Lambda)}$,

it follows from Theorem 3.2(2) that there are index sets Λ and Λ' such that the functor $u_L K$ is a retract of the functor u_F^Λ and the functor u_F is a retract of the functor $(u_L K)^{(\Lambda')}$ if and only if the right F-module (L, α) is a retract of the right F-module $(F, \mu)^\Lambda$ and the right F-module (F, μ) is a retract of the right F-module $(L, \alpha)^{(\Lambda')}$. Thus, the functor $\phi_F : \mathbb{A} \to \widetilde{\mathbb{A}}_F$ is QF if and only if the functor $\phi_F : \mathbb{A} \to \mathbb{A}_F$ is QF.

(2) The proof is dual to that of (1). $\square$

3.17. **Proposition.** *For a monad (F, μ, η) on $\mathbb{A}$ admitting a left adjoint comonad (G, δ, ε), the following are equivalent:*

(a) *the functor $\phi_F : \mathbb{A} \to \mathbb{A}_F$ is QF;*

(b) *the functor $\phi_F : \mathbb{A} \to \widetilde{\mathbb{A}}_F$ is QF;*

(c) *the functor $\phi^G : \mathbb{A} \to \mathbb{A}^G$ is QF;*

(d) *the functor $\phi^G : \mathbb{A} \to \widetilde{\mathbb{A}}^G$ is QF.*

Proof. (a) and (b) are equivalent by Proposition 3.16(1), while (c) and (d) are equivalent by Proposition 3.16(2). The equivalence of (b) and (d) follows from the commutativity of the diagram

$$\begin{tikzcd} \mathbb{A} \arrow[r, "\phi_F"] \arrow[dr, "\phi^G"'] & \widetilde{\mathbb{A}}_F \arrow[d, "T"] \\ & \widetilde{\mathbb{A}}^G \end{tikzcd}$$

(see the proof of Proposition 3.7(1)). $\square$

4. Applications

We illustrate the definitions from Section 1 in the case of module categories over associative unital rings R and S. By ${}_R\mathbb{M}$ we denote the category of left R-modules.

For commutative rings R, Proposition 2.3 reads as follows.

4.1. **Proposition.** *For any module M over a commutative ring R, the following are equivalent:*

(a) *$M \otimes_R - : \mathbb{M}_R \to \mathbb{M}_R$ is a QF functor;*

(b) *$M \otimes_R - : \mathbb{M}_R \to \mathbb{M}_R$ is a Frobenius functor;*

(c) *the canonical morphism $\mathrm{Hom}_R(M, R) \otimes_R M \to \mathrm{End}_R(M)$ is an isomorphism.*

4.2. **Functors between module categories.** Any functor ${}_R\mathbb{M} \to {}_S\mathbb{M}$ which allows for a right adjoint is given by an (R, S)-bimodule ${}_R P_S$ and the adjoint functor pair

$$P \otimes_S - : {}_S\mathbb{M} \to {}_R\mathbb{M}, \quad \mathrm{Hom}_R(P, -) : {}_R\mathbb{M} \to {}_S\mathbb{M}.$$

We define the full subcategories of ${}_R\mathbb{M}$ and ${}_S\mathbb{M}$, respectively (see [33]),

$$\begin{array}{rcl} \mathrm{Gen}({}_RP) &=& \{N \in {}_R\mathbb{M} \,|\, N \text{ is } P\text{-generated}\}, \\ \sigma[P] &=& \{N \in {}_R\mathbb{M} \,|\, N \text{ is a submodule of some } P\text{-generated module}\}, \\ \mathrm{Stat}(P) &=& \{N \in {}_R\mathbb{M} \,|\, P \otimes_S \mathrm{Hom}_R(P,N) \simeq N\}, \\ \mathrm{Adst}(P) &=& \{X \in {}_S\mathbb{M} \,|\, X \simeq \mathrm{Hom}_R(P, P \otimes_S X)\}. \end{array}$$

By restriction and corestriction we obtain the following pairs of adjoint functors (keeping the symbols for the functors) where Q denotes a cogenerator in $\sigma[{}_RP]$:

$$\begin{array}{ll} L_1 = P \otimes_S - : {}_S\mathbb{M} \to {}_R\mathbb{M}, & R_1 = \mathrm{Hom}_R(P,-) : {}_R\mathbb{M} \to {}_S\mathbb{M}, \\ L_2 = P \otimes_S - : {}_S\mathbb{M} \to \sigma[P], & R_2 = \mathrm{Hom}_R(P,-) : \sigma[P] \to {}_S\mathbb{M}, \\ L_3 = P \otimes_S - : \mathrm{Adst}(P) \to \mathrm{Stat}(P), & R_3 = \mathrm{Hom}_R(P,-) : \mathrm{Stat}(P) \to \mathrm{Adst}(P). \end{array}$$

For all these adjunctions one may ask if they are left or right quasi-Frobenius.

4.3. **Proposition.** *Let P be an (R,S)-bimodule.*

1. *L_1 is a QF functor if and only if both ${}_RP$ and P_S are finitely generated and projective and the functors $\mathrm{Hom}_R(P,R) \otimes_R -$ and $\mathrm{Hom}_S(P,S) \otimes_R -$ are similar (compare* [7, Definition 3.6]*).*
2. *If L_2 is a QF functor, then P_S is a Mittag-Leffler module and ${}_RP$ is finitely generated and self-projective.*
3. *L_3 is always a (quasi-) Frobenius functor.*

Proof. (1) Let L_1 be a QF functor. Then, by Proposition 2.6, L_1 and R_1 preserve all limits and colimits and this implies that ${}_RP$ and P_S have to be finitely generated and projective.

Putting $N = \mathrm{Hom}_S(P,S)$, we get that the functor $N \otimes_R - : {}_R\mathbb{M} \to {}_S\mathbb{M}$ is left adjoint to $P \otimes_S -$ by the isomorphisms

$$\begin{array}{rcl} \mathrm{Hom}_S(N \otimes_R X, Y) &\simeq& \mathrm{Hom}_R(X, \mathrm{Hom}_S(\mathrm{Hom}_S(P,S), Y)) \\ &\simeq& \mathrm{Hom}_R(X, \mathrm{Hom}_S(\mathrm{Hom}_S(P,S),S) \otimes_S Y) \\ &\simeq& \mathrm{Hom}_R(X, P \otimes_S Y), \end{array}$$

where the first isomorphism follows from the tensor-hom adjunction, while the others follow from the fact that P is a finitely generated and projective right S-module.

By Proposition 2.2, $\mathrm{Hom}_R(P,-)$ is a retract of $(N \otimes_R -)^{(\Lambda)}$ and $(N \otimes_R -)$ is a retract of $\mathrm{Hom}_R(P,-)^{\Lambda'}$, for some index sets Λ, Λ'. In particular there are retractions of (S,R)-modules

$$\mathrm{Hom}_R(P,R) \to \mathrm{Hom}_S(P,S)^{(\Lambda)}, \quad \varphi : \mathrm{Hom}_S(P,S) \to \mathrm{Hom}_R(P,R)^{\Lambda'}.$$

Since $\mathrm{Hom}_R(P,R)$ is finitely generated as right R-module, Λ can be chosen to be finite. This implies that $\mathrm{Hom}_R(P,R)$ is also finitely generated and projective as left S-module.

As a consequence, $\mathrm{Hom}_S(\mathrm{Hom}_S(P,S), \mathrm{Hom}_R(P,R))$ is finitely generated as a left S-module, say by $g_1, \ldots, g_k$. Then for any $\lambda \in \Lambda'$, the canonical projection $\pi_\lambda : \mathrm{Hom}_R(P,R)^{\Lambda'} \to \mathrm{Hom}_R(P,R)$ can be written as $\pi_\lambda \circ \varphi = \sum_{i=1}^k s_i^\lambda g_i$ for some $s_i^\lambda \in S$ and

$$\bigcap\nolimits_{i=1}^k \mathrm{Ke}\, g_i \subseteq \bigcap\nolimits_{\Lambda'} \mathrm{Ke}\, \pi_\lambda \circ \varphi = 0.$$

From this it follows that Λ' can also be chosen to be finite.

This shows that ${}_RP_S \otimes_S -$ is a QF functor (in our sense) if and only if it is a *quasi-Frobenius bimodule* in the sense of [7, Definition 3.6] which means that both ${}_RP$ and

P_S are finitely generated and projective, and moreover, $\mathrm{Hom}_R(P,R)$ and $\mathrm{Hom}_S(P,S)$ are similar and so are the related functors.

(2) Similar arguments as in (1) show that ${}_RP$ is finitely generated and projective in $\sigma[{}_RP]$. L_2 preserves products means that for a family $\{X_i\}_I$ of S-modules,

$$P \otimes_S \prod_I X_i \simeq \mathrm{Tr}^{\sigma[P]} \prod_I (P \otimes_S X_i) \subset \prod_I (P \otimes_S X_i),$$

where the middle term denotes the product of the $P \otimes_S X_i$ in $\sigma[{}_RP]$. This shows that the canonical map $P \otimes_S \prod_I X_i \to \prod_I (P \otimes_S X_i)$ is injective, that is, P_S is Mittag-Leffler (e.g. [33]).

Notice that if ${}_RP$ is faithful, P_S is only finitely generated provided $\sigma[{}_RP] = {}_R\mathbb{M}$.

(3) Obviously (L_3, R_3) is an equivalence of categories (e.g. [33, 2.4]) and hence L_3 is a (quasi-) Frobenius functor. □

4.4. **QF ring extensions.** A ring extension $R \to A$ provides a monad $A \otimes_R -$ on ${}_R\mathbb{M}$ and adjoint functor pairs

$$\begin{tikzcd}[column sep=huge] {}_A\mathbb{M} \arrow[r, "U_A = A\otimes_A -"'] & {}_R\mathbb{M}, \arrow[l, bend right, "A\otimes_R -"'] \arrow[l, bend left, "{}_R\mathrm{Hom}(A,-)"] \end{tikzcd}$$

where $A \otimes_R -$ is left adjoint to U_A and U_A is in turn left adjoint to ${}_R\mathrm{Hom}(A,-)$. $A \otimes_R -$ is a *QF monad* provided U_A is a QF functor (see Definition 3.6). Applying 4.3(1) to the above diagram gives that the (A,R)-bimodules A and ${}^*A = {}_R\mathrm{Hom}(A,R)$ are similar and the notion coincides with the usual QF extensions (e.g. [25], [24, §5], [7, Corollary 4.2]). In particular, ${}_RA$ and A_R are finitely generated and projective.

4.5. **Theorem.** *For any ring extension $R \to A$, the functor $A \otimes_R - : {}_R\mathbb{M} \to {}_A\mathbb{M}$ is QF if and only if the monad $A \otimes_R - : {}_R\mathbb{M} \to {}_R\mathbb{M}$ is QF.*

Proof. Applying 4.3(1) to the bimodule ${}_AA_R$ gives that the functor

$$A \otimes_R - : {}_R\mathbb{M} \to {}_A\mathbb{M}$$

is QF if and only if ${}_RA_R$ is finitely generated and projective on both sides and the (R,A)-bimodules A and $A^* = \mathrm{Hom}_R(A,R)$ are similar. By [25, Satz 2], this is equivalent to saying that the (A,R)-bimodules A and ${}^*A = {}_R\mathrm{Hom}(A,R)$ are similar. 4.4 now completes the proof. □

4.6. **QF corings.** An A-coring $(\mathcal{C}, \delta, \varepsilon)$ is called a *QF coring* if $\mathcal{C} \otimes_A - : {}_A\mathbb{M} \to {}_A\mathbb{M}$ is a QF comonad (as defined in 3.6) and is called *Frobenius coring* if $\mathcal{C} \otimes_A -$ is a Frobenius comonad.

For results about Frobenius corings we refer to [5, 27.8].

The following characterisations show that this notion coincides with the one given in [7, Definition 7.4] and generalise parts of [7, Theorem 7.5] (without a priori conditions on the A-module structure of $\mathcal{C}$).

4.7. **Theorem.** *The following are equivalent for an A-coring $\mathcal{C}$ and $\mathcal{C}^* = \mathrm{Hom}_A(\mathcal{C}, A)$.*

(a) *The functor ${}^{\mathcal{C}}U : {}^{\mathcal{C}}\mathbb{M} \to {}_A\mathbb{M}$ is QF;*

(b) *the functor $\mathcal{C} \otimes_A - : {}_A\mathbb{M} \to {}^{\mathcal{C}}\mathbb{M}$ is QF;*

(c) *$\mathcal{C}_A$ is finitely generated and projective and the functor $U_{\mathcal{C}^*} : {}_{\mathcal{C}^*}\mathbb{M} \to {}_A\mathbb{M}$ is QF;*

(d) $\mathcal{C}_A$ *is finitely generated and projective and the functor* $\mathcal{C}^* \otimes_A - : {}_A\mathbb{M} \to {}_{\mathcal{C}^*}\mathbb{M}$ *is QF;*

(e) $\mathcal{C}_A$ *is finitely generated and projective and the ring extension* $A \to \mathcal{C}^*$ *is a QF extension (in the sense of* [25]*).*

Proof. (c)$\Leftrightarrow$(d) by Theorem 4.5, while (c)$\Leftrightarrow$(e) follows by 4.4.

To show the equivalence (a)$\Leftrightarrow$(c), note first that the functor $\mathcal{C} \otimes_A - : {}_A\mathbb{M} \to {}_A\mathbb{M}$ admits as a left adjoint the functor $\mathcal{C}^* \otimes_A - : {}_A\mathbb{M} \to {}_A\mathbb{M}$ if and only if $\mathcal{C}_A$ is finitely generated and projective. Now, if the functor ${}^{\mathcal{C}}U : {}^{\mathcal{C}}\mathbb{M} \to {}_A\mathbb{M}$ is QF, then it admits a left adjoint and then the functor $\mathcal{C} \otimes_A - : {}_A\mathbb{M} \to {}_A\mathbb{M}$ does so. Applying Theorem 3.7(1) gives that (a) and (c) are equivalent.

If the functor $\mathcal{C} \otimes_A - : {}_A\mathbb{M} \to {}^{\mathcal{C}}\mathbb{M}$ is QF, the functor ${}^{\mathcal{C}}U : {}^{\mathcal{C}}\mathbb{M} \to {}_A\mathbb{M}$ preserves limits by Proposition 2.2(2). Then the functor $\mathcal{C} \otimes_A - : {}_A\mathbb{M} \to {}_A\mathbb{M}$ also preserves limits and hence $\mathcal{C}_A$ is finitely generated and projective and ${}^{\mathcal{C}}\mathbb{M} \simeq {}_{\mathcal{C}^*}\mathbb{M}$ (e.g. [5]). Thus the functor $\mathcal{C} \otimes_A - : {}_A\mathbb{M} \to {}_{\mathcal{C}^*}\mathbb{M}$ is QF. It then follows from Proposition 4.3 that ${}_{\mathcal{C}^*}\mathcal{C}_A$ (and hence also ${}_A\mathcal{C}^*{}_{\mathcal{C}^*}$) is a quasi-Frobenius bimodule. But to say that the bimodule ${}_A\mathcal{C}^*{}_{\mathcal{C}^*}$ is quasi-Frobenius is to say that the $(A, \mathcal{C}^*)$-bimodules ${}_A\mathrm{Hom}(\mathcal{C}^*, A)$ and $\mathcal{C}^* \simeq {}_{\mathcal{C}^*}\mathrm{Hom}(\mathcal{C}^*, \mathcal{C}^*)$ are similar. Since $\mathcal{C}$ is finitely generated and projective as a right A-module, $\mathcal{C}^*$ is finitely generated and projective as a left A-module. Thus ${}_A\mathrm{Hom}(\mathcal{C}^*, A) \simeq \mathcal{C}$, and hence the $(A, \mathcal{C}^*)$-bimodules $\mathcal{C}$ and $\mathcal{C}^*$ are similar, which by 4.4 just means that the ring extension $A \to \mathcal{C}^*$ is a QF extension (in the sense of [25]). Thus (b)$\Rightarrow$(e).

If the ring extension $A \to \mathcal{C}^*$ is a QF extension, then $(\mathcal{C}^*, A)$-bimodules $\mathcal{C}^*$ and ${}_A\mathrm{Hom}(\mathcal{C}^*, A)$ are similar. If, in addition, $\mathcal{C}_A$ is finitely generated and projective, then $\mathcal{C}^*$ is finitely generated and projective as a left A-module, and thus the $(A, \mathcal{C}^*)$-bimodule $\mathcal{C}^*$ is quasi-Frobenius. Then the $(\mathcal{C}^*, A)$-bimodule ${}_A\mathrm{Hom}(\mathcal{C}^*, A)$ is also quasi-Frobenius. Since $\mathcal{C}_A$ is finitely generated and projective, ${}_A\mathrm{Hom}(\mathcal{C}^*, A) \simeq \mathcal{C}$. Thus the $(\mathcal{C}^*, A)$-bimodule $\mathcal{C}$ is quasi-Frobenius. Applying now Proposition 4.3, we obtain that the functor $\mathcal{C} \otimes_A - : {}_A\mathbb{M} \to {}_{\mathcal{C}^*}\mathbb{M}$ is QF. Since $\mathcal{C}_A$ is finitely generated and projective, ${}^{\mathcal{C}}\mathbb{M} \simeq {}_{\mathcal{C}^*}\mathbb{M}$, and thus the functor $\mathcal{C} \otimes_A - : {}_A\mathbb{M} \to {}^{\mathcal{C}}\mathbb{M}$ is also QF, showing that (e) implies (b). This completes the proof of the theorem. $\square$

4.8. **Frobenius categories.** We use [17] as a reference for *exact categories*. An *exact category* is an additive category $\mathbb{A}$ endowed with a class $\mathcal{E}$ of exact pairs (i, p) of morphisms satisfying certain axioms (i is called *inflation*, p is called *deflation*). An exact category $(\mathbb{A}, \mathcal{E})$ is said to be *Frobenius* provided it has enough $\mathcal{E}$-projectives and $\mathcal{E}$-injectives and, moreover, the classes of $\mathcal{E}$-projectives and $\mathcal{E}$-injectives coincide. Frobenius categories are of interest in homological algebra because they give rise to *algebraic triangulated categories* by passing to the stable category $\underline{\mathbb{A}}$ of $\mathbb{A}$.

An additive category is said to be *weakly idempotent complete* if retracts have kernels (equivalently, coretracts have cokernels).

4.9. **Theorem.** *Let* $(\mathbb{A}, \mathcal{E}_{\mathbb{A}})$ *and* $(\mathbb{B}, \mathcal{E}_{\mathbb{B}})$ *be exact categories and* (L, F, R) *a QF triple of functors* $F : \mathbb{A} \to \mathbb{B}$ *and* $L, R : \mathbb{B} \to \mathbb{A}$ *. Suppose* $\mathbb{A}$ *to be weakly idempotent complete and the unit* $\eta^R : I_{\mathbb{A}} \to RF$ *(resp. counit* $\varepsilon^L : LF \to I_{\mathbb{A}}$*) to be a componentwise inflation (resp. deflation). Define* $\mathcal{E}_F$ *as the class of those* $\mathcal{E}_{\mathbb{A}}$*-exact pairs in* $\mathbb{A}$ *that become split short exact sequences upon applying* F*. Then the pair* $(\mathbb{A}, \mathcal{E}_F)$ *is a Frobenius category.*

Proof. Since the F-injectives and F-projectives in $\mathbb{A}$ coincide (see Proposition 2.6), by [13, Theorem 3.3] it is enough to show that the subcategories of $\mathbb{A}$ generated by all summands of the images of L and R coincide. But since $\mathbb{A}$ is assumed to be weakly idempotent complete, this follows from Proposition 2.11. □

Suppose now that (F, μ, ε) is a QF monad on an abelian category $\mathbb{A}$. Since the functor F has a right adjoint, it is additive. Using that it preserves all limits and colimits (see Proposition 3.7), it is not hard to show that the category $\mathbb{A}_F$ is also abelian.

4.10. **Theorem.** *Let (F, μ, ε) be a QF monad on an abelian category $\mathbb{A}$. Write $\mathcal{E}_F$ for the class of short exact sequences that become split short exact upon applying F. Then $(\mathbb{A}, \mathcal{E}_F)$ is a Frobenius exact category.*

Proof. Since the forgetful functor $U_F : \mathbb{A}_F \to \mathbb{A}$ is faithful, the result can be derived by combining Proposition 3.7 and Proposition 2.11 with Grime [13, Theorem 3.4]. □

For any exact functor $H : (\mathbb{A}, \mathcal{E}_{\mathbb{A}}) \to (\mathbb{B}, \mathcal{E}_{\mathbb{B}})$, we write $\mathcal{E}_H^\circ$ for the class of those $\mathcal{E}_{\mathbb{A}}$-exact pairs whose image under H is a split exact sequence in $\mathbb{B}$.

4.11. **Theorem.** *Let $(\mathbb{A}, \mathcal{E}_{\mathbb{A}})$ and $(\mathbb{B}, \mathcal{E}_{\mathbb{B}})$ be exact categories and (L, F, R) a QF triple of exact functors $F : \mathbb{A} \to \mathbb{B}$ and $L, R : \mathbb{B} \to \mathbb{A}$. Suppose that the following conditions are satisfied:*

- *$\mathbb{A}$ is weakly idempotent complete;*
- *every morphism in $\mathbb{B}$, whose image under L is a coretraction, is an inflation of $(\mathbb{A}, \mathcal{E}_{\mathbb{A}})$;*
- *every morphism in $\mathbb{B}$, whose image under R is a retraction, is an deflation of $(\mathbb{A}, \mathcal{E}_{\mathbb{A}})$.*

Then $\mathcal{E}_L^\circ = \mathcal{E}_R^\circ := \overline{\mathcal{E}}$ and the pair $(\mathbb{B}, \overline{\mathcal{E}})$ is Frobenius.

Proof. According to Beck [2, Proposition 0.2], we have only to show that $\mathcal{E}_L^\circ = \mathcal{E}_R^\circ$ and this follows easily from 2.11. □

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