Weak Frobenius monads and Frobenus bimodules

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Overview

- Frobenius algebras and monads

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- Weak monads and comonads

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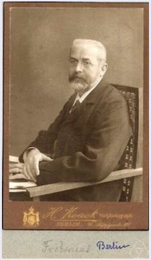
Weak Frobenius monads

Ferdinand Frobenius, *Theorie der hyperkomplexen Größen, 1903*



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A finite dimensional K-algebra
$A^* = \mathrm{Hom}_K(A, K)$ left A-module
$A \simeq A^*$ as left A-modules

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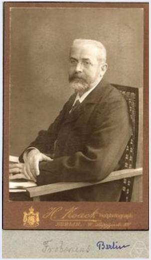

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$A^* = \mathrm{Hom}_K(A, K)$ left A-module
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$\sigma : A \times A \to K$, nondegenerate, associative $\sigma(ab, c) = \sigma(a, bc)$

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named and studied by Brauer and Nesbitt (1937)

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Frobenius monads in categories

Frobenius algebras, L. Abrams, 1999

Coalgebra structure, $\lambda : A \to A^*$, $\varepsilon := \lambda(1_A) : A \to K$

$$
\begin{tikzcd}
A \arrow[r, "\delta"] \arrow[d, "\lambda"'] & A \otimes_K A \\
A^* \arrow[r, "m^*"] & A^* \otimes_K A^* \arrow[u, "\lambda^{-1}\otimes\lambda^{-1}"']
\end{tikzcd}
,
\qquad
\begin{tikzcd}
A & A \otimes_R A \arrow[l, "A\otimes\varepsilon"'] \arrow[r, "\varepsilon\otimes A"] & A \\
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satisfies Frobenius conditions

$$
\begin{tikzcd}
A \otimes A \arrow[r, "m"] \arrow[d, "I \otimes \delta"'] & A \arrow[d, "\delta"] \\
A \otimes A \otimes A \arrow[r, "m \otimes I"] & A \otimes A
\end{tikzcd},
\qquad
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A-modules $\simeq$ A-comodules: $\mathbb{M}_A \simeq \mathbb{M}^A$

S. Eilenberg - J.C. Moore, *Adjoint functors and triples, 1965*

Adjoint endofunctors (F, G)

$F(A) \xrightarrow{\rho} A \qquad \longmapsto$

$A \xrightarrow{\eta_A} GF(A) \xrightarrow{G(\rho)} G(A)$

Equivalence $\quad \mathbb{A}_F \to \mathbb{A}^G$

General categories

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Frobenius: $F = G$, Equivalence $\quad \mathbb{A}_F \to \mathbb{A}^F$

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(1) $F : \mathbb{A} \to \mathbb{A}$ is a monad;

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Monad (F, m, η) on category $\mathbb{A}$. Equivalent (2013):

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Monad (F, m, η) on category $\mathbb{A}$. Equivalent (2013):

(a) F is a Frobenius monad;

(b) $\overline{F} = (F, \delta, \varepsilon)$ comonad with isomorphism $K : \mathbb{A}_F \to \mathbb{A}^F$ and commutative diagram

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "\phi_F"] \arrow[d, "="'] & \mathbb{A}_F \arrow[r, "U_F"] \arrow[d, "K"] & \mathbb{A} \arrow[d, "="] \\
\mathbb{A} \arrow[r, "\phi^F"'] & \mathbb{A}^F \arrow[r, "U^F"'] & \mathbb{A}\,.
\end{tikzcd}$$

Pairing of functors $L : \mathbb{A} \to \mathbb{B}$ and $R : \mathbb{B} \to \mathbb{A}$, covariant

Maps natural in $A \in \mathbb{A}$ and $B \in \mathbb{B}$

$$\alpha_{A,B} : \mathrm{Mor}_{\mathbb{B}}(L(A), B) \to \mathrm{Mor}_{\mathbb{A}}(A, R(B)), \qquad \eta : I_{\mathbb{A}} \to RL,$$

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Composing α and β on $I_L : L \to L$, $I_R : R \to R$, identities

$$\begin{aligned} \alpha(I_L) &= I_{\mathbb{A}} \xrightarrow{\eta} RL, \\ \beta \cdot \alpha(I_L) &= L \xrightarrow{L\eta} LRL \xrightarrow{\varepsilon L} L, \\ \alpha \cdot \beta \cdot \alpha(I_L) &= I_{\mathbb{A}} \xrightarrow{\eta} RL \xrightarrow{RL\eta} RLRL \xrightarrow{R\varepsilon L} RL. \end{aligned}$$

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$$\begin{aligned} \beta(I_R) &= LR \xrightarrow{\varepsilon} I_{\mathbb{B}}, \\ \alpha \circ \beta(I_R) &= R \xrightarrow{\eta R} RLR \xrightarrow{R\varepsilon} R, \\ \beta \circ \alpha \circ \beta(I_R) &= LR \xrightarrow{L\eta R} LRLR \xrightarrow{LR\varepsilon} LR \xrightarrow{\varepsilon} I_{\mathbb{B}}. \end{aligned}$$

Pairing (L, R, α, β)

Natural endomorphisms

$$\vartheta : RL \xrightarrow{RL\eta} RLRL \xrightarrow{R\varepsilon L} RL, \quad \underline{\vartheta} : RL \xrightarrow{\eta RL} RLRL \xrightarrow{R\varepsilon L} RL,$$

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Regular pairing

$$\alpha = \alpha \cdot \beta \cdot \alpha \quad (\eta = \gamma \cdot \eta), \qquad \beta = \beta \cdot \alpha \cdot \beta \quad (\varepsilon = \varepsilon \cdot \vartheta).$$

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(ii) if $\gamma = \underline{\gamma}$, then $(LR, L\eta R, \varepsilon)$ is a weak comonad;

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(iii) if $\vartheta = \underline{\vartheta}$, then $(RL, R\varepsilon L, \eta)$ is a weak monad.

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$$\widetilde{\alpha}_{A,B} : \mathrm{Mor}_{\mathbb{A}}(R(B), A) \to \mathrm{Mor}_{\mathbb{B}}(B, L(A)), \qquad \psi : I_{\mathbb{B}} \to LR,$$
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Regular pairing

$$\widetilde{\alpha} = \widetilde{\alpha} \cdot \widetilde{\beta} \cdot \widetilde{\alpha} \quad (\psi = \widetilde{\gamma} \cdot \psi), \qquad \widetilde{\beta} = \widetilde{\beta} \cdot \widetilde{\alpha} \cdot \widetilde{\beta} \quad (\varphi = \varphi \cdot \widetilde{\vartheta}).$$

$\mathbb{K}$ a class of morphisms in $\mathbb{A}$ closed under composition

$\mathbb{K}$-*equaliser* : a cofork

$$B \xrightarrow{k} C \underset{f}{\overset{g}{\rightrightarrows}} D$$

with $k \in \mathbb{K}$ and, for any $h : Q \to C$ in $\mathbb{K}$ with $f \cdot h = g \cdot h$, there is a unique $q : Q \to B$ in $\mathbb{K}$ such that $h = k \cdot q$.

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$\mathbb{K}$-*coequaliser* : a fork

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$\mathbb{K}$ is *ideal class*

if for any $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathbb{A}$, f or g in $\mathbb{K}$ implies $g \cdot f$ in $\mathbb{K}$.

Related comonads

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$\mathbb{K}$ class of morphisms in $\underrightarrow{\mathbb{B}}^{LR}$

(B, ω) is a $\mathbb{K}$-*cofirm comodule*, provided the defining cofork

$$B \xrightarrow{\omega} LR(B) \underset{LR(\omega)}{\overset{L\eta R_B}{\rightrightarrows}} LRLR(B)$$

is a $\mathbb{K}$-equaliser.

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If $\mathbb{K} = \mathrm{Mor}(\underrightarrow{\mathbb{B}}^{LR})$, a $\mathbb{K}$-cofirm comodule is just called *cofirm*.

(L, R, α, β) regular pairing with β symmetric

γ-compatible comodule morphisms, $\quad \gamma : LR \xrightarrow{L\eta R} LRLR \xrightarrow{LR\varepsilon} LR$

are morphism h between LR-comodules (B, ω) and (B', ω') with commutative diagram

$$\begin{tikzcd} B \arrow[r, "\omega"] \arrow[d, "h"'] \arrow[drr, "h"] & LR(B) \arrow[r, "\varepsilon_B"] & B \arrow[d, "h"] \\ B' \arrow[r, "\omega'"'] & LR(B') \arrow[r, "\varepsilon_{B'}"'] & B'. \end{tikzcd}$$

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(1) $\mathbb{K}_\gamma := \{$all γ-compatible morphisms in $\overline{\mathbb{B}}^{LR}\}$ is an ideal class.

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LR(Q) \arrow[r, "LR(h)"'] & LRLR(B) \arrow[r, shift left, "\varepsilon LR"] \arrow[r, shift right, "LR\varepsilon"'] & LR(B)
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It is well-known that any counital LR-comodule is cofirm.

Related monads

$\varphi : RL \to I_{\mathbb{A}}$

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(B, ϱ) is a $\mathbb{K}$-*firm module*, provided the defining fork

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$\widetilde{\vartheta}$-compatible module morphisms, $\quad \widetilde{\vartheta} : LR \xrightarrow{LR\psi} LRLR \xrightarrow{L\varphi R} LR$

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Frobenius property

$$\begin{tikzcd}
 & LRLRLR \arrow[dr, "LRL\varphi R"] & \\
LRLR \arrow[ur, "L\eta RLR"] \arrow[r, "L\varphi R"] \arrow[dr, "LRL\eta R"'] & LR \arrow[r, "L\eta R"] & LRLR \\
 & LRLRLR \arrow[ur, "L\varphi RLR"'] &
\end{tikzcd}$$

Frobenius bimodules: $\varrho : LR(B) \to B$ and $\omega : B \to LR(B)$

$$\begin{tikzcd}
LRLR(B) \arrow[r, "LR(\varrho)"] \arrow[d, "L\varphi R"'] & LR(B) \arrow[r, "LR(\omega)"] \arrow[d, "\varrho"] & LRLR(B) \arrow[d, "L\varphi R"] \\
LR(B) \arrow[r, "\varrho"] \arrow[d, "L\eta R"'] & B \arrow[r, "\omega"] \arrow[d, "\omega"] & LR(B) \arrow[d, "L\eta R"] \\
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Natural transformation

$$\theta : LR \xrightarrow{L\eta R} LRLR \xrightarrow{L\varphi R} LR$$

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$$\varrho \cdot \omega \cdot \varrho = \varrho \cdot \theta_B \quad \text{and} \quad \omega \cdot \varrho \cdot \omega = \theta_B \cdot \omega$$

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Weak separability: (η, φ) regular $\quad (\Rightarrow \theta$ idempotent$)$

(i) If $\eta \cdot \varphi \cdot \eta = \eta$, then $\varrho \cdot \omega$ is an idempotent morphisms.

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θ-compatible bimodule morphisms

morphisms h between Frobenius modules (B, ϱ, ω) and (B', ϱ', ω') with commutative diagram

$$
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B \arrow[r, "\omega"] \arrow[d, "h"'] \arrow[drr, "h"] & LR(B) \arrow[r, "\varrho"] & B \arrow[d, "h"] \\
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(3) (B, ω, ϱ) Frobenius bimodule:

$$\omega \text{ is } \theta\text{-compatible iff} \quad \omega = \omega \cdot \varrho \cdot \omega,$$

Frobenius monads: $\eta : I_{\mathbb{A}} \to RL$ and $\varphi : RL \to I_{\mathbb{A}}$

θ-compatible bimodule morphisms

morphisms h between Frobenius modules (B, ϱ, ω) and (B', ϱ', ω') with commutative diagram

$$\begin{tikzcd} B \arrow[r, "\omega"] \arrow[d, "h"'] \arrow[drr, "h"] & LR(B) \arrow[r, "\varrho"] & B \arrow[d, "h"] \\ B' \arrow[r, "\omega'"'] & LR(B') \arrow[r, "\varrho'"'] & B'. \end{tikzcd}$$

(1) If $\varphi \cdot \eta \cdot \varphi = \varphi$, then $L\varphi R$ is θ-compatible.

(2) If $\eta \cdot \varphi \cdot \eta = \eta$, then $L\eta R$ is θ-compatible.

(3) (B, ω, ϱ) Frobenius bimodule:

ω is θ-compatible iff $\quad \omega = \omega \cdot \varrho \cdot \omega$,

ϱ is θ-compatible iff $\quad \varrho = \varrho \cdot \omega \cdot \varrho$.

Firm bimodules (B, ϱ, ω)

(i) If ω is θ-compatible, then (B, ω) is $\mathbb{K}_\theta$-cofirm.

(ii) If ϱ is θ-compatible, then (B, ϱ) is $\mathbb{K}_\theta$-firm.

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If (η, φ) is a regular pair, then $L\varphi R_B$ and $L\eta R_B$ are $\mathbb{K}_\theta$-compatible bimodule morphisms and

$$\mathbb{B} \to \widetilde{\mathbb{B}}^{LR}_{LR}, \quad B \mapsto (LR(B), L\varphi R_B, L\eta R_B)$$

is a functor from $\mathbb{B}$ to Frobenius bimodules with θ-regular structure morphisms.

Frobenius monads: $\eta : I_{\mathbb{A}} \to RL$ and $\varphi : RL \to I_{\mathbb{A}}$

$\mathbb{K}$-cofirm comodules, $\mathbb{K}$ ideal class in $\mathbb{B}^{LR}$, $L\varphi R_B$ in $\mathbb{K}$

If (B, ω) in $\underrightarrow{\mathbb{B}}^{LR}$ is a $\mathbb{K}$-cofirm comodule, then there is some $\varrho : LR(B) \to B$ in $\mathbb{K}$ making (B, ϱ, ω) a Frobenius module.

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Morphisms between $\mathbb{K}$-cofirm LR-comodules (B, ω) and (B', ω') are morphisms of the Frobenius modules (B, ω, ϱ) and (B', ω', ϱ').

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$$\widetilde{\mathbb{B}}^{LR} \to \widetilde{\mathbb{B}}^{LR}_{LR} \to \underset{\rightarrow}{\mathbb{B}}_{LR}, \quad (B, \omega) \mapsto (B, \varrho, \omega) \mapsto (B, \varrho),$$

functor from $\mathbb{K}$-cofirm comodules to Frobenius bimodules with (B, ω) $\mathbb{K}$-cofirm, and to (B, ϱ), $\varrho \in \mathbb{K}$.

Frobenius monads: $\eta : I_{\mathbb{A}} \to RL$ and $\varphi : RL \to I_{\mathbb{A}}$

Proof

$$\begin{tikzcd}
LRLR(B) \arrow[r, dotted, "LR(\varrho)"] \arrow[d, "L\varphi R"'] \arrow[dr, phantom, "\mathrm{(I)}"] & LR(B) \arrow[r, "LR(\omega)"] \arrow[d, dotted, "\varrho"] \arrow[dr, phantom, "\mathrm{(II)}"] & LRLR(B) \arrow[d, "L\varphi R"] \\
LR(B) \arrow[r, dotted, "\varrho"] \arrow[d, "L\eta R"'] \arrow[dr, phantom, "\mathrm{(III)}"] & B \arrow[r, "\omega"] \arrow[d, "\omega"] \arrow[dr, phantom, "\mathrm{(IV)}"] & LR(B) \arrow[d, "L\eta R"] \\
LRLR(B) \arrow[r, dotted, "LR(\varrho)"'] & LR(B) \arrow[r, "LR(\omega)"'] & LRLR(B),
\end{tikzcd}$$

Frobenius monads: $\eta : I_{\mathbb{A}} \to RL$ and $\varphi : RL \to I_{\mathbb{A}}$

$\mathbb{K}'$-firm modules, $\mathbb{K}'$ ideal class in $\mathbb{B}_{LR}$, $L\eta R_B$ in $\mathbb{K}'$

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$$\widetilde{\mathbb{B}}_{LR} \to \widetilde{\mathbb{B}}_{LR}^{LR} \to \underrightarrow{\mathbb{B}}^{LR}, \quad (B, \varrho) \mapsto (B, \varrho, \omega) \mapsto (B, \varrho),$$

functor from $\mathbb{K}'$-firm modules to Frobenius bimodules with (B, ϱ) $\mathbb{K}'$-cofirm, and to (B, ω), $\omega \in \mathbb{K}'$.

Regular pairings and Frobenius monads

(L, R, α, β) regular pairing with β symmetric, $\quad \varphi : RL \to I_{\mathbb{A}}$

(i) $\underline{\vartheta} : RL \to RL$ and $\gamma : LR \to LR$ are idempotent;

(ii) $\eta = \underline{\vartheta} \cdot \eta$;

(iii) for $\widehat{\varphi} := \varphi \cdot \underline{\vartheta} : RL \to I_{\mathbb{A}}$, $\ \widehat{\varphi} = \widehat{\varphi} \cdot \underline{\vartheta}$.

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Commutative diagram

$$\begin{tikzcd}
LRLR \arrow[r, "\gamma LR"] \arrow[d, "L\widehat{\varphi}R"'] \arrow[dr, "L\widehat{\varphi}R"] & LRLR \arrow[d, "L\widehat{\varphi}R"] \\
LR \arrow[r, "\gamma"'] & LR
\end{tikzcd}$$

that is, $L\widehat{\varphi}R$ is in $\mathbb{K}_{\gamma}$.

So we may assume that $L\varphi R$ is γ-compatible.

Regular pairings and Frobenius monads

(L, R, α, β) regular pairing, β symmetric, $\varphi : RL \to I_{\mathbb{A}}$, $L\varphi R \in \mathbb{K}_\gamma$

For compatible LR-comodule $\omega : B \to LR(B)$, there is $\varrho : LR(B) \to B$ in $\mathbb{K}_\gamma$ making (B, ϱ, ω) a Frobenius module,

$$\varrho : LR(B) \xrightarrow{LR(\omega)} LRLR(B) \xrightarrow{L\varphi R_B} LR(B) \xrightarrow{\varepsilon_B} B.$$

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Functors, $\overline{\mathbb{B}}^{LR}_{LR}$ – Frobenius modules compatible as LR-comodules

$$\overline{\mathbb{B}}^{LR} \xrightarrow{\Psi} \overline{\mathbb{B}}^{LR}_{LR} \xrightarrow{U_{LR}} \overline{\mathbb{B}}^{LR}, \quad (B, \omega) \mapsto (B, \varrho, \omega),$$

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Regular pairings and Frobenius monads

(L, R, α, β) adjunction, $\varphi : RL \to I_{\mathbb{A}}$ (Böhm, Gómez-Torrecillas)

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(3) If $\varphi \cdot \eta = I_{\mathbb{A}}$, then, for Frobenius module (B, ϱ, ω), (B, ϱ) is a firm LR-module.

Regular pairings and Frobenius monads

$(R, L, \widetilde{\alpha}, \widetilde{\beta})$ regular pairing, $\widetilde{\alpha}$ symmetric, $\eta : I_{\mathbb{A}} \to RL$, $L\eta R \in \mathbb{K}_\theta$

(1) For compatible LR-module $\varrho : LR(B) \to B$, there is some $\omega : B \to LR(B)$ in $\mathbb{K}_{\widetilde{\vartheta}}$ making (B, ϱ, ω) a Frobenius module,

$$\omega : B \xrightarrow{\psi_B} LR(B) \xrightarrow{L\eta R_B} LRLR(B) \xrightarrow{LR(\varrho)} LR(B).$$

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Functors, $\mathbb{B}^{LR}_{LR}$ – Frobenius modules compatible as LR-modules

$$\mathbb{B}_{LR} \xrightarrow{\Phi} \mathbb{B}^{LR}_{LR} \xrightarrow{U^{LR}} \mathbb{B}_{LR}, \quad (B, \varrho) \mapsto (B, \varrho, \omega) \mapsto (B, \varrho).$$

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$$\mathbb{B}_{LR} \xrightarrow{\Phi} \mathbb{B}_{LR}^{LR} \xrightarrow{U_{LR}} \mathbb{B}^{LR}, \quad (B, \varrho) \mapsto (B, \varrho, \omega) \mapsto (B, \omega).$$

Regular pairings and Frobenius monads

Theorem

(L, R, α, β) regular pairing, β symmetric, $L\varphi R$ γ-compatible,

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$(R, L, \widetilde{\alpha}, \widetilde{\beta})$ regular pairing, $\widetilde{\alpha}$ symmetric, $L\eta R$ ϑ-compatible.

Then $\gamma = \widetilde{\vartheta}$ and the functors above yield category equivalences

$$\overline{\mathbb{B}}^{LR} \xrightarrow{\Psi} \underline{\overline{\mathbb{B}}}^{LR}_{LR} \xrightarrow{U^{LR}} \underline{\mathbb{B}}_{LR}, \qquad \underline{\mathbb{B}}_{LR} \xrightarrow{\Phi} \underline{\overline{\mathbb{B}}}^{LR}_{LR} \xrightarrow{U_{LR}} \overline{\mathbb{B}}^{LR}$$

$\underline{\overline{\mathbb{B}}}^{LR}_{LR}$ – Frobenius modules comp. as LR-modules, LR-comodules.

Weak Frobenius monads

Definition

(F, μ) non-unital monad, (F, δ) non-counital comonad,
$(B, \varrho) \in \underrightarrow{\mathbb{B}}_F$ and $(B, \omega) \in \underrightarrow{\mathbb{B}}^F$.

(F, μ, δ) satisfies the *Frobenius property* and (B, ϱ, ω) is a *Frobenius bimodule*, provided the diagrams

$$\begin{tikzcd}
 & FFF \arrow[dr, "F\mu"] & \\
FF \arrow[ur, "\delta F"] \arrow[r, "\mu"] \arrow[dr, "F\delta"'] & F \arrow[r, "\delta"] & FF \\
 & FFF \arrow[ur, "\mu F"'] &
\end{tikzcd},$$

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 & FFF \arrow[ur, "\mu F"'] &
\end{tikzcd}
\quad , \qquad
\begin{tikzcd}
 & FF(B) \arrow[dr, "F(\varrho)"] & \\
F(B) \arrow[ur, "\delta_B"] \arrow[r, "\varrho"] \arrow[dr, "F(\omega)"'] & B \arrow[r, "\omega"] & F(B) \\
 & FF(B) \arrow[ur, "\mu_B"'] &
\end{tikzcd}
\quad .
$$

commute, respectively.

Weak Frobenius monads

Definition

$(F, \mu, \eta; \delta, \varepsilon)$ is called a *weak Frobenius bimonad* provided

$$(F, \mu, \eta) \text{ is a weak monad,} \quad (F, \delta, \varepsilon) \text{ is a weak comonad,}$$

Weak Frobenius monads

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$$\begin{array}{rl} (F, \mu, \eta) \text{ is a weak monad,} & (F, \delta, \varepsilon) \text{ is a weak comonad,} \\ (F, \mu, \delta) \text{ has Frobenius property,} & \mu \cdot F\eta = F\varepsilon \cdot \delta, \text{ i.e. } \gamma = \delta. \end{array}$$

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Theorem

There are category equivalences

$$\overline{\mathbb{B}}^F \xrightarrow{\Psi} \overline{\underline{\mathbb{B}}}^F_F \xrightarrow{U^F} \underline{\mathbb{B}}_F \,, \qquad \underline{\mathbb{B}}_F \xrightarrow{\Phi} \overline{\underline{\mathbb{B}}}^F_F \xrightarrow{U_F} \overline{\mathbb{B}}^F \,.$$

$\overline{\mathbb{B}}^F$ – compatible F-comodules, $\underline{\mathbb{B}}_F$ – compatible F-modules,
$\overline{\underline{\mathbb{B}}}^F_F$ – Frob-bimodules compatible as modules and comodules.

Separable monads

Separable monad on category $\mathbb{A}$

(1) $F : \mathbb{A} \to \mathbb{A}$ is a monad;

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Monad (F, m, η) on category $\mathbb{A}$. Equivalent:

(a) F is a separable monad;

(b) $m : FF \to F$ has a natural section $\delta : F \to FF$ with commutative diagrams

$$\begin{tikzcd} FF \arrow[r, "\delta F"] \arrow[d, "m"'] & FFF \arrow[d, "Fm"] \\ F \arrow[r, "\delta"] & FF \end{tikzcd} \qquad \begin{tikzcd} FF \arrow[r, "F\delta"] \arrow[d, "m"'] & FFF \arrow[d, "mF"] \\ F \arrow[r, "\delta"] & FF \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "\delta"] \arrow[dr, "="'] & FF \arrow[d, "m"] \\ & F. \end{tikzcd}$$

Azumaya monads

Azumaya monad $(F, m, e; \lambda)$ on category $\mathbb{A}$

distributive law $\lambda : FF \to FF$ satisfying *Yang-Baxter equation*

$$\begin{tikzcd} FFF \arrow[r, "F\lambda"] \arrow[d, "\lambda F"'] & FFF \arrow[r, "\lambda F"] & FFF \arrow[d, "F\lambda"] \\ FFF \arrow[r, "F\lambda"] & FFF \arrow[r, "\lambda F"] & FFF. \end{tikzcd}$$

(1) $\mathcal{F}^\lambda = (F, m \cdot \lambda, e)$ is a monad on $\mathbb{A}$;

(2) $\mathcal{F}\mathcal{F}^\lambda = (FF, mm \cdot FF\lambda \cdot F\lambda F, ee)$ is a monad:

(3) comparison functor $K : \mathbb{A} \to \mathbb{A}_{\mathcal{F}\mathcal{F}^\lambda}$ sending $A \in \mathbb{A}$ to

$$(F(A),\ FFF(A) \xrightarrow{F(\lambda_A)} FFF(A) \xrightarrow{F(m_A)} FF(A) \xrightarrow{m_A} F(A)).$$

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Azumaya monad – if K is an equivalence of categories.

Bimonads

Bimonad on category $\mathbb{A}$

$F : \mathbb{A} \to \mathbb{A}$ a monad (F, m, e) and a comonad (F, δ, ε), double entwining $\tau : FF \to FF$, $\varepsilon \cdot e = 1$, commutative diagrams

$$\begin{tikzcd} FF \arrow[r, "m"] \arrow[d, "\delta\delta"'] & F \arrow[r, "\delta"] & FF \\ FFFF \arrow[rr, "F\tau F"] & & FFFF, \arrow[u, "mm"'] \end{tikzcd} \quad \begin{tikzcd} FF \arrow[r, "F\varepsilon"] \arrow[d, "m"'] & F \arrow[d, "\varepsilon"] \\ F \arrow[r, "\varepsilon"] & 1, \end{tikzcd} \quad \begin{tikzcd} 1 \arrow[r, "e"] \arrow[d, "e"'] & F \arrow[d, "\delta"] \\ F \arrow[r, "eF"] & FF. \end{tikzcd}$$

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$F : \mathbb{A} \to \mathbb{A}$ a monad (F, m, e) and a comonad (F, δ, ε),
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$$
\begin{tikzcd}
FF \arrow[r, "m"] \arrow[d, "\delta\delta"'] & F \arrow[r, "\delta"] & FF \\
FFFF \arrow[rr, "F\tau F"] & & FFFF, \arrow[u, "mm"']
\end{tikzcd}
\quad
\begin{tikzcd}
FF \arrow[r, "F\varepsilon"] \arrow[d, "m"'] & F \arrow[d, "\varepsilon"] \\
F \arrow[r, "\varepsilon"] & 1,
\end{tikzcd}
\quad
\begin{tikzcd}
1 \arrow[r, "e"] \arrow[d, "e"'] & F \arrow[d, "\delta"] \\
F \arrow[r, "eF"] & FF.
\end{tikzcd}
$$

Hopf monad

$$FF \xrightarrow{F\delta} FFF \xrightarrow{mF} FF \quad \text{is an isomorphism.}$$

Bimonads

Bimonad on category $\mathbb{A}$

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\end{tikzcd}
\qquad
\begin{tikzcd}
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F \arrow[r, "\varepsilon"] & 1,
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "e"] \arrow[d, "e"'] & F \arrow[d, "\delta"] \\
F \arrow[r, "eF"] & FF.
\end{tikzcd}$$

Hopf monad

$$FF \xrightarrow{F\delta} FFF \xrightarrow{mF} FF$$ is an isomorphism.

$$\overline{K} : \mathbb{A} \to \mathbb{A}_F^F(\tau),\ A \mapsto (F(A), \delta_A, m_A)$$ is an equivalence.

Various algebras

Algebras and coalgebras, A R-module

$$A \otimes_R A \xrightarrow{m} A,\ R \xrightarrow{\eta} A; \quad A \xrightarrow{\delta} A \otimes_R A,\ A \xrightarrow{\varepsilon} R$$

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Frobenius algebra $(A, m, \eta; \delta, \varepsilon)$

m is right A-module and right A-comodule morphism

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Separable algebra $(A, m, \eta; \delta)$

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Azumaya algebra $(A, m, \eta; \tau)$

equivalence $\mathbb{M}_R \to {}_A\mathbb{M}_A$ (separable and central)

Various algebras

Algebras and coalgebras, A R-module

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m is right A-module and right A-comodule morphism

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Azumaya algebra $(A, m, \eta; \tau)$

equivalence $\mathbb{M}_R \to {}_A\mathbb{M}_A$ (separable and central)

Hopf algebra $(A, m, \eta, \delta, \varepsilon)$

Bialgebra: m is a coalgebra morphism (δ is an algebra morphism)
Hopf algebra: $(m \otimes I) \cdot (I \otimes \delta) = I$, equivalence $\mathbb{M}_R \to \mathbb{M}_A^A$

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