# On coseparable and biseparable corings

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ABSTRACT. A relationship between coseparable corings and separable non-unital rings is established. In particular it is shown that a coseparable $A$-coring $\mathcal{C}$ has an associative $A$-balanced product. A Morita context is constructed for a coseparable coring with a grouplike element. Biseparable corings are defined, and a conjecture relating them to Frobenius corings is proposed.

## 1. Introduction

Corings were introduced by Sweedler in [22] as a generalisation of coalgebras and a means for dualising the Jacobson-Bourbaki theorem. Recently, corings have resurfaced in the theory of Hopf-type modules, in particular it has been shown in [5] that the category of entwined modules is an example of a category of comodules of a coring. Since entwined modules appear to be the most general of Hopf-type modules studied since the mid-seventies, the theory of corings provides one with a uniform and general approach to studying all such modules. This simple observation renewed interest in general theory of corings.

Corings appear naturally in the theory of ring extensions. Indeed, they provide an equivalent description of certain types of extensions (cf. [6]). In this paper we study properties of corings associated to extensions. In particular, we study *coseparable corings* introduced by Guzman [13] (and recently studied in [12] from a different point of view) and we reveal an intriguing duality between such corings and a non-unital generalisation of separable ring extensions. We also show that to any grouplike element in a coseparable coring one can associate a Morita context. This leads to a pair of adjoint functors. One of these functors turns out to be fully faithful. Furthermore, we introduce the notion of a *biseparable coring* and study

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its relationship to Frobenius corings introduced in [6]. This allows us to consider a conjecture from [9], concerning biseparable and Frobenius extensions in a new framework.

Our paper is organised as follows. In the next section, apart from recalling some basic facts about corings and comodules, we introduce a non-unital generalisation of separable extensions, which we term *separable A-rings*. We show that any coseparable coring is an example of such a separable $A$-ring, and conversely, that every separable $A$-ring leads to a non-unital coring. We then proceed in Section 3 to construct a Morita context associated to a grouplike element in a coseparable coring. We consider some examples coming from ring extensions and bialgebroids. Finally in Section 4 we introduce the notion of *biseparable corings*. These are closely related to biseparable extensions, and may serve as a means for settling the question put forward in [9] of whether biseparable extensions are Frobenius.

Throughout the paper, $A$ denotes an associative ring with unit $1_A$, and we use the standard notation for right (resp. left) $A$-modules $\mathbf{M}_A$ (resp. $_A\mathbf{M}$), bimodules, such as $\mathrm{Hom}_A(-,-)$ for right $A$-module maps, $_A\mathrm{Hom}(-,-)$ for left $A$-module maps etc. For any $(A,A)$-bimodule $M$ the centraliser of $A$ in $M$ is denoted by $M^A$, i.e., $M^A := \{m \in M \mid \forall a \in A, \ am = ma\}$.

## 2. Coseparable $A$-corings and separable $A$-rings

### 2.1. Coseparable corings

We begin by recalling the definition of a coring from [22]. An $(A,A)$-bimodule $\mathcal{C}$ is said to be a *non-counital A-coring* if there exists an $(A,A)$-bimodule map $\Delta_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}$ rendering the following diagram commutative

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ \Delta_{\mathcal{C}}\ } & \mathcal{C}\otimes_A\mathcal{C} \\
{\scriptstyle \Delta_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle I_{\mathcal{C}}\otimes\Delta_{\mathcal{C}}} \\
\mathcal{C}\otimes_A\mathcal{C} & \xrightarrow{\ \Delta_{\mathcal{C}}\otimes I_{\mathcal{C}}\ } & \mathcal{C}\otimes_A\mathcal{C}\otimes_A\mathcal{C}\,.
\end{array}
$$

The map $\Delta_{\mathcal{C}}$ is termed a *coproduct*. Given a non-counital $A$-coring $\mathcal{C}$ with a coproduct $\Delta_{\mathcal{C}}$, an $(A,A)$-bimodule map $\epsilon_{\mathcal{C}} : \mathcal{C} \to A$ such that

$$(\epsilon_{\mathcal{C}}\otimes I_{\mathcal{C}}) \circ \Delta_{\mathcal{C}} = I_{A\otimes_A\mathcal{C}} = I_{\mathcal{C}}, \quad (I_{\mathcal{C}}\otimes\epsilon_{\mathcal{C}}) \circ \Delta_{\mathcal{C}} = I_{\mathcal{C}\otimes_A A} = I_{\mathcal{C}}.$$

is called a *counit* of $\mathcal{C}$. A non-counital $A$-coring with a counit is called an $A$-coring. If $\mathcal{C}$ is an (non-counital) $A$-coring, a right $A$-module $M$ is called a *non-counital right $\mathcal{C}$-comodule* if there exists a right $A$-module map $\varrho^M : M \to M \otimes_A \mathcal{C}$ rendering the following diagram commutative

$$
\begin{array}{ccc}
M & \xrightarrow{\ \varrho^M\ } & M\otimes_A\mathcal{C} \\
{\scriptstyle \varrho^M}\downarrow & & \downarrow{\scriptstyle \varrho^M\otimes I_{\mathcal{C}}} \\
M\otimes_A\mathcal{C} & \xrightarrow{\ I_M\otimes\Delta_{\mathcal{C}}\ } & M\otimes_A\mathcal{C}\otimes_A\mathcal{C}\,.
\end{array}
$$

The map $\varrho^M$ is called a $\mathcal{C}$*-coaction*. If, in addition, a $\mathcal{C}$-coaction satisfies the condition

$$(I_M\otimes\epsilon_{\mathcal{C}}) \circ \varrho^M = I_{M\otimes_A A} = I_M,$$

then $M$ is called a *right $\mathcal{C}$-comodule*. Similarly one defines left $\mathcal{C}$-comodules, and $(\mathcal{C}, \mathcal{C})$-bicomodules. Given right $\mathcal{C}$-comodules $M$, $N$, a right $A$-linear map $f : M \to N$ is called a *morphism of right $\mathcal{C}$-comodules* provided the following diagram

$$
\begin{array}{ccc}
M & \xrightarrow{\;\;f\;\;} & N \\
{\scriptstyle \varrho^M}\downarrow & & \downarrow{\scriptstyle \varrho^N} \\
M \otimes_A \mathcal{C} & \xrightarrow{\;f \otimes I_{\mathcal{C}}\;} & N \otimes_A \mathcal{C}
\end{array}
$$

is commutative. The category of right $\mathcal{C}$-comodules is denoted by $\mathbf{M}^{\mathcal{C}}$. We use Sweedler notation to denote the action of a coproduct or a coaction on elements,

$$
\Delta_{\mathcal{C}}(c) = \sum c_{(1)} \otimes c_{(2)}, \qquad \varrho^M(m) = \sum m_{(0)} \otimes m_{(1)}.
$$

An immediate example of a left and right $\mathcal{C}$-comodule is provided by $\mathcal{C}$ itself. In both cases coaction is given by the coproduct $\Delta_{\mathcal{C}}$. Also, for any right (resp. left) $A$-module $M$, the tensor product $M \otimes_A \mathcal{C}$ (resp. $\mathcal{C} \otimes_A M$) is a right (resp. left) $\mathcal{C}$-comodule with the coaction $I_M \otimes \Delta_{\mathcal{C}}$ (resp. $\Delta_{\mathcal{C}} \otimes I_M$). This defines a functor which is the right adjoint of a forgetful functor from the category of $\mathcal{C}$-comodules to the category of $A$-modules. This functor can be defined for non-counital corings and non-counital comodules, and adjointness holds for corings with a counit.

In particular $\mathcal{C} \otimes_A \mathcal{C}$ is a $(\mathcal{C}, \mathcal{C})$-bicomodule, and $\Delta_{\mathcal{C}}$ is a $(\mathcal{C}, \mathcal{C})$-bicomodule map, and following [13] we have

DEFINITION 2.1. A (non-counital) coring $\mathcal{C}$ is said to be *coseparable* if there exists a $(\mathcal{C}, \mathcal{C})$-bicomodule splitting of the coproduct $\Delta_{\mathcal{C}}$.

Although Definition 2.1 makes sense for non-counital corings, it is much more meaningful in the case of corings with a counit. In this case $(\mathcal{C}, \mathcal{C})$-bicomodule splittings of $\Delta_{\mathcal{C}}$, $\pi : \mathcal{C} \otimes_A \mathcal{C} \to \mathcal{C}$ are in bijective correspondence with $(A, A)$-bimodule maps $\gamma : \mathcal{C} \otimes_A \mathcal{C} \to A$ such that for all $c, c' \in \mathcal{C}$,

$$
\sum \gamma(c \otimes c'_{(1)}) c'_{(2)} = \sum c_{(1)} \gamma(c_{(2)} \otimes c'), \qquad \sum \gamma(c_{(1)} \otimes c_{(2)}) = \epsilon_{\mathcal{C}}(c).
$$

Such a map $\gamma$ is termed a *cointegral* in $\mathcal{C}$, and the first of the above equations is said to express a *colinearity* of a cointegral. The correspondence is given by $\gamma = \epsilon_{\mathcal{C}} \circ \pi$ and $\pi(c \otimes c') = \sum c_{(1)} \gamma(c_{(2)} \otimes c')$. Furthermore, $\mathcal{C}$ is a coseparable $A$-coring if and only if the forgetful functor $\mathbf{M}^{\mathcal{C}} \to \mathbf{M}_A$ is separable (cf. [5, Theorem 3.5]).

Corings appear naturally in the context of ring extensions. A ring extension $B \to A$ determines the *canonical* Sweedler $A$-coring $\mathcal{C} := A \otimes_B A$ with coproduct $\Delta_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}$ given by $\Delta_{\mathcal{C}}(a \otimes a') = a \otimes 1_A \otimes a'$ and counit $\epsilon_{\mathcal{C}} : \mathcal{C} \to A$ given by $\epsilon_{\mathcal{C}}(a \otimes a') = aa'$ for all $a, a' \in A$. Recall from [19] that an extension $B \to A$ is said to be *split* if there exists a $(B, B)$-bimodule map $E : A \to B$ such that $E(1_A) = 1_B$. The map $E$ is known as a *conditional expectation*. The canonical Sweedler coring associated to a split ring extension is coseparable. A cointegral $\gamma$ coincides with the splitting map $E$ via the natural isomorphisms

$$
{}_B\mathrm{Hom}_B(A, B) \subset {}_B\mathrm{Hom}_B(A, A) \cong {}_A\mathrm{Hom}_A(A \otimes_B A \otimes_A A \otimes_B A, A)
$$

(cf. [5, Corollary 3.7]).

## 2.2. Separable $A$-rings

Corings can be seen as a dualisation of $A$-*rings* and coseparable corings turn out to be closely related to a generalisation of separable extensions of rings. In this subsection we describe this generalisation.

DEFINITION 2.2. An $(A, A)$-bimodule $B$ is called an $A$-*ring* provided there exists an $(A, A)$-bimodule map $\mu : B \otimes_A B \to B$ rendering commutative the diagram

$$
\begin{array}{ccc}
B\otimes_A B\otimes_A B & \xrightarrow{I_B \otimes \mu} & B\otimes_A B \\
{\scriptstyle \mu \otimes I_B}\downarrow & & \downarrow{\scriptstyle \mu} \\
B\otimes_A B & \xrightarrow{\quad \mu \quad} & B\,.
\end{array}
$$

This means that $\mu$ is associative. Note that an $A$-ring is necessarily a (non-unital) ring in the usual sense. Equivalently, an $A$-ring can be defined as a ring and an $(A, A)$-bimodule $B$ with product that is an $A$-balanced $(A, A)$-bimodule map.

Note further that the notion of an $A$-ring in Definition 2.2 is a non-unital generalisation of ring extensions. Indeed, it is only natural to call an $(A, A)$-bilinear map $\iota : A \to B$ a *unit* (for $(B, \mu)$) if it induces a commutative diagram

$$
\begin{array}{ccc}
B & \xrightarrow{I_B \otimes \iota} & B\otimes_A B \\
{\scriptstyle \iota \otimes I_B}\downarrow & \overset{=}{\searrow} & \downarrow{\scriptstyle \mu} \\
B\otimes_A B & \xrightarrow{\quad \mu \quad} & B\,.
\end{array}
$$

If this holds then $\iota(1_A) = 1_B$ is a unit of $B$ in the usual sense. One can then easily show that $\iota$ is a ring map, hence a unital $A$-ring is simply a ring extension.

DEFINITION 2.3. Given an $A$-ring $B$, a right $A$-module $M$ is said to be a *right $B$-module* provided there exists a right $A$-module map $\varrho_M : M\otimes_A B \to M$ making the following diagram

$$
\begin{array}{ccc}
M\otimes_A B\otimes_A B & \xrightarrow{\varrho_M \otimes I_B} & M\otimes_A B \\
{\scriptstyle I_M \otimes \mu}\downarrow & & \downarrow{\scriptstyle \varrho_M} \\
M\otimes_A B & \xrightarrow{\quad \varrho_M \quad} & M
\end{array}
$$

commute. The map $\varrho_M$ is called a *right $B$-action*. On elements the action is denoted by a dot in a standard way, i.e., $m \cdot b = \varrho_M(m\otimes b)$. Remember that for all $a \in A$, $(ma) \cdot b = m \cdot (ab)$.

A morphism $f : M \to N$ between two $B$-modules is an $A$-linear map which makes the following diagram

$$
\begin{array}{ccc}
M\otimes_A B & \xrightarrow{f \otimes I_B} & N\otimes_A B \\
{\scriptstyle \varrho_M}\downarrow & & \downarrow{\scriptstyle \varrho_N} \\
M & \xrightarrow{\quad f \quad} & N
\end{array}
$$

commute. A right $B$-module $M$ is said to be *firm* provided the induced map $M \otimes_B B \to M$, $m \otimes b \mapsto m \cdot b$ is a right $B$-module isomorphism. The category of firm right $B$-modules is denoted by $\mathbf{M}_B$.

Obviously left $B$-modules are defined in a symmetric way. Similarly one defines $(B, B)$-bimodules, $(A, B)$-bimodules etc.

Dually to the definition of coseparable corings we can define separable $A$-rings.

DEFINITION 2.4. An $A$-ring $B$ is said to be *separable* if the product map $\mu : B \otimes_A B \to B$ has a $(B, B)$-bimodule section $\delta : B \to B \otimes_A B$.

If $B$ is a separable $A$-ring then clearly $\mu$ is surjective and the induced map $B \otimes_B B \to B$ is an isomorphism. Therefore $B$ is a firm left and right $B$-module, i.e., $B$ is a firm ring.

Note that if $B$ has a unit $\iota : A \to B$ then $B$ is a separable $A$-ring if and only if $B$ is a separable extension of $A$. Thus Definition 2.4 extends the notion of a separable extension to non-unital rings. Note, however, that in general this is not an extension, since there is no (ring) map $A \to B$.

*Remark* 2.5. In consistency with $A$-corings, we use the terminology of [3] in Definition 2.2. In [19, 11.7] $A$-rings are termed *multiplicative $A$-bimodules*. Following [23] one might call a separable $A$-ring (as defined in Definition 2.4) an *$A$-ring with a splitting map*.

## 2.3. Coseparable $A$-corings are separable $A$-rings

The main result of this section is contained in the following

THEOREM 2.6. *If $\mathcal{C}$ is a coseparable $A$-coring then $\mathcal{C}$ is a separable $A$-ring.*

*Proof.* Let $\pi : \mathcal{C} \otimes_A \mathcal{C} \to \mathcal{C}$ be a bicomodule retraction of the coproduct $\Delta_{\mathcal{C}}$, and let $\gamma = \epsilon_{\mathcal{C}} \circ \pi$ be the corresponding cointegral. We claim that $\mathcal{C}$ is an associative $A$-ring with product $\mu = \pi$. Indeed, since the alternative expressions for product are $cc' = \sum \gamma(c \otimes c'_{(1)})c'_{(2)} = \sum c_{(1)}\gamma(c_{(2)} \otimes c')$, for all $c, c', c'' \in \mathcal{C}$ we obtain, using the left $A$-linearity of $\gamma$ and $\Delta_{\mathcal{C}}$,

$$(cc')c'' = \sum (\gamma(c \otimes c'_{(1)})c'_{(2)})c'' = \sum \gamma(c \otimes c'_{(1)})\gamma(c'_{(2)} \otimes c''_{(1)})c''_{(2)}.$$

On the other hand, the colinearity and right $A$-linearity of $\gamma$, and the left $A$-linearity of $\Delta_{\mathcal{C}}$ imply

$$\begin{aligned} c(c'c'') &= \sum c(\gamma(c' \otimes c''_{(1)})c''_{(2)}) = \sum \gamma(c \otimes \gamma(c' \otimes c''_{(1)})c''_{(2)})c''_{(3)} \\ &= \sum \gamma(c \otimes c'_{(1)}\gamma(c'_{(2)} \otimes c''_{(1)}))c''_{(2)} = \sum \gamma(c \otimes c'_{(1)})\gamma(c'_{(2)} \otimes c''_{(1)})c''_{(2)}. \end{aligned}$$

This explicitly proves that the product in $\mathcal{C}$ is associative. Clearly this product is $(A, A)$-bilinear. Note that $\Delta_{\mathcal{C}}$ is a $(\mathcal{C}, \mathcal{C})$-bimodule map since

$$c\Delta_{\mathcal{C}}(c') = \sum cc'_{(1)} \otimes c'_{(2)} = \sum \pi(c \otimes c'_{(1)}) \otimes c'_{(2)} = \Delta_{\mathcal{C}} \circ \pi(c \otimes c') = \Delta_{\mathcal{C}}(cc'),$$

from the right colinearity of $\pi$. Similarly for the left $\mathcal{C}$-linearity. Finally $\pi$ is split by $\Delta_{\mathcal{C}}$ since $\pi$ is a retraction of $\Delta_{\mathcal{C}}$. This proves that $\mathcal{C}$ is a separable $A$-ring. $\square$

PROPOSITION 2.7. *Let $\mathcal{C}$ be a coseparable $A$-coring with a cointegral $\gamma$. View $\mathcal{C}$ as an $A$-ring with product $\pi$ as in Theorem 2.6. Then any right $\mathcal{C}$-comodule $M$ is a firm right $\mathcal{C}$-module with the action $\varrho_M = (I_M \otimes \gamma) \circ (\varrho^M \otimes I_{\mathcal{C}})$.*

*Proof.* Take any $m \in M$ and $c, c' \in \mathcal{C}$. Then explicitly the action reads $m \cdot c = \sum m_{(0)} \gamma(m_{(1)} \otimes c)$, and we can compute

$$
\begin{aligned}
(m \cdot c) \cdot c' &= \sum (m_{(0)} \gamma(m_{(1)} \otimes c)) \cdot c' = \sum m_{(0)} \gamma(m_{(1)} \gamma(m_{(2)} \otimes c) \otimes c') \\
&= \sum m_{(0)} \gamma(\gamma(m_{(1)} \otimes c_{(1)}) c_{(2)} \otimes c') = \sum m_{(0)} \gamma(m_{(1)} \otimes c_{(1)}) \gamma(c_{(2)} \otimes c') \\
&= \sum m_{(0)} \gamma(m_{(1)} \otimes c_{(1)} \gamma(c_{(2)} \otimes c')) = \sum m_{(0)} \gamma(m_{(1)} \otimes cc') = m \cdot (cc'),
\end{aligned}
$$

as required. We used the following properties of a cointegral: colinearity to derive the third equality and $A$-bilinearity to derive the fourth and fifth equalities. Obviously the action is right $A$-linear. Thus $M$ is a $\mathcal{C}$-module. We need to show that it is firm.

Note that $M \otimes_{\mathcal{C}} \mathcal{C}$ is defined as a cokernel of the following right $\mathcal{C}$-linear map

$$
\lambda : M \otimes_A \mathcal{C} \otimes_A \mathcal{C} \to M \otimes_A \mathcal{C}, \quad m \otimes c \otimes c' \mapsto mc \otimes c' - m \otimes cc'.
$$

Since $\gamma$ is a cointegral, $\varrho_M$ is a right $\mathcal{C}$-linear retraction of $\varrho^M$, hence, in particular it is a surjection and there is the following sequence of right $\mathcal{C}$-module maps

$$
M \otimes_A \mathcal{C} \otimes_A \mathcal{C} \xrightarrow{\;\lambda\;} M \otimes_A \mathcal{C} \xrightarrow{\;\varrho_M\;} M \longrightarrow 0.
$$

We need to show that this sequence is exact. Clearly the associativity of the action of $\mathcal{C}$ on $M$ implies that $\varrho_M \circ \lambda = 0$, so that $\mathrm{Im}\,\lambda \subseteq \ker \varrho_M$. Furthermore, for all $m \in M$ and $c \in \mathcal{C}$,

$$
\begin{aligned}
(\varrho^M \circ \varrho_M - \lambda \circ (I_M \otimes \Delta_{\mathcal{C}}))(m \otimes c) &= \sum m_{(0)} \otimes m_{(1)} \gamma(m_{(2)} \otimes c) \\
&\quad - \sum m \cdot c_{(1)} \otimes c_{(2)} + \sum m \otimes \pi(c_{(1)} \otimes c_{(2)}) \\
&= \sum m_{(0)} \gamma(m_{(1)} \otimes c_{(1)}) \otimes c_{(2)} \\
&\quad - \sum m_{(0)} \gamma(m_{(1)} \otimes c_{(1)}) \otimes c_{(2)} + m \otimes c \\
&= m \otimes c,
\end{aligned}
$$

where we used the colinearity of a cointegral. This implies that $\ker \varrho_M \subseteq \mathrm{Im}\,\lambda$, i.e., the above sequence is exact as required. $\square$

As an example of a coseparable coring one can take the canonical Sweedler coring associated to a split ring extension $B \to A$. In this case the product in $A \otimes_B A$ comes out as

$$
(a \otimes a')(a'' \otimes a''') = a E(a' a'') \otimes a''', \qquad \forall a, a', a'', a''' \in A
$$

where $E$ is a splitting map. This is known as the *E-multiplication*. Since a comodule of the canonical coring is a *descent datum* for a ring extension $B \to A$, Proposition 2.7 implies that every descent datum is a firm module of the $A$-ring $A \otimes_B A$ with the $E$-multiplication (cf. [5, Example 2.1], [8, Section 25] or [10, Section 4.8] for a discussion of the correspondence between descent data and comodules of a canonical coring).

Theorem 2.6 has the following (part-) converse.

PROPOSITION 2.8. *Let $B$ be a separable $A$-ring. Then $B$ is a coseparable non-counital coring.*

*Proof.* Let $\Delta : B \to B \otimes_A B$ be a $(B, B)$-bimodule map splitting the product $\mu$ in $B$. The $B$-linearity of $\Delta$ implies that the following diagram

$$
\begin{array}{ccccc}
B \otimes_A B \otimes_A B & \xleftarrow{\;I_B \otimes \Delta\;} & B \otimes_A B & \xrightarrow{\;\Delta \otimes I_B\;} & B \otimes_A B \otimes_A B \\
{\scriptstyle \mu \otimes I_B} \downarrow & & {\scriptstyle \mu} \downarrow & & \downarrow {\scriptstyle I_B \otimes \mu} \\
B \otimes_A B & \xleftarrow{\quad \Delta \quad} & B & \xrightarrow{\quad \Delta \quad} & B \otimes_A B
\end{array}
$$

is commutative. For all $b \in B$ we write $(\Delta \otimes I_B) \circ \Delta(b) = \sum b_{(1)(1)} \otimes b_{(1)(2)} \otimes b_{(2)}$ and $(I_B \otimes \Delta) \circ \Delta(b) = \sum b_{(1)} \otimes b_{(2)(1)} \otimes b_{(2)(2)}$, and use the above diagram to obtain

$$
\begin{aligned}
\Delta(b) = (\Delta \circ \mu \circ \Delta)(b) \quad &= (I_B \otimes \mu) \circ (\Delta \otimes I_B) \circ \Delta(b) = \sum b_{(1)(1)} \otimes \mu(b_{(1)(2)} \otimes b_{(2)}) \\
&= (\mu \otimes I_B) \circ (I_B \otimes \Delta) \circ \Delta(b) = \sum \mu(b_{(1)} \otimes b_{(2)(1)}) \otimes b_{(2)(2)}.
\end{aligned}
$$

Using these identities we can compute

$$
\begin{aligned}
(I_B \otimes \Delta) \circ \Delta(b) \quad &= \sum b_{(1)(1)} \otimes (\Delta \circ \mu)(b_{(1)(2)} \otimes b_{(2)}) \\
&= \sum b_{(1)(1)} \otimes ((\mu \otimes I_B) \circ (I_B \otimes \Delta))(b_{(1)(2)} \otimes b_{(2)}) \\
&= \sum b_{(1)(1)} \otimes \mu(b_{(1)(2)} \otimes b_{(2)(1)}) \otimes b_{(2)(2)},
\end{aligned}
$$

and

$$
\begin{aligned}
(\Delta \otimes I_B) \circ \Delta(b) \quad &= \sum (\Delta \circ \mu)(b_{(1)} \otimes b_{(2)(1)}) \otimes b_{(2)(2)} \\
&= \sum ((I_B \otimes \mu) \circ (\Delta \otimes I_B))(b_{(1)} \otimes b_{(2)(1)}) \otimes b_{(2)(2)} \\
&= \sum b_{(1)(1)} \otimes \mu(b_{(1)(2)} \otimes b_{(2)(1)}) \otimes b_{(2)(2)},
\end{aligned}
$$

i.e., $(\Delta \otimes I_B) \circ \Delta = (I_B \otimes \Delta) \circ \Delta$. This proves that $B$ is a non-counital $A$-coring with coproduct $\Delta$.

Next note that the above diagram can also be understood as a statement that $\mu$ is a $(B, B)$-bicomodule map. Since $\mu$ is a retraction for $\Delta$, $B$ is a coseparable non-counital coring as required. This completes the proof. $\square$

## 3. Morita contexts for coseparable corings

Although the Morita theory is usually developed for rings with unit, it can be extended to firm rings without units (cf. [4, Exercise 4.1.4]). Recall that a right module $M$ of a non-unital ring $R$ is said to be *firm* if the map $M \otimes_R R \to M$ induced from the $R$-product in $M$ is an $R$-module isomorphism. Similarly one defines left firm modules. A non-unital ring is a firm ring if it is firm as a left and right $R$-module.

DEFINITION 3.1. Given a pair of firm non-unital rings $R, S$, a Morita context consists of a firm $(R, S)$-bimodule $V$ and a firm $(S, R)$-bimodule $W$ and a pair of bimodule maps

$$
\sigma : W \otimes_R V \to S, \qquad \tau : V \otimes_S W \to R,
$$

such that the following diagrams

$$W\otimes_R V\otimes_S W \xrightarrow{\ \sigma\otimes I_W\ } S\otimes_S W$$

with vertical maps $I_W\otimes\tau$ and $\cong$,

$$W\otimes_R R \xrightarrow{\ \cong\ } W$$

$$V\otimes_S W\otimes_R V \xrightarrow{\ \tau\otimes I_V\ } R\otimes_R V$$

with vertical maps $I_V\otimes\sigma$ and $\cong$,

$$V\otimes_S S \xrightarrow{\ \cong\ } V$$

commute. A Morita context is denoted by $(R,S,V,W,\tau,\sigma)$. A Morita context is said to be *strict* provided $\sigma$ and $\tau$ are isomorphisms.

The Morita theory for non-unital rings can be developed along the same lines as the usual Morita theory [18]. The aim of this section is to show that there is a Morita context associated to any coseparable coring with a grouplike element. To construct such a context we employ techniques developed in recent papers [1], [11]. First recall that an element $g$ of an $A$-coring $\mathcal{C}$ is said to be a *grouplike element*, provided $\Delta_{\mathcal{C}}(g) = g\otimes g$ and $\epsilon_{\mathcal{C}}(g) = 1$. Obviously, not every coring has grouplike elements. The results of this section are contained in the following

THEOREM 3.2. *Let $\mathcal{C}$ be a coseparable $A$-coring with a cointegral $\gamma$ and a grouplike element $g$. View $\mathcal{C}$ as a separable $A$-ring as in Theorem 2.6 and let $\mathbf{M}_{\mathcal{C}}$ denote the category of firm right modules of the $A$-ring $\mathcal{C}$ (cf. Definition 2.3). For any $M \in \mathbf{M}_{\mathcal{C}}$ define*

$$M_{g,\gamma}^{\mathcal{C}} = \{m \in M \mid \forall c \in \mathcal{C},\ m\cdot c = m\gamma(g\otimes c)\}.$$

*Then:*

(1) *$B = A_{g,\gamma}^{\mathcal{C}} = \{b \in A \mid \forall c \in \mathcal{C},\ \gamma(gb\otimes c) = b\gamma(g\otimes c)\}$, is a subring of $A$.*
(2) *The assignment $(-)_{g,\gamma}^{\mathcal{C}} : \mathbf{M}_{\mathcal{C}} \to \mathbf{M}_B$, $M \mapsto M_{g,\gamma}^{\mathcal{C}}$ is a covariant functor which has a left adjoint $-\otimes_B A : \mathbf{M}_B \to \mathbf{M}_{\mathcal{C}}$.*
(3) *$Q = \mathcal{C}_{g,\gamma}^{\mathcal{C}}$ is a firm left ideal in $\mathcal{C}$ and hence a $(\mathcal{C},B)$-bimodule.*
(4) *For every $M \in \mathbf{M}_{\mathcal{C}}$, the additive map*

$$\omega_M : M \otimes_{\mathcal{C}} Q \to M_{g,\gamma}^{\mathcal{C}}, \qquad m\otimes q \mapsto m\cdot q,$$

*is bijective.*
(5) *Define two maps*

$$\sigma : Q\otimes_B A \to \mathcal{C},\ q\otimes a \mapsto qa \quad \text{and} \quad \tau : A\otimes_{\mathcal{C}} Q \to B,\ a\otimes q \mapsto \gamma(ga\otimes q).$$

*Then $(B,\mathcal{C},A,Q,\tau,\sigma)$ is a Morita context, in which $\tau$ is surjective (hence an isomorphism).*

*Proof.* (1) First note that $A$ is a right $\mathcal{C}$-comodule with the coaction $a \mapsto 1_A\otimes ga$. Thus by Proposition 2.7, $A$ is a firm $\mathcal{C}$-module with the action $a\cdot c = \gamma(ga\otimes c)$, for all $a \in A$ and $c \in \mathcal{C}$. Therefore the definition of $B$ makes sense and takes the form stated. Obviously $1_A \in B$. Furthermore, for all $b, b' \in B$ and $c \in \mathcal{C}$,

$$\gamma(gbb'\otimes c) = \gamma(gb\otimes b'c) = b\gamma(g\otimes b'c) = bb'\gamma(g\otimes c),$$

so that $bb' \in B$ as required. Alternatively, we note that $B \cong \mathrm{End}_{\mathcal{C}}(A)$ and is therefore a ring.

(2) We note that $M_{g,\gamma}^{\mathcal{C}} \cong \mathrm{Hom}_{\mathcal{C}}(A, M)$ via $f \mapsto f(1_A)$ has the left adjoint $-\otimes_B A$. In more detail, take any $M \in \mathbf{M}_{\mathcal{C}}$, $m \in M_{g,\gamma}^{\mathcal{C}}$, $b \in B$ and $c \in \mathcal{C}$, and use the definitions of $B$ and $M_{g,\gamma}^{\mathcal{C}}$ to compute

$$(mb) \cdot c = m \cdot (bc) = m\gamma(g \otimes bc) = mb\gamma(g \otimes c).$$

This shows that $M_{g,\gamma}^{\mathcal{C}}$ is a right $B$-module, hence $(-)_{g,\gamma}^{\mathcal{C}}$ is a functor as stated. In the opposite direction, for any right $B$-module $N$, $N \otimes_B A$ is a firm right $\mathcal{C}$-module with the action $(n \otimes a) \cdot c = n \otimes \gamma(ga \otimes c)$. Note that this action is well-defined by the construction of $B$. More precisely, $A$ is a left $B$-module and a firm right $\mathcal{C}$-module (since it is a right $\mathcal{C}$-comodule). It is a $(B, \mathcal{C})$-bimodule, since for every $b \in B$, $a \in A$ and $c \in \mathcal{C}$,

$$(ba) \cdot c = \gamma(gba \otimes c) = \gamma(gb \otimes ac) = b\gamma(g \otimes ac) = b(a \cdot c),$$

by (1). The above action is simply induced from the action of $\mathcal{C}$ on $A$ and thus well-defined. Therefore there is a functor as required. Now, one can easily check that the unit and counit of the adjunction are given by

$$\eta_N : N \to (N \otimes_B A)_{g,\gamma}^{\mathcal{C}}, \qquad n \mapsto n \otimes 1_A,$$

$$\epsilon_M : M_{g,\gamma}^{\mathcal{C}} \otimes_B A \to M, \qquad m \otimes a \mapsto ma,$$

for all $N \in \mathbf{M}_B$ and $M \in \mathbf{M}_{\mathcal{C}}$.

(3) Note that $Q \cong \mathrm{Hom}_{\mathcal{C}}(A, \mathcal{C})$ and is therefore a natural $(\mathcal{C}, B)$-bimodule. In more detail, $(cq)c' = c(qc') = cq\gamma(g \otimes c')$ for any $c, c' \in \mathcal{C}$ and $q \in Q$ , so $cq$ is an element of $Q$, hence $Q$ is a left $\mathcal{C}$-ideal. By (2) $Q$ is also a right $B$-module, and since the product in $\mathcal{C}$ is an $(A, A)$-bimodule map, $Q$ is a $(\mathcal{C}, B)$-bimodule. We only need to show that $Q$ is firm as a left $\mathcal{C}$-module. This can be shown by the same technique as in the proof of Proposition 2.7. $\mathcal{C} \otimes_{\mathcal{C}} Q$ is defined as a cokernel of the following left $\mathcal{C}$-linear map

$$\lambda : \mathcal{C} \otimes_A \mathcal{C} \otimes_A Q \to \mathcal{C} \otimes_A Q, \quad \lambda = I_{\mathcal{C}} \otimes \pi - \pi \otimes I_Q,$$

where $\pi$ is the product map in $\mathcal{C}$ (i.e., the splitting of $\Delta_{\mathcal{C}}$) corresponding to the cointegral $\gamma$. Observe that the product map $\pi : \mathcal{C} \otimes_A Q \to Q$ is a surjection. Indeed, first note that since for all $c \in \mathcal{C}$, $\pi(g \otimes c) = g\gamma(g \otimes c)$ by the relationship between $\pi$ and $\gamma$, the grouplike element $g$ is in $Q$. For any $q \in Q$ take $q \otimes g \in \mathcal{C} \otimes_A Q$. Then $\pi(q \otimes g) = q\gamma(g \otimes g) = q$, by the properties of the cointegral $\gamma$. Thus $\pi$ is a surjection as claimed.

Consider the following sequence of left $\mathcal{C}$-module maps

$$\mathcal{C} \otimes_A \mathcal{C} \otimes_A Q \xrightarrow{\lambda} \mathcal{C} \otimes_A Q \xrightarrow{\pi} Q \longrightarrow 0.$$

We need to show that this sequence is exact. Clearly the associativity of $\pi$ implies that $\pi \circ \lambda = 0$, so that $\mathrm{Im}\lambda \subseteq \ker \pi$. Furthermore,

$$\Delta_{\mathcal{C}} \circ \pi - \lambda \circ (\Delta_{\mathcal{C}} \otimes I_Q) = (I_{\mathcal{C}} \otimes \pi) \circ (\Delta_{\mathcal{C}} \otimes I_Q) - (I_{\mathcal{C}} \otimes \pi) \circ (\Delta_{\mathcal{C}} \otimes I_Q) + (\pi \circ \Delta_{\mathcal{C}}) \otimes I_Q = I_{\mathcal{C}} \otimes I_Q,$$

where we used the colinearity of $\pi$ and the fact that $\pi$ is a splitting of $\Delta_{\mathcal{C}}$. This implies that $\ker \pi \subseteq \mathrm{Im}\lambda$, i.e., the above sequence is exact as required.

(4) Note that $\omega_M$ is the natural map $M \otimes_{\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(A, \mathcal{C}) \to \mathrm{Hom}_{\mathcal{C}}(A, M) \cong M_{g,\gamma}^{\mathcal{C}}$, $m \otimes f \mapsto [a \mapsto f(1)m]$, and hence it is well-defined. Explicitly, for all $m \in M$, $q \in Q$

and $c \in \mathcal{C}$ we compute $(m \cdot q) \cdot c = m \cdot (qc) = m \cdot q\gamma(g{\otimes}c)$, as required. We need to show that $\omega_M$ is bijective. Consider a map

$$\theta_M : M^{\mathcal{C}}_{g,\gamma} \to M{\otimes}_{\mathcal{C}}Q, \qquad m \mapsto m{\otimes}g.$$

This is well-defined, since as shown in the proof of (3), $g \in Q$. Take any $m \in M^{\mathcal{C}}_{g,\gamma}$. Then

$$\omega_M(\theta_M(m)) = m \cdot g = m\gamma(g{\otimes}g) = m,$$

by the definition of $M^{\mathcal{C}}_{g,\gamma}$ and properties of a cointegral. Conversely, for any simple tensor $m{\otimes}q \in M{\otimes}_{\mathcal{C}}Q$,

$$\theta_M(\omega_M(m{\otimes}q)) = m \cdot q{\otimes}g = m{\otimes}qg = m{\otimes}q\gamma(g{\otimes}g) = m{\otimes}q,$$

again by the definition of $Q$ and properties of cointegrals.

(5) Note that $\sigma$ is the evaluation mapping $\mathrm{Hom}_{\mathcal{C}}(A,\mathcal{C}){\otimes}_B A \to \mathcal{C}$, while $\tau$ is the canonical map $A{\otimes}_{\mathcal{C}}\mathrm{Hom}_{\mathcal{C}}(A,\mathcal{C}) \to \mathrm{End}_{\mathcal{C}}(A)$. In more detail, we show that the maps $\sigma$ and $\tau$ are well-defined as bimodule maps. Obviously, $\sigma$ is left $\mathcal{C}$-linear. Take any $q \in Q$, $a \in A$ and $c \in \mathcal{C}$ and compute

$$\sigma(q{\otimes}a \cdot c) = q\gamma(ga{\otimes}c) = q\gamma(g{\otimes}ac) = \pi(q{\otimes}ac) = \pi(qa{\otimes}c) = (qa)c.$$

This shows that $\sigma$ is $(\mathcal{C},\mathcal{C})$-bilinear as required. Note that $\tau = \omega_A$, and since $B = A^{\mathcal{C}}_{g,\gamma}$ it is well defined and surjective. Clearly $\tau$ is right $B$-linear. An easy computation which involves the definition of $B$ confirms that $\tau$ is $(B,B)$-bilinear. Next we need to check the commutativity of diagrams in Definition 3.1. The commutativity of the second diagram follows immediately from $A$-linearity of the cointegral. Now take any $a \in A$, and $q,q' \in Q$ and compute

$$\sigma(q{\otimes}a)q' = (qa)q' = q(aq') = q\gamma(g{\otimes}aq') = q\tau(a{\otimes}q'),$$

where the definition of $Q$ was used to derive the third equality. Thus there is a Morita context as required. A standard argument in Morita theory confirms that $\tau$ is an isomorphism (cf. [2, II (3.4) Theorem]). $\qquad\square$

COROLLARY 3.3. *With the assumptions and notation as in Theorem 3.2 we have:*

(1) *$Q$ is a subring of $\mathcal{C}$ with a right unit $g$.*

(2) *The functor $- {\otimes}_B A : \mathbf{M}_B \to \mathbf{M}_{\mathcal{C}}$ is fully faithful, i.e., the unit of adjunction $\eta$ is an isomorphism.*

(3) *$A_{\mathcal{C}}$ and ${}_{\mathcal{C}}Q$ are direct summands of $\mathcal{C}^n$ for some (finite) $n \in \mathbb{N}$.*

(4) *$A$ and $Q$ are generators as left resp. right $B$-modules.*

*Proof.* (1) follows immediately from the proof of Theorem 3.2(3), while (2)-(4) follow from Morita theory with surjective $\tau$, and can be proven by the same methods as in the unital case (cf. [2, Ch. II.3]). In particular, in the case of (3) the "dual bases" of $A$ and $Q$ can be constructed as follows. Let $\{a_i \in A, q^i \in Q\}_{i=1,\ldots,n}$ be such that $1_B = \sum_i \tau(a_i{\otimes}q^i)$. Define $\sigma^i = \sigma(q^i{\otimes}-) \in \mathrm{Hom}_{\mathcal{C}}(A,\mathcal{C})$. Then for every $a \in A$,

$$\sum_i a_i \cdot \sigma^i(a) = \sum_i a_i \cdot \sigma(q^i{\otimes}a) = \sum_i \tau(a_i{\otimes}q^i)a = a,$$

so that $\{a_i, \sigma^i\}_{i=1,\ldots,n}$ is a dual basis for $A_{\mathcal{C}}$. Similarly a dual basis for ${}_{\mathcal{C}}Q$ can be constructed as $\{q^i, \sigma_i\}$ with $\sigma_i = \sigma(-{\otimes}a_i)$. $\qquad\square$

One can easily find a sufficient condition for the Morita context of Theorem 3.2 to be strict.

PROPOSITION 3.4. *Let $\mathcal{C}$ be a coseparable $A$-coring with a grouplike element $g$. Then the Morita context $(B, \mathcal{C}, A, Q, \tau, \sigma)$ in Theorem 3.2 is strict, provided $g$ is a left unit in $\mathcal{C}$.*

*Proof.* In this case $\gamma(g \otimes c) = \epsilon_{\mathcal{C}}(c)$, hence $B$ and $Q$ are characterised by relations $\epsilon_{\mathcal{C}}(bc) = b\epsilon_{\mathcal{C}}(c)$ and $qc = q\epsilon_{\mathcal{C}}(c)$, respectively, for all $c \in \mathcal{C}$. We need to show that $\sigma$ is an isomorphism. For any $c \in \mathcal{C}$, $\sigma(g \otimes \epsilon_{\mathcal{C}}(c)) = g\epsilon_{\mathcal{C}}(c) = gc = c$, since $g \in Q$. Thus $\sigma$ is surjective. Suppose now that $\sum_i q_i \otimes a_i \in \ker \sigma$, i.e., $\sum_i q_i a_i = 0$. This implies that $\sum_i \epsilon_{\mathcal{C}}(q_i) a_i = 0$, so that

$$0 = g \otimes_B \sum_i \epsilon_{\mathcal{C}}(q_i) a_i = \sum_i g\epsilon_{\mathcal{C}}(q_i) \otimes a_i = \sum_i gq_i \otimes a_i = \sum_i q_i \otimes a_i.$$

Here we used that for all $q \in Q$, $\epsilon_{\mathcal{C}}(q) = \gamma(g \otimes q) \in B$, the fact that $g \in Q$ and that $g$ is a left unit in $\mathcal{C}$. This completes the proof. $\square$

Finally, we consider two examples of Theorem 3.2.

EXAMPLE 3.5. Consider a split extension $\bar{B} \xrightarrow{\iota} A$ with splitting map $E : A \to \bar{B}$. Then the canonical Sweedler coring $\mathcal{C} = A \otimes_{\bar{B}} A$ is an $A$-ring with the $E$-multiplication, $1_A \otimes 1_A \in \mathcal{C}$ is a grouplike element, and the Morita context constructed in Theorem 3.2 comes out as follows. The ring $B$ is just

$$B = \iota(\bar{B}),$$

while the $(\mathcal{C}, B)$-bimodule $Q \subset A \otimes_B A$ is

$$Q \cong A$$

via $a \mapsto a \otimes 1_A$ and the left module action of the $A$-ring $\mathcal{C}$ on $A$ given by $c \cdot a = \sum c^1 E(c^2 a)$ (suppressing a possible summation in $c = \sum c^1 \otimes c^2 \in A \otimes_B A$). The module $A_{\mathcal{C}}$ is similarly given by $a \cdot c = \sum E(ac^1) c^2$ for each $a \in A, c \in \mathcal{C}$. The Morita maps read $\sigma(a \otimes a') = a \otimes a'$ and $\tau(a \otimes a') = E(aa')$. This context is obviously strict.

The proof of this involves applying the theorem, noting that

$$B = \{b \in A \mid \forall a \in A, \ 1_A E(ba) = bE(a)\} = \iota(\bar{B})$$

since $\supseteq$ is clear and $\subseteq$ follows from letting $a = 1_A$. Next one notes that

$$Q = \{q = \sum_i q_i \otimes \bar{q}_i \in A \otimes_{\bar{B}} A \mid \forall a \in A, \ \sum_i q_i E(\bar{q}_i a) \otimes 1_A = qE(a)\} = A \otimes 1_A$$

since $\supseteq$ is clear and $\subseteq$ follows from taking $a = 1_A$.

EXAMPLE 3.6. Hopf algebroids over a noncommutative base $k$-algebra $A$, where $k$ is a commutative ring, provide examples of $A$-corings with grouplike elements; in particular, the canonical bialgebroids $\mathrm{End}_k A$ and $A \otimes_k A^{\mathrm{op}}$ do (cf. [16] [7] for the definition and examples of Hopf algebroids). They can be extended to ring extensions via an algebraic formulation of depth two for subfactors [15]: a ring extension $B \to A$ is of *depth two* (D2) if $A \otimes_B A$ is isomorphic to a direct summand of $A \oplus \cdots \oplus A$ as a $(B, A)$-bimodule, and similarly as an $(A, B)$-bimodule. The two

conditions are equivalent respectively to the existence of finitely many elements $c_j, b_i \in (A \otimes_B A)^B$ and $\gamma_j, \beta_i \in {}_B\mathrm{End}_B(A)$ such that for all $a, a' \in A$,

$$a \otimes a' = \sum_i b_i \beta_i(a) a' = \sum_j a \gamma_j(a') c_j. \tag{1}$$

Denoting the centraliser $A^B$ of a D2 extension $B \to A$ by $R$, the following $R$-coring structure for $\mathcal{C} := {}_B\mathrm{End}_B(A)$ is considered in [15]. The $(R, R)$-bimodule structure is $r\alpha r' = r\alpha(-)r'$ ($\alpha \in \mathcal{C}$). The coproduct is given most simply by noting [15, 3.10]: $\mathcal{C} \otimes_R \mathcal{C} \cong {}_B\mathrm{Hom}_B(A \otimes_B A, A)$ via

$$\alpha \otimes \beta \longmapsto [a \otimes a' \mapsto \alpha(a)\beta(a')].$$

Then

$$\Delta_{\mathcal{C}}(\alpha)(a \otimes a') = \alpha(aa'),$$

with counit

$$\epsilon_{\mathcal{C}}(\alpha) = \alpha(1_A).$$

We also have the alternative formulae for the coproduct [15, Eqs. (66), (68)]:

$$\Delta_{\mathcal{C}}(\alpha) = \sum_j \gamma_j \otimes_R c_j^1 \alpha(c_j^2 -) = \sum_i \alpha(-b_i^1) b_i^2 \otimes_R \beta_i,$$

where $c_j = \sum c_j^1 \otimes c_j^2$ and $b_i = \sum b_i^1 \otimes b_i^2$ is a notation suppressing a possible summation index. We note the grouplike element $I_A$.

Suppose the D2 extension $B \to A$ is separable with separability element $e = \sum e^1 \otimes e^2 \in A \otimes_B A$ (summation index suppressed). Then the $R$-coring $\mathcal{C}$ is coseparable with cointegral $\gamma : \mathcal{C} \otimes_R \mathcal{C} \to R$ given by $\gamma(\alpha \otimes \beta) = \sum \alpha(e^1)\beta(e^2)$. The corresponding $R$-ring structure on $\mathcal{C}$ is given by ($x \in A$)

$$(\alpha * \beta)(x) = \sum \alpha(xe^1)\beta(e^2) = \sum \alpha(e^1)\beta(e^2 x)$$

with the $R$-bimodule structure above.

The Morita context in the theorem applied to $\mathcal{C}$ turns out as follows:

PROPOSITION 3.7. *The centre $Z$ of $A$ and the non-unital ring $(\mathcal{C}, *)$ are related by the Morita context $(Z, \mathcal{C}, {}_Z R_{\mathcal{C}}, {}_{\mathcal{C}} R_Z, \tau, \sigma)$ where ${}_Z R_{\mathcal{C}}$ is given by $zr \cdot \alpha = \sum ze^1 r\alpha(e^2)$, ${}_{\mathcal{C}} R_Z$ by $\alpha \cdot rz = \sum \alpha(e^1) re^2 z$,*

$$\sigma : R \otimes_Z R \to \mathcal{C}, \quad r \otimes r' \longmapsto \lambda(r)\rho(r') = \rho(r')\lambda(r),$$

*where $\lambda(r), \rho(r) \in \mathcal{C}$ denote left and right multiplication by $r \in R$, respectively, and*

$$\tau : R \otimes_{\mathcal{C}} R \xrightarrow{\cong} \mathcal{C}, \quad r \otimes r' \longmapsto \sum e^1 rr' e^2.$$

*The Morita context is strict if $B \to A$ is H-separable.*

*Proof.* We check that $\gamma$ is a cointegral. Take any $\alpha \in \mathcal{C}$ and compute

$$\sum \gamma(\alpha_{(1)} \otimes \alpha_{(2)}) = \sum \alpha(e^1 e^2) = \epsilon_{\mathcal{C}}(\alpha).$$

Thus $\gamma$ is normalised. Furthermore, for all $\alpha, \beta \in \mathcal{C}$,

$$\sum \gamma(\alpha \otimes \beta_{(1)})\beta_{(2)} = \sum_i \alpha(e^1)\beta(e^2 b_i^1) b_i^2 \beta_i = \sum \alpha(e^1)\beta(e^2 -).$$

On the other hand

$$\sum \alpha_{(1)} \gamma(\alpha_{(2)} \otimes \beta) = \sum_j \gamma_j(-) c_j^1 \alpha(c_j^2 e^1) \beta(e^2) = \sum \alpha(-e^1) \beta(e^2),$$

so that $\gamma$ is colinear and hence a cointegral.
The subring of $R$ in Theorem 3.2 is

$$R_{I_A, \gamma}^{\mathcal{C}} = \{r \in R \mid \forall a \in A, \sum e^1 r \alpha(e^2) = \sum r e^1 \alpha(e^2)\} = Z,$$

since $\supseteq$ is clear, and $\subseteq$ follows from taking $\alpha = I_A$ and observing $\sum e^1 r e^2 \in Z$.
The Morita context bimodule

$$Q = \{q \in \mathcal{C} \mid \forall \alpha \in \mathcal{C}, \sum q(-e^1) \alpha(e^2) = \sum q(-) e^1 \alpha(e^2)\} = \lambda(R),$$

since $\supseteq$ is clear, and $\subseteq$ follows from taking $\alpha = I_A$, whence $q = \sum q(-e^1) e^2 = \sum \lambda(q(e^1) e^2) \in \lambda(R)$.
That $(Z, \mathcal{C}, Q, R, \sigma, \tau)$ is a Morita context is now straightforward; $\tau$ being epi by an old lemma of Hirata and Sugano [14].
Recall that $B \to A$ is H-separable (after Hirata) if there are (Casimir) elements $e_i \in (A \otimes_B A)^A$ and $r_i \in R$ (the centraliser) such that $1_A \otimes 1_A = \sum_i e_i r_i (= \sum_i r_i e_i)$ (a very strong version of Eqs. (1) above). It is well-known that $A$ is a separable extension of $B$. Moreover,

$$R \otimes_Z R^{\mathrm{op}} \xrightarrow{\cong} {}_B \mathrm{End}_B(A),$$

via $r \otimes r' \mapsto \lambda(r) \rho(r')$ with inverse $\alpha \mapsto \sum_i \alpha(e_i^1) e_i^2 \otimes r_i$. Whence $\sigma$ is an isomorphism if we begin with an H-separable extension. $\qquad\square$

If $A$ is a separable algebra over a (commutative) ground ring $B$, then the proposition shows that the center $Z$ and $\mathrm{End}_B(A)$ (with the exotic multiplication above) are related by a Morita context, which is strict if $Z = B 1_A$, i.e. $A$ is Azumaya.
As a third example, we may instead work with the dual bialgebroid in [15] and prove that a split D2 extension $B \to A$ has a coseparable $R$-coring structure on $(A \otimes_B A)^B$ which is essentially a restriction of $\mathcal{C}$ in Example 3.5.

## 4. Are biseparable corings Frobenius?

In this section we will show that a one-sided, slightly stronger version of the problem in [9] is equivalent to the problem if cosplit, coseparable corings with a condition of finite projectivity are Frobenius. Given the techniques developed for corings and the many examples coming from entwined structures [5], we expect this equivalence to be useful in solving this problem.
As recalled in Section 2, an $A$-coring $\mathcal{C}$ is coseparable if the forgetful functor $F : \mathbf{M}^{\mathcal{C}} \to \mathbf{M}_A$ is separable (cf. [17] for the definition of a separable functor). Dually, we say that $\mathcal{C}$ is *cosplit* if the functor $- \otimes_A \mathcal{C}$ is a separable functor from the category of right $A$-modules $\mathbf{M}_A$ into the category of right $\mathcal{C}$-comodules $\mathbf{M}^{\mathcal{C}}$. (Recall that $F$ is the left adjoint of $- \otimes_A \mathcal{C}$ [13].)
An $A$-coring $\mathcal{C}$ determines two ring extensions $\iota^* : A \to \mathcal{C}^*$ and ${}^*\iota : A \to {}^*\mathcal{C}$ where $\mathcal{C}^* := \mathrm{Hom}_A(\mathcal{C}, A)$ and ${}^*\mathcal{C} := {}_A\mathrm{Hom}(\mathcal{C}, A)$, i.e., the right and left duals of $\mathcal{C}$. The ring structure on $\mathcal{C}^*$ is given by $(\xi\xi')(c) = \sum \xi(\xi'(c_{(1)}) c_{(2)})$ $(\xi, \xi' \in \mathcal{C}^*, c \in \mathcal{C})$ with unity $\epsilon_{\mathcal{C}}$ and the natural Abelian group structure, while the ring structure

on $^*\mathcal{C}$ is given by $(\xi\xi')(c) = \sum \xi'(c_{(1)}\xi(c_{(2)}))$. The mappings $\iota^*$ and $^*\iota$ are given by $\iota^*(a) = \epsilon_{\mathcal{C}}(a-)(= a\epsilon_{\mathcal{C}})$ and $^*\iota(a) = \epsilon_{\mathcal{C}}(-a)$. We note by short calculations that the induced $(A, A)$-bimodule structures on $\mathcal{C}^*$ and $^*\mathcal{C}$ coincide with the usual structures, which we recall are given by $(a\xi a')(c) := a\xi(a'c)$ for $\xi \in \mathcal{C}^*$ and $a, a' \in A$ and $(a\xi a')(c) = \xi(ca)a'$ for $\xi \in {^*\mathcal{C}}$. Also note that $^*\iota$ and $\iota^*$ are monomorphisms if $\epsilon_{\mathcal{C}} : \mathcal{C} \to A$ is surjective.

Recall from [9] that a ring extension $B \to A$ is biseparable if it is split, separable and the natural modules $A_B$ and $_BA$ are finitely generated projective. We will say that $B \to A$ is *left or right biseparable* if $B \to A$ is split, separable but only one of $_BA$ or $A_B$, respectively, need be finitely generated projective. This motivates the following

DEFINITION 4.1. An $A$-coring $\mathcal{C}$ is said to be *biseparable* if $\mathcal{C}_A$ and $_A\mathcal{C}$ are finitely generated projective and $\mathcal{C}$ is cosplit as well as coseparable.

PROPOSITION 4.2. *If $B \to A$ is a biseparable extension, then the canonical Sweedler $A$-coring $\mathcal{C} := A \otimes_B A$ is a biseparable coring.*

*Proof.* Since $B \to A$ is separable, the induction functor $- \otimes_A \mathcal{C}$ from $\mathbf{M}_A$ into $\mathbf{M}^{\mathcal{C}}$ is separable by [5, Corollary 3.4]. Since $B \to A$ is split, and $A_B$ is a projective generator (therefore faithfully flat), the forgetful functor $F : \mathbf{M}^{\mathcal{C}} \to \mathbf{M}_A$ is a separable functor by [5, Corollary 3.7]. It follows by definition that the canonical coring $\mathcal{C}$ is cosplit and coseparable.

Finally we note that $_BA$ finitely generated projective implies $_AA \otimes_B A$ finitely generated projective. Similarly, $\mathcal{C}_A$ is finitely generated projective, and we conclude that $\mathcal{C}$ is biseparable.                    $\square$

Recall that an $A$-coring $\mathcal{C}$ is said to be *Frobenius* if the forgetful functor $F : \mathbf{M}^{\mathcal{C}} \to \mathbf{M}_A$ is a Frobenius functor (has the same left and right adjoint), i.e. $- \otimes_A \mathcal{C}$ is also a left adjoint of $F$ [5, 6]. Motivated by the question in [9] let us make

CONJECTURE 4.3. *A biseparable $A$-coring $\mathcal{C}$ is Frobenius.*

PROPOSITION 4.4. *If Conjecture 4.3 is true, then biseparable extensions are Frobenius.*

*Proof.* Given a biseparable extension $B \to A$, its canonical coring $\mathcal{C} = A \otimes_B A$ is biseparable by the previous proposition. If $\mathcal{C}$ is then a Frobenius coring by hypothesis, it follows from [6, Theorem 2.7] that $B \to A$ is a Frobenius extension, since $_BA$ is faithfully flat.                    $\square$

We now proceed to establish a converse to this proposition.

PROPOSITION 4.5. *If $\mathcal{C}$ is a cosplit $A$-coring, then $\iota^* : A \to \mathcal{C}^*$ and $^*\iota : A \to {^*\mathcal{C}}$ are both split extensions.*

*Proof.* By [5, Theorem 3.3], $\mathcal{C}$ is cosplit if and only if there is $e \in \mathcal{C}^A$ such that $\epsilon_{\mathcal{C}}(e) = 1_A$. (In other words, $\epsilon_{\mathcal{C}} : \mathcal{C} \to A$ is a split $(A, A)$-epimorphism.) We now define a "conditional expectation" or bimodule projection $E^* : \mathcal{C}^* \to A$, respectively $^*E : {^*\mathcal{C}} \to A$ simply by

$$E^*(\xi) = \xi(e), \quad {^*E}(\xi') = \xi'(e) \quad (\xi \in \mathcal{C}^*, \xi' \in {^*\mathcal{C}}).$$

Note that $E^*(\epsilon_{\mathcal{C}}) = 1_A = {}^*E(1_{*\mathcal{C}})$ and

$$E^*(a\xi a') = a\xi(a'e) = a\xi(ea') = a\xi(e)a' = aE^*(\xi)a'$$

for $a, a' \in A$, whence $E^*$ and similarly ${}^*E$ give $(A, A)$-bimodule splittings of $\iota^*$ and ${}^*\iota$. $\qquad\square$

For example, the canonical Sweedler coring $\mathcal{C}$ of a ring extension $B \to A$ is cosplit if and only if $B \to A$ is a separable extension. Now $\mathcal{C}^* \cong \mathrm{End}_B(A)$ as rings via $\xi \mapsto \xi(-\otimes 1_A)$ with inverse

$$f \longmapsto [a \otimes a' \mapsto f(a)a'].$$

Since $\iota^*$ corresponds to the left regular representation $\lambda : A \to \mathrm{End}_B(A)$, we recover results by Müller and Sugano that $\lambda$ is a split extension if $B \to A$ is separable.

PROPOSITION 4.6. *Let $\mathcal{C}$ be an $A$-coring. If ${}_A\mathcal{C}$ is finitely generated projective, then $\mathcal{C}$ is coseparable if and only if ${}^*\iota : A \to {}^*\mathcal{C}$ is a separable extension. If $\mathcal{C}_A$ is finitely generated projective, then $\mathcal{C}$ is a coseparable $A$-coring if and only if $\iota^* : A \to \mathcal{C}^*$ is a separable extension.*

*Proof.* We will prove the first statement, the second follows similarly. If ${}_A\mathcal{C}$ is finitely generated projective, then the category of right comodules $\mathbf{M}^{\mathcal{C}}$ is isomorphic to the category $\mathbf{M}_{*\mathcal{C}}$ of right modules over ${}^*\mathcal{C}$ [5, Lemma 4.3]. Recall that given a coaction $\varrho^M : M_A \to M \otimes_A \mathcal{C}$, we define an action of $\xi \in {}^*\mathcal{C}$ on $m \in M \in \mathbf{M}^{\mathcal{C}}$ by $m \cdot \xi = \sum m_{(0)}\xi(m_{(1)})$. It is trivial to check that $(M, \cdot) \in \mathbf{M}_{*\mathcal{C}}$. Inversely, given dual bases $\{\xi_i \in {}^*\mathcal{C}\}$ and $\{c_i \in \mathcal{C}\}$ such that $c = \sum_i \xi_i(c)c_i$ for each $c \in \mathcal{C}$, and right action of ${}^*\mathcal{C}$ on $M \in \mathbf{M}_{*\mathcal{C}}$, we define a coaction

$$\varrho^M(m) = \sum_i m \cdot \xi_i \otimes_A c_i.$$

It is easily checked that $(M, \varrho^M) \in \mathbf{M}^{\mathcal{C}}$, and that the two operations are natural and inverses to one another, so that $\mathbf{M}_{*\mathcal{C}} \cong \mathbf{M}^{\mathcal{C}}$.

Now $\mathcal{C}$ is coseparable if and only if the forgetful functor $F : \mathbf{M}^{\mathcal{C}} \to \mathbf{M}_A$ is a separable functor. Since

$$m \cdot {}^*\iota(a) = \sum m_{(0)}\epsilon_{\mathcal{C}}(m_{(1)})a = ma$$

for each $a \in A, m \in M \in \mathbf{M}^{\mathcal{C}}$, $F$ corresponds under the isomorphism of categories above to the forgetful functor $G : \mathbf{M}_{*\mathcal{C}} \to \mathbf{M}_A$ induced by ${}^*\iota : A \to {}^*\mathcal{C}$. But $G$ is a separable functor if and only if ${}^*\iota$ is a separable extension [17]. $\qquad\square$

PROPOSITION 4.7. *Suppose $\mathcal{C}$ is an $A$-coring which is reflexive as a left and right $A$-module. Then $\mathcal{C}$ is a Frobenius coring if and only if ${}^*\iota : A \to {}^*\mathcal{C}$ is a Frobenius extension if and only if $\iota^* : A \to \mathcal{C}^*$ is a Frobenius extension.*

*Proof.* The proof is quite similar to the proof of Proposition 4.6 (cf. [5, Theorem 4.1]). If ${}^*\iota$ is Frobenius, it follows that ${}^*\mathcal{C}_A$ is finitely generated projective, so ${}_A({}^*\mathcal{C})^*$ is finitely generated projective, whence by reflexivity ${}_A\mathcal{C}$ is finitely generated projective. Then the categories $\mathbf{M}_{*\mathcal{C}}$ and $\mathbf{M}^{\mathcal{C}}$ are isomorphic. But the forgetful functor $G : \mathbf{M}_{*\mathcal{C}} \to \mathbf{M}_A$ has equal left and right adjoint if ${}^*\iota$ is Frobenius, in which case $F$ is Frobenius and $\mathcal{C}$ is a Frobenius coring. The other case is entirely similar.

Conversely, if $\mathcal{C}$ is a Frobenius $A$-coring, then both $\mathcal{C}_A$ and $_A\mathcal{C}$ are finitely generated projective [6, Corollary 2.3]. The rest follows from the functorial definitions of Frobenius extension and coring applied to either isomorphism of left or right module and comodule categories.                                            □

THEOREM 4.8. *Biseparable corings are Frobenius if and only if left or right biseparable extensions are Frobenius.*

*Proof.* Without one-sidedness, we saw $\Rightarrow$ in Proposition 4.4. ($\Leftarrow$) Suppose $\mathcal{C}$ is a biseparable $A$-coring. Then $^*\iota : A \to {}^*\mathcal{C}$ and $\iota^* : A \to \mathcal{C}^*$ are split, separable extensions by Propositions 4.5 and 4.6. Since $_A\mathcal{C}$ and $\mathcal{C}_A$ are finitely generated projective, it follows that $^*\mathcal{C}_A$ and $_A\mathcal{C}^*$ are finitely generated projective. If either left or right biseparable extensions are Frobenius, then either $\iota^*$ or $^*\iota$ is a Frobenius extension. In either case, Proposition 4.7 shows $\mathcal{C}$ to be a Frobenius coring.        □

We note the following special "depth one" case for which there is a solution to our conjecture. If $\mathcal{C}$ is a centrally projective $A$-bimodule, i.e., as $(A, A)$-bimodules $\mathcal{C} \oplus W \cong \oplus^n A$ for some $(A, A)$-bimodule $W$, and $\mathcal{C}$ is moreover biseparable, then $\mathcal{C}$ is Frobenius by a classical result of Sugano:

PROPOSITION 4.9. *If $\mathcal{C}$ is a centrally projective, cosplit, coseparable $A$-coring, then $\mathcal{C}$ is Frobenius.*

*Proof.* We easily obtain $\mathcal{C}^* \oplus W^* \cong \oplus^n A$ as $(A, A)$-bimodules, whence $\iota^* : A \to \mathcal{C}^*$ is a centrally projective separable extension by Proposition 4.6, and monomorphism since $\epsilon_\mathcal{C}$ is surjective. By [21, Theorem 2] $\iota^*$ is a Frobenius extension. Then by Proposition 4.7, $\mathcal{C}$ is a Frobenius coring.                        □

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