Adjoint Functors and Equivalences

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Abstract

For any ring R, an R-module P with $S = \mathrm{End}_R(P)$ is called *self-tilting* provided $\mathrm{Hom}_R(P, -)$ is exact on short exact sequences of P-generated modules and all P-generated modules are P-presented. We extend the theory of self-tilting modules to any additive functor $\mathbf{R} : \mathcal{A} \to \mathcal{B}$ which has a left adjoint $\mathbf{L} : \mathcal{B} \to \mathcal{A}$, together with an adstatic generator B of $\mathcal{B}$. $(\mathbf{L}, \mathbf{R}; B)$ is called a *right pointed tilting pair* if $\mathbf{R}$ is exact on short exact sequences of $\mathbf{L}(B)$-generated modules and all $\mathbf{L}(B)$-generated modules are $\mathbf{L}(B)$-presented. It is shown that for a right pointed tilting pair $(\mathbf{L}, \mathbf{R}; B)$, $\mathbf{R} : \mathrm{Gen}(\mathbf{L}(B)) \to \mathrm{Cog}(\mathbf{R}(A))$ is an equivalence (with inverse $\mathbf{L}$), where A is any cogenerator in $\mathcal{A}$, provided $\mathbf{R}$ preserves coproducts of $\mathbf{L}(B)$. Basic properties of right pointed tilting pairs are investigated and various applications of the results are outlined.

Introduction

The classical Morita Theory decribes an equivalence between two module categories R-Mod and S-Mod, for unital rings R, S, by a functor $\mathrm{Hom}_R(P, -)$, where P is a finitely generated projective generator in R-Mod and $\mathrm{End}_R(P) \simeq S$. This setting was generalized in various directions.

One may ask for which proerties of $P \in R$-Mod, $\mathrm{Hom}_R(P, -)$ induces an equivalence between certain subcategories of R-Mod and S-Mod, respectively. For example, this functor induces an equivalence between the category of all P-generated modules $\mathrm{Gen}(P) \subset R$-Mod and the category of all Q-cogenerated modules $\mathrm{Cog}(Q)$ for $Q = \mathrm{Hom}_R(P, U)$, where U is any cogenerator in R-Mod, if and only if P is a finitely generated self-tilting module (in the terminology of [22]) or a $*$-*module* (in the terminology of Menini-Orsatti, see [6]). Variations of this were studied by many authors (e.g., Sato [18]) and the formulation of this setting in Grothendieck categories was given in Colpi [5].

Without any special conditions on $P \in R$-Mod and $S = \mathrm{End}_R(P)$, $\mathrm{Hom}_R(P, -)$ always defines an equivalence between the category $\mathrm{Stat}(P)$ of all P-*static* modules $X \in R$-Mod, i.e., $P \otimes_S \mathrm{Hom}_R(P, X) \simeq X$, and the category $\mathrm{Adst}(P)$ of P-*adstatic* modules $Y \in S$-Mod, i.e., $Y \simeq \mathrm{Hom}_R(P, P \otimes_S Y)$. Depending on the properties of P these classes may have special properties like being closed under submodules, factor modules, etc. These notions were considered by Naumann [14, 15] and a comprehensive treatment is given in [23].

The present paper was stimulated by the simple fact that *static* objects $X \in \mathcal{A}$ and *adstatic* objects $Y \in \mathcal{B}$ can be defined with respect to any pair of adjoint functors $\mathbf{R} : \mathcal{A} \to \mathcal{B}$ and $\mathbf{L} : \mathcal{B} \to \mathcal{A}$ between (complete and cocomplete abelian) categories by the conditions $\mathbf{LR}(X) \simeq X$, reps. $Y \simeq \mathbf{RL}(Y)$. Essential for building up the theory in this general setting is the existence of a static generator $B \in \mathcal{B}$ or - dually - a static cogenerator $A \in \mathcal{A}$. We call the data $(\mathbf{L}, \mathbf{R}; B)$ (resp., $(A; \mathbf{L}, \mathbf{R})$) a *right (left) pointed pair of adjoint funtors.*

After recalling some general facts needed for our investigation in the first section, the formal theory of a right pointed pair of adjoint funtors is presented in Section 2. The close connection to suitable Hom-functors is outlined in Section 3.

The application of our results to Doi-Koppinen modules defined for a right H-comodule algebra A and a right H-module coalgebra, where H is a bialgebra over a ring K, is presented in Section 4. Another interesting case is given by graded rings and modules and this is considered in Section 5. Similar constructions are studied in Marcus [11].

The dual situation is subject of Section 6. Here the elementary properties of left pointed pairs of adjoint funtors are sketched. These results are applied in Section 7 to the category of comodules thus obtaining generalizations of the Takeuchi equivalences between comodule categories.

Our techniques subsume a number of results about equivalences of (subcategories) of module categories and the tilting theory for Grothendieck categories. Moreover they extend related constructions to other situations, in particular to important categories in comodule theory.

1 Equivalences related to adjunctions

Let $\mathcal{C}$ be a complete and cocomplete abelian category and consider an object C of $\mathcal{C}$. For any set I, we denote by C^I the product, and by $C^{(I)}$ the coproduct of I copies of C in the category $\mathcal{C}$. An object X of $\mathcal{C}$ is called *C-cogenerated* (resp. *C-copresented*) if there are sets I,J and exact sequences $0 \to X \to C^I$ (resp. $0 \to X \to C^I \to C^J$). The subcategories of all C-cogenerated and all C–copresented objects will be denoted by $\mathrm{Cog}(C)$ and $\mathrm{Cop}(C)$, respectively. Dually, X is said to be *C-generated* (resp. *C-presented*) if there are sets I, J and exact sequences $C^{(I)} \to X \to 0$ (resp. $C^{(J)} \to C^{(I)} \to X \to 0$). When both I and J are finite sets, we say that X is *finitely C–presented*. The subcategories of all C–generated and all C–presented objects will be denoted by $\mathrm{Gen}(C)$ and $\mathrm{Pres}(C)$, respectively. The subcategory of all finitely C–presented objects is denoted by $\mathrm{Pres}_f(C)$. We start by fixing our notation and recalling basic facts on adjunctions which will be used in the sequel. The notation $X \in \mathcal{C}$ means in this framework "X is an object of $\mathcal{C}$".

1.1. Adjoint pairs of functors. Let $\mathbf{R} : \mathcal{A} \to \mathcal{B}$ be an additive functor between complete and cocomplete abelian categories which has a left adjoint $\mathbf{L} : \mathcal{B} \to \mathcal{A}$. Notice that $\mathbf{R}$

always preserves inverse limits and $\mathbf{L}$ preserves direct limits.

Let $\eta : 1_{\mathcal{B}} \to \mathbf{RL}$, $\delta : \mathbf{LR} \to 1_{\mathcal{A}}$ be the unit and the counit of the adjunction, respectively, and consider any object Y of $\mathcal{B}$. From the exact sequence

$$\mathbf{L}(Y) \xrightarrow{\mathbf{L}(\eta_Y)} \mathbf{LRL}(Y) \longrightarrow \mathbf{L}(\operatorname{coker}(\eta_Y)) \longrightarrow 0$$

we get that $\operatorname{coker}(\mathbf{L}(\eta_Y)) \cong \mathbf{L}(\operatorname{coker}(\eta_Y))$, and by properties of the unit η,

$$\delta_{\mathbf{L}(Y)}\mathbf{L}(\eta_Y) = 1_{\mathbf{L}(Y)}. \tag{1}$$

From equality (1) it follows that $\mathbf{LRL}(Y) \cong \ker \delta_{\mathbf{L}(Y)} \oplus Im(\mathbf{L}(\eta_Y))$.
So, $\operatorname{coker}(\mathbf{L}(\eta_Y)) \cong \ker \delta_{\mathbf{L}(Y)}$ and we have an isomorphism

$$\ker \delta_{\mathbf{L}(Y)} \cong \mathbf{L}(\operatorname{coker}(\eta_Y)). \tag{2}$$

Analogously, let X be any object of $\mathcal{A}$. From the exact sequence

$$0 \longrightarrow \mathbf{R}(\ker \delta_X) \longrightarrow \mathbf{RLR}(X) \xrightarrow{\mathbf{R}(\delta_X)} \mathbf{R}(X)$$

we get that $\ker \mathbf{R}(\delta_X) \cong \mathbf{R}(\ker \delta_X)$, and by properties of the counit δ,

$$\mathbf{R}(\delta_X)\eta_{\mathbf{R}(X)} = 1_{\mathbf{R}(X)}. \tag{3}$$

From equality (3) it follows that $\mathbf{RLR}(X) \cong \ker \mathbf{R}(\delta_X) \oplus Im(\eta_{\mathbf{R}(X)})$.
So, $\operatorname{coker}(\eta_{\mathbf{R}(X)}) \cong \ker \mathbf{R}(\delta_X)$ and we have the isomorphism

$$\operatorname{coker}(\eta_{\mathbf{R}(X)}) \cong \mathbf{R}(\ker \delta_X). \tag{4}$$

1.2. Static and adstatic objects. An object A of $\mathcal{A}$ is said to be $\mathbf{R}$–*static* if δ_A is an isomorphism. By Stat($\mathbf{R}$) we will denote the full subcategory of $\mathcal{A}$ consisting of all $\mathbf{R}$–static objects. Analogously, an object B of $\mathcal{B}$ is $\mathbf{L}$–static if η_B is an isomorphism. If we think of $\mathbf{R}$ as the basic object of our study, then we will use $\mathbf{R}$-*adstatic* as synonymous for $\mathbf{L}$-static. The full subcategory of $\mathcal{B}$ consisting of all $\mathbf{R}$–adstatic objects will be denoted by Adst($\mathbf{R}$). Obviously the functor $\mathbf{R}$ induces an equivalence between the categories Stat($\mathbf{R}$) and Adst($\mathbf{R}$) with inverse $\mathbf{L}$. The term 'static' comes from the equivalence theory for modules as expounded in the papers [1], [14] and [23]. It was also used in the framework of abstract localization theory in Grothendieck categories in [9].

1.3. Related subclasses. Let B be a generator for $\mathcal{B}$ and put $P = \mathbf{L}(B)$. For any object Y in $\mathcal{B}$, there is an exact sequence

$$B^{(I)} \longrightarrow B^{(J)} \longrightarrow Y \longrightarrow 0\,,$$

for some sets I, J. Apply the right exact and coproducts preserving functor $\mathbf{L}$ to obtain the exact sequence

$$P^{(I)} \longrightarrow P^{(J)} \longrightarrow \mathbf{L}(Y) \longrightarrow 0\,.$$

This implies that $\mathbf{L}(\mathcal{B}) \subseteq \mathrm{Pres}(P)$, where $\mathbf{L}(\mathcal{B})$ denotes the full subcategory of $\mathcal{A}$ consisting of all objects isomorphic to $\mathbf{L}(Y)$ for some object Y of $\mathcal{B}$. So we have the chain

$$\mathrm{Stat}(\mathbf{R}) \subseteq \mathbf{L}(\mathcal{B}) \subseteq \mathrm{Pres}(P) \subseteq \mathrm{Gen}(P) \subseteq \mathcal{A}. \tag{5}$$

Assume that $\mathcal{A}$ has a cogenerator A, and put $Q = \mathbf{R}(A)$. By dualizing the preceding arguments, we have

$$\mathrm{Adst}(\mathbf{R}) \subseteq \mathbf{R}(\mathcal{A}) \subseteq \mathrm{Cop}(Q) \subseteq \mathrm{Cog}(Q) \subseteq \mathcal{B}. \tag{6}$$

By right exactness of $\mathbf{L}$ and left exactness of $\mathbf{R}$, it follows that $\mathrm{Gen}(\mathbf{L}(B))$ and $\mathrm{Cog}(\mathbf{R}(A))$ are independent of the choice of the generator B of $\mathcal{B}$ and the cogenerator A of $\mathcal{A}$.

1.4. Lemma. *1. If $P^{(I)}$ is $\mathbf{R}$–static for every set I, then $\mathbf{L}(\mathcal{B}) = \mathrm{Pres}(P)$. Moreover, δ_X is an epimorphism for every $X \in \mathrm{Gen}(P)$.*

2. If Q^I is $\mathbf{R}$–adstatic for every set I, then $\mathbf{R}(\mathcal{A}) = \mathrm{Cop}(Q)$. Moreover, η_X is a monomorphism for every $X \in \mathrm{Cog}(Q)$.

Proof. (1). Let $X \in \mathrm{Pres}(P)$. We have an exact sequence $P^{(I)} \to P^{(J)} \to X \to 0$, for some sets I, J. Let $f : \mathbf{R}(P^{(I)}) \to C$ be the co-kernel of $\mathbf{R}(P^{(J)}) \to \mathbf{R}(P^{(I)})$. Applying the functor $\mathbf{L}$ to the diagram

$$\begin{tikzcd}
\mathbf{R}(P^{(J)}) \arrow[r] & \mathbf{R}(P^{(I)}) \arrow[rr, "f"] \arrow[dr] & & C \arrow[r] \arrow[dl, "g"] & 0\,, \\
& & \mathbf{R}(X) & &
\end{tikzcd}$$

we get the commutative diagram

$$\begin{tikzcd}
\mathbf{LR}(P^{(J)}) \arrow[r] \arrow[dd, "\cong"] & \mathbf{LR}(P^{(I)}) \arrow[r, "\mathbf{L}(f)"] \arrow[dd, "\cong"] \arrow[drr] & \mathbf{L}(C) \arrow[r] \arrow[dd, "\alpha"] \arrow[dr, "\mathbf{L}(g)"] & 0 \\
& & & \mathbf{LR}(X) \arrow[dl, "\delta_X"] \\
P^{(J)} \arrow[r] & P^{(I)} \arrow[r] & X \arrow[r] & 0.
\end{tikzcd}$$

Therefore, α is an isomorphism which proves that $X \in \mathbf{L}(\mathcal{B})$. Now, if X is P–generated, then we have an epimorphism $P^{(I)} \to X \to 0$. This gives the commutative square with exact rows

$$\begin{tikzcd}
\mathbf{LR}(P^{(I)}) \arrow[r] \arrow[d, "\cong"] & \mathbf{LR}(X) \arrow[d, "\delta_X"] & \\
P^{(I)} \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

whence δ_X is an epimorphism.

The proof of (2) is dual to that of (1). $\square$

1.5. Remark. Let $\mathrm{Pres}_f(P)$ denote the full subcategory of $\mathcal{A}$ whose objects are the finitely P–presented objects. Then the proof of part (1) in Lemma 1.4 shows that $\mathrm{Pres}_f(P) \subseteq \mathbf{R}(\mathcal{A})$, under the assumption that P is $\mathbf{R}$–static. Moreover, the counit morphism δ_X is an epimorphism for every finitely P–generated object X. The dual statements corresponding to part (2) in the lemma are also true.

1.6. Theorem. *Let $\mathbf{R} : \mathcal{A} \to \mathcal{B}$ be an additive functor with left adjoint $\mathbf{L} : \mathcal{B} \to \mathcal{A}$. Let B be any generator in $\mathcal{B}$ and A any cogenerator in $\mathcal{A}$. Putting $P = \mathbf{L}(B)$ and $Q = \mathbf{R}(A)$, the following conditions are equivalent:*

(i) $\mathbf{R}(\mathcal{A}) = \mathrm{Adst}(\mathbf{R})$*;*

(ii) $\mathbf{L}(\mathcal{B}) = \mathrm{Stat}(\mathbf{R})$*;*

(iii) $\mathbf{R}(\ker \delta_X) = 0$ *for every X in $\mathcal{A}$;*

(iv) $\mathbf{L}(\mathrm{coker}(\eta_Y)) = 0$ *for every Y in $\mathcal{B}$;*

(v) $\mathrm{Pres}(P) = \mathrm{Stat}(\mathbf{R})$*;*

(vi) $\mathrm{Cop}(Q) = \mathrm{Adst}(\mathbf{R})$*;*

(vii) $\mathbf{R} : \mathrm{Pres}(P) \to \mathrm{Cop}(Q)$ *is an equivalence of categories with inverse* $\mathbf{L}$*.*

Proof. (i) $\Rightarrow$ (iii). Let X be an object of $\mathcal{A}$. Then $\mathbf{R}(X) \in \mathbf{R}(\mathcal{A}) = \mathrm{Adst}(\mathbf{R})$, which means that $\eta_{\mathbf{R}(X)}$ is an isomorphism. By 1.1.(4), $\mathbf{R}(\ker \delta_X) = 0$.

(iii) $\Rightarrow$ (iv). Let Y be any object in $\mathcal{B}$. Then $\mathbf{L}(Y) \in \mathcal{B}$, so that $\mathbf{R}(\ker \delta_{\mathbf{L}(Y)}) = 0$. By 1.1.(2) we have that $\mathbf{LR}(\mathrm{coker}(\eta_Y)) = 0$ and this implies, by 1.1.(3) that $\mathbf{R}(\mathrm{coker}(\eta_Y)) = 0$.

(iv) $\Rightarrow$ (ii). Let $\mathbf{L}(Y)$ be an object of $\mathbf{L}(\mathcal{B})$. Since $Y \in \mathcal{B}$ then $\mathbf{L}(\mathrm{coker}(\eta_Y)) = 0$. By 1.1.(2) we have that $\ker \delta_{\mathbf{L}(Y)} = 0$, which implies that $\delta_{\mathbf{L}(Y)}$ is a monomorphism, and, hence, an isomorphism.

(ii) $\Rightarrow$ (iv) is similar to (i) $\Rightarrow$ (iii), and (iv) $\Rightarrow$ (iii) is similar to (iii) $\Rightarrow$ (iv).

(ii) $\Leftrightarrow$ (v). Since $P^{(I)} \in \mathbf{L}(\mathcal{B}) \cap \mathrm{Pres}(P)$ for any set I, this follows from Lemma 1.4.

(i) $\Leftrightarrow$ (vi). Since $Q^I \in \mathbf{R}(\mathcal{A}) \cap \mathrm{Cop}(Q)$ for any set I, this follows from Lemma 1.4.

(v) and (vi) $\Leftrightarrow$ (vii). This is a direct consequence of the definition of $\mathrm{Stat}(\mathbf{R})$ and $\mathrm{Adst}(\mathbf{R})$. □

1.7. Remark. The equivalence between (i)-(iv) in Theorem 1.6 holds without the assumption of the existence of a generator B in $\mathcal{B}$ or a cogenerator A in $\mathcal{A}$.

1.8. Corollary. *Let $\mathbf{R} : \mathcal{A} \to \mathcal{B}$ be an additive functor with left adjoint $\mathbf{L} : \mathcal{B} \to \mathcal{A}$, and let B be any generator of $\mathcal{B}$. Then the following conditions are equivalent.*

1. $\eta : 1_{\mathcal{B}} \to \mathbf{RL}$ *is a natural isomorphism.*
2. *The functor* $\mathbf{R} : \mathrm{Pres}(\mathbf{L}(B)) \to \mathcal{B}$ *is an equivalence of categories with inverse* $\mathbf{L}$*.*

Proof. If η is a natural isomorphism, then clearly $\mathrm{Adst}(\mathbf{R}) = \mathcal{B}$ and, obviously, $\mathrm{coker}(\eta_Y) = 0$ for every object Y of $\mathcal{B}$. It follows from Theorem 1.6 that $\mathrm{Pres}(\mathbf{L}(B)) = \mathrm{Stat}(\mathbf{R})$. This gives the equivalence $\mathrm{Pres}(\mathbf{L}(B)) \sim \mathcal{B}$. The converse is obvious. □

2 Right pointed (tilting) pairs

Definition. A *right pointed pair of adjoint functors* $(\mathbf{L}, \mathbf{R}; B)$ for the categories $\mathcal{A}$ and $\mathcal{B}$ consists of an additive functor $\mathbf{R} : \mathcal{A} \to \mathcal{B}$ with a left adjoint $\mathbf{L} : \mathcal{B} \to \mathcal{A}$, and a generator B for $\mathcal{B}$ which is $\mathbf{R}$-adstatic, i.e., $B \simeq \mathbf{RL}(B)$.

Denote by $\mathrm{Add}(P)$ (resp., $\mathrm{add}(P)$) the full subcategory of $\mathcal{A}$ consisting of all direct summands of (finite) direct sums of copies of P. Let $(\mathbf{L}, \mathbf{R}; B)$ be a right pointed pair of adjoint functors and $P = \mathbf{R}(B)$. Then clearly $\mathrm{add}(P) \subseteq \mathrm{Stat}(\mathbf{R})$ and also $\mathrm{add}(B) \subseteq \mathrm{Adst}(\mathbf{R})$. In general we have neither $\mathrm{Add}(P) \subseteq \mathrm{Stat}(\mathbf{R})$ nor $\mathrm{Add}(B) \subseteq \mathrm{Adst}(\mathbf{R})$.

2.1. Coproduct preserving functors. Let $\mathbf{R} : \mathcal{A} \to \mathcal{B}$ be an additive functor. Given a coproduct $P^{(I)}$ of copies of $P \in \mathcal{A}$, consider the canonical injection $\iota_i : P \to P^{(I)}$ for each $i \in I$. The family of morphisms

$$\{\mathbf{R}(\iota_i) : \mathbf{R}(P) \to \mathbf{R}(P^{(I)}) \mid i \in I\}$$

induces a canonical homomorphism $\Psi : \mathbf{R}(P)^{(I)} \to \mathbf{R}(P^{(I)})$. The functor $\mathbf{R}$ is said to *respect coproducts of* P if Ψ is an isomorphism for any set I.
Assume $(\mathbf{L}, \mathbf{R}; B)$ to be a right pointed pair of adjoint functors and put $P = \mathbf{L}(B)$. Since $\mathbf{L}$ preserves coproducts, $P^{(I)} \simeq \mathbf{L}(B^{(I)})$. If $j_i : B \to B^{(I)}$ denotes the i-th canonical injection, then $\mathbf{L}(j_i) : \mathbf{L}(B) \to \mathbf{L}(B^{(I)})$ is the corresponding injection for the coproduct $P^{(I)}$. From the commutative diagram

$$\begin{tikzcd} B^{(I)} \arrow[r, "\eta_{B^{(I)}}"] & \mathbf{RL}(B^{(I)}) \\ B^{(I)} \arrow[u, "Id"] \arrow[r, "\eta_B^{(I)}"] & \mathbf{RL}(B)^{(I)}, \arrow[u, "\Psi"'] \end{tikzcd} \tag{7}$$

we get that $\mathbf{R}$ respects coproducts of P if and only if $B^{(I)} \in \mathrm{Adst}(\mathbf{R})$, for any set I. This implies that if $\mathbf{R}$ respects coproducts of P then any coproduct of copies of P is $\mathbf{R}$-static.

2.2. Theorem. *Let* $(\mathbf{L}, \mathbf{R}; B)$ *be a right pointed pair of adjoint functors for* $\mathcal{A}$ *and* $\mathcal{B}$, *and let* A *be any cogenerator for* $\mathcal{A}$. *Put* $P = \mathbf{L}(B)$, $Q = \mathbf{R}(A)$ *and assume the following conditions.*

(1) $\mathbf{R}$ *respects coproducts of* P.

(2) The functor $\mathbf{R}$ *respects the exactness of the sequences* $0 \to K \to P^{(I)} \to X \to 0$, *where* $K \in \mathrm{Gen}(P)$.

Then $\mathbf{R} : \mathrm{Pres}(P) \to \mathrm{Cog}(Q)$ *is an equivalence of categories with inverse* $\mathbf{L}$.

Proof. Assume the conditions (1) and (2) hold. By Theorem 1.6, if we prove that $\mathrm{Pres}(P) = \mathrm{Stat}(\mathbf{R})$, then $\mathbf{R}$ gives the equivalence $\mathrm{Pres}(P) \sim \mathrm{Cop}(Q)$. Since $\mathrm{Stat}(\mathbf{R}) \subseteq \mathrm{Pres}(P)$ is always true, let us prove that every P–presented object X is $\mathbf{R}$–static. An exact sequence

$$P^{(J)} \longrightarrow P^{(I)} \longrightarrow X \longrightarrow 0$$

yields an exact sequence

$$0 \longrightarrow K \longrightarrow P^{(I)} \longrightarrow X \longrightarrow 0, \tag{8}$$

where $0 \to K \to P^{(I)}$ is the kernel of $P^{(I)} \to X$. By condition (2), we get an exact sequence

$$0 \longrightarrow \mathbf{R}(K) \longrightarrow \mathbf{R}(P^{(I)}) \longrightarrow \mathbf{R}(X) \longrightarrow 0.$$

Next, apply $\mathbf{L}$ to obtain the diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] & P^{(I)} \arrow[r] & X \arrow[r] & 0 \\
 & \mathbf{LR}(K) \arrow[r] \arrow[u, "\delta_K"] & \mathbf{LR}(P^{(I)}) \arrow[r] \arrow[u, "\delta_{P^{(I)}}"] & \mathbf{LR}(X) \arrow[r] \arrow[u, "\delta_X"] & 0
\end{tikzcd}$$

Now, $\delta_{P^{(I)}}$ is an isomorphism by hypothesis (1) and 2.1, implying that δ_X is an epimorphism. Clearly, $K \in \mathrm{Gen}(P)$ and this implies, by Lemma 1.4, that δ_K is also an epimorphism. From this we deduce that δ_X is a monomorphism and, thus, X is $\mathbf{R}$–static.

Finally, we have to show that $\mathrm{Cop}(Q) = \mathrm{Cog}(Q)$. Let Y be a Q–cogenerated object of $\mathcal{B}$. Since we have proved that $\mathrm{Gen}(P) = \mathrm{Stat}(\mathbf{R})$, it follows from Theorem 1.6 that $\mathrm{Cop}(Q) = \mathrm{Adst}(\mathbf{R})$. In particular, $Q^I \in \mathrm{Adst}(\mathbf{R})$ for every set I. By Lemma 1.4 we get that η_Y is a monomorphism. Let us prove that it is an epimorphism. Since B is a generator, there exists an exact sequence

$$0 \longrightarrow Y_1 \longrightarrow B^{(I)} \longrightarrow Y \longrightarrow 0\,.$$

Apply the right exact functor $\mathbf{L}$ to obtain the exact sequence

$$\mathbf{L}(Y_1) \longrightarrow P^{(I)} \longrightarrow \mathbf{L}(Y) \longrightarrow 0.$$

This last sequence yields a new exact sequence

$$0 \longrightarrow K \longrightarrow P^{(I)} \longrightarrow \mathbf{L}(Y) \longrightarrow 0\,,$$

for some object K which is an epimorphic image of $\mathbf{L}(Y_1)$. Since $\mathbf{L}(Y_1) \in \mathrm{Gen}(P)$, it follows that $K \in \mathrm{Gen}(P)$. Therefore, condition (2) gives the diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{R}(K) \arrow[r] & \mathbf{R}(P^{(I)}) \arrow[r] & \mathbf{RL}(Y) \arrow[r] & 0 \\
 & & B^{(I)} \arrow[r] \arrow[u, "\eta_{B^{(I)}}"] & Y \arrow[r] \arrow[u, "\eta_Y"] & 0.
\end{tikzcd}$$

By 2.1, $\eta_{B^{(I)}}$ is an isomorphism, whence η_Y is an epimorphism. □

The above observations motivate the following

2.3. Definition. A right pointed pair of adjoint functors $(\mathbf{L}, \mathbf{R}; B)$ is said to be a *right pointed tilting pair* provided $\mathrm{Gen}(P) = \mathrm{Pres}(P)$ and $\mathbf{R}$ is exact on short exact sequences in $\mathrm{Gen}(P)$.

As an immediate consequence of the preceding theorem we have:

2.4. Proposition. *Let $(\mathbf{L}, \mathbf{R}; B)$ be a right pointed tilting pair for $\mathcal{A}$ and $\mathcal{B}$ and assume that $\mathbf{R}$ preserves coproducts of P. Then $\mathbf{R} : \mathrm{Gen}(P) \to \mathrm{Cog}(Q)$ is an equivalence of categories with inverse $\mathbf{L}$, where $P = \mathbf{L}(B)$ and $Q = \mathbf{R}(A)$, for some cogenerator $A \in \mathcal{A}$.*

By definition, an R-module P with $S = \mathrm{End}_R(P)$ is self-tilting if and only if the right pointed pair of adjoint functors $(\mathrm{Hom}_R(P, -), P \otimes_S -; S)$ is a right pointed tilting pair for ${}_R\mathcal{M}$ and ${}_S\mathcal{M}$ (see [22]). For P finitely generated and self-tilting, 2.4 yields the equivalence $\mathrm{Hom}_R(P, -) : \mathrm{Gen}(P) \to \mathrm{Cog}(Q)$ shown in [22, 5.5].

2.5. Corollary. *Let $(\mathbf{L}, \mathbf{R}; B)$ be a right pointed pair of adjoint functors and assume that $\mathbf{R}$ is exact and respects coproducts of $P = \mathbf{L}(B)$. Then the functor $\mathbf{R} : \mathrm{Pres}(P) \to \mathcal{B}$ is an equivalence of categories with inverse $\mathbf{L}$.*

Proof. We know from Theorem 2.2 that $\mathbf{R} : \mathrm{Pres}(P) \to \mathrm{Cog}(Q)$ gives an equivalence with inverse $\mathbf{L}$. Notice that, in this case, $\mathrm{Cog}(Q) = \mathrm{Adst}(\mathbf{R})$. Let Y be any object of $\mathcal{B}$. Since B is a generator and $\mathbf{R}$ is exact, there exists a commutative exact diagram of the form

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, "\eta_K"] & B^{(I)} \arrow[r] \arrow[d, "\eta_{B^{(I)}}"] & Y \arrow[r] \arrow[d, "\eta_Y"] & 0 \\
 & \mathbf{RL}(K) \arrow[r] & \mathbf{RL}(B^{(I)}) \arrow[r] & \mathbf{RL}(Y) \arrow[r] & 0.
\end{tikzcd} \tag{9}$$

By 2.1, $\eta_{B^{(I)}}$ is an isomorphism, whence η_K is a monomorphism. Combined with the fact that $\mathbf{RL}(K) \in \mathrm{Cog}(Q)$ this implies that $K \in \mathrm{Cog}(Q)$. Therefore $K \in \mathrm{Adst}(\mathbf{R})$ and η_K is an isomorphism. We get from (9) that η_Y is an isomorphism and, thus, $Y \in \mathrm{Adst}(\mathbf{R})$. We have shown that $\mathcal{B} = \mathrm{Adst}(\mathbf{R})$, which finishes the proof. □

2.6. Theorem. *Assume $\mathcal{B}$ to be a Grothendieck category with generator B and let $(\mathbf{L}, \mathbf{R}; B)$ be a right pointed pair of adjoint functors, $P = \mathbf{L}(B)$ and $Q = \mathbf{R}(A)$ for some cogenerator A of $\mathcal{A}$. Then the following are equivalent:*

(i) $\mathbf{R} : \mathrm{Gen}(P) \to \mathrm{Cog}(Q)$ *is an equivalence;*

(ii) $\mathrm{Gen}(P) = \mathrm{Pres}(P)$ *and* $\mathbf{R}$ *respects coproducts of* P *and the exactness of short exact sequences in* $\mathrm{Gen}(P)$.

Proof. $(ii) \Rightarrow (i)$ is shown in 2.4. Assume (i). Clearly, $\mathrm{Gen}(P) = \mathrm{Pres}(P) = \mathrm{Stat}(\mathbf{R})$. Since $B \in \mathrm{Adst}(\mathbf{R}) = \mathrm{Cog}(Q)$ and this last subcategory of $\mathcal{B}$ is closed under products and subobjects, it follows that $B^{(I)} \in \mathrm{Adst}(\mathbf{R})$ for every index set I (here we use that $B^{(I)}$ is isomorphic to a subobject of B^I by [16, Corollary 2.8.10]). This implies that $\mathbf{R}$ respects coproducts of P (see 2.1). The fact that $\mathbf{R}$ respects exactness of short exact sequences in $\mathrm{Gen}(P)$ follows from the proof of [6, Proposition 1.1] (see also [5, Lemma 1.3]), since $\mathrm{Gen}(P) = \mathbf{L}(\mathrm{Cop}(Q))$ and $\mathrm{Cop}(Q) = \mathrm{Adst}(\mathbf{R})$. □

2.7. Remark. Let P be any object of a complete and co-complete abelian category $\mathcal{A}$ and let $S = \mathrm{End}_{\mathcal{A}}(P)$. The functor $\mathrm{Hom}_{\mathcal{A}}(P,-) : \mathcal{A} \to {}_S\mathcal{M}$ has a left adjoint denoted by $P \otimes_S -$ such that $P \otimes_S S \cong P$ (see, e.g., [17, p.301]). Therefore, if we apply Theorem 2.6 to this situation, we obtain a generalization of [5, Theorem 3.1, $(b) \Leftrightarrow (d)$].

3 Relation with the Hom adjunctions

Let $(\mathbf{L}, \mathbf{R}; B)$ be a right pointed pair of adjoint functors for the categories $\mathcal{A}$ and $\mathcal{B}$. For $P = \mathbf{L}(B)$ we have a ring isomorphism

$$S := \mathrm{End}_{\mathcal{A}}(P) \simeq \mathrm{Hom}_{\mathcal{B}}(B, \mathbf{RL}(B)) \simeq \mathrm{End}_{\mathcal{B}}(B) =: T,$$

and a commutative diagram of functors

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\mathbf{R}"] \arrow[d, "\mathrm{Hom}_{\mathcal{A}}(P,-)"'] & \mathcal{B} \arrow[d, "\mathrm{Hom}_{\mathcal{B}}(B,-)"] \\ \mathcal{M}_S \arrow[r, "\cong"] & \mathcal{M}_T. \end{tikzcd}$$

Recall that an object $P \in \mathcal{A}$ is said to be *self-small* provided $\mathrm{Hom}_{\mathcal{A}}(P,-)$ preserves coproducts of P, and P is *w-Σ–quasi-projective* if $\mathrm{Hom}_{\mathcal{A}}(P,-)$ preserves exactness of sequences

$$0 \longrightarrow K \longrightarrow P^{(\Lambda)} \longrightarrow N \longrightarrow 0 \qquad (*)$$

in $\mathcal{A}$, for any set Λ, where $K \in \mathrm{Gen}(P)$.

3.1. Proposition. *If P is w-Σ–quasi-projective (in $\mathcal{A}$), then $\mathbf{R}$ respects exactness of sequences of type $(*)$.*

The converse is true if B is a projective generator in $\mathcal{B}$.

Proof. This is easily deduced from the fact that $\mathrm{Hom}_{\mathcal{B}}(B,-)$ is a faithful functor (B being a generator). When B is projective, the functor $\mathrm{Hom}_{\mathcal{B}}(B,-)$ is, in addition, exact, which gives the converse. $\square$

3.2. Proposition. *Assume that $\mathcal{B}$ is a Grothendieck category.*

(a) If $\mathbf{R}$ preserves coproducts of P and B is a self-small object, then P is a self-small object.

(b) If P is a self-small object, then $\mathbf{R}$ preserves coproducts of P.

Proof. From the diagram (7) we get that $\eta_{B^{(\Lambda)}}$ is a monomorphism for any set Λ. This gives a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A}}(P, P^{(\Lambda)}) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathcal{B}}(B, \mathbf{RL}(B^{(\Lambda)})) \\
\mathrm{Hom}_{\mathcal{A}}(P, P)^{(\Lambda)} \arrow[u] & \mathrm{Hom}_{\mathcal{B}}(B, B^{(\Lambda)}) \arrow[u, "\mathrm{Hom}_{\mathcal{B}}(B, \eta_{B^{(\Lambda)}})"'] \\
\mathrm{Hom}_{\mathcal{B}}(B, \mathbf{RL}(B))^{(\Lambda)} \arrow[u, "\cong"'] \arrow[r, "\cong"] & \mathrm{Hom}_{\mathcal{B}}(B, B)^{(\Lambda)}, \arrow[u]
\end{tikzcd}$$

which yields our assertions. □

3.3. Theorem. *Let A be a cogenerator for $\mathcal{A}$ and assume $\mathcal{B}$ to be a Grothendieck category. Let $Q = \mathbf{R}(A)$ and $P^* = \mathrm{Hom}_{\mathcal{A}}(P, A)$. If P is w-Σ-projective and self-small (in $\mathcal{A}$), then we have the commutative diagram of equivalences of categories*

$$\begin{tikzcd}
\mathrm{Pres}(P) \arrow[r, "\mathbf{R}", "\cong"'] \arrow[d, "\mathrm{Hom}_{\mathcal{A}}(P,-)"', "\cong"] & \mathrm{Cog}(Q) \arrow[dl, "\cong"', "\mathrm{Hom}_{\mathcal{B}}(B,-)"] \\
\mathrm{Cog}(P^*) &
\end{tikzcd}$$

Proof. Applied to the right pointed pair $(P \otimes_S -, \mathrm{Hom}_{\mathcal{A}}(P, -); S)$, Theorem 2.2 yields that $\mathrm{Hom}_{\mathcal{A}}(P, -) : \mathrm{Pres}(P) \to \mathrm{Cog}(P^*)$ is an equivalence.

In view of Proposition 3.1 and 3.2, we can apply Theorem 2.2 also to the right pointed pair $(\mathbf{L}, \mathbf{R}; B)$ to obtain the equivalence $\mathbf{R} : \mathrm{Pres}(P) \to \mathrm{Cog}(Q)$. This finishes the proof. □

4 Doi-Koppinen modules

In this section we outline some applications of our general results to the theory of generalised Hopf modules, which were introduced in [7] (see also [10]). Let K be a commutative ring and $(H, \Delta_H, \varepsilon)$ a bialgebra over K. Consider a right H-comodule K-algebra A, i.e., the H–comodule structure morphism

$$\varrho_A : A \longrightarrow A \otimes H \tag{10}$$

is a homomorphism of K-algebras. Dually, let (C, Δ_C) be a right H-module K–coalgebra, i.e., the H–module structure morphism

$$\nu_C : C \otimes H \longrightarrow C \tag{11}$$

is a homomorphism of coalgebras.

4.1. Doi-Koppinen modules. To every right A–module N we associate the right $A \otimes H$–module $N \otimes C$ with action given by

$$(n \otimes c)(a \otimes h) = na \otimes ch.$$

Restricting scalars by the algebra morphism (10), we endow $N \otimes C$ with the structure of a right A–module given by

$$(n \otimes c)a = (n \otimes c)\varrho_A(a). \tag{12}$$

Dually, given a right C–comodule L with structure map $\varrho_L : L \longrightarrow L \otimes C$, we have the right $C \otimes H$-comodule structure given by

$$\overline{\varrho}_{L\otimes A} : \ L \otimes A \xrightarrow{\varrho_L \otimes \varrho_A} L \otimes C \otimes A \otimes H \xrightarrow{id \otimes \tau \otimes id} L \otimes A \otimes C \otimes H,$$

where τ denotes, as usual, the 'flip' map. The corestriction functor associated to the coalgebra morphism (11) gives $L \otimes A$ the structure of a C–comodule

$$\varrho_{L\otimes A} : \ L \otimes A \xrightarrow{\overline{\varrho}_{L\otimes A}} L \otimes A \otimes C \otimes H \xrightarrow{id \otimes id \otimes \nu_C} L \otimes A \otimes C \ , \tag{13}$$

where $\nu_C : C \otimes H \to C$ is the right H–module structure map of C.

4.2. Definition. A right (H, A, C)–*Doi-Koppinen module* is a K–module M which is a right C–comodule and a right A–module by the structure maps

$$\varrho_M : M \longrightarrow M \otimes C \ , \quad \nu_M : M \otimes A \longrightarrow M \ ,$$

such that ϱ_M is A–linear (or, equivalently, ν_M is C-colinear). The Doi-Koppinen right modules form an additive category (with arbitrary coproducts) $\mathcal{M}(H)_A^C$, where the morphisms are the K–module homomorphisms which are both A–module and C–comodule homomorphisms. The set (which is in fact a K–module) of all morphisms between two objects M and N in this category will be denoted by $\mathrm{Hom}_A^C(M, N)$.

4.3. $-\otimes A$ as a left adjoint. Let L be a right C–comodule. The K–module $L \otimes A$ has the right A–module structure inherited from A, and the right C–comodule structure

$$\varrho_{L\otimes A} : L \otimes A \longrightarrow L \otimes A \otimes C$$

given in (13). It is easy to see that $L \otimes A$ becomes a Doi-Koppinen module. This leads to an additive functor

$$- \otimes A : \mathcal{M}^C \longrightarrow \mathcal{M}(H)_A^C \ .$$

This functor is left adjoint to the forgetful functor $\mathcal{M}(H)_A^C \longrightarrow \mathcal{M}^C$ (see [3]). In fact, it is easy to see that the mapping

$$\eta_L : L \longrightarrow L \otimes A, \quad l \mapsto l \otimes 1, \tag{14}$$

is a right C–comodule homomorphism. Moreover, if M is a Doi-Koppinen module, then its right A–module map

$$\nu_M : M \otimes A \longrightarrow M \tag{15}$$

is a morphism in the category $\mathcal{M}(H)_A^C$. Some straightforward computations show that (14) and (15) are the unit and counit, respectively, for the mentioned adjunction.

4.4. $\mathcal{M}(H)_A^C$ is a Grothendieck category. Assume C to be flat as a K–module. Then for any morphism $f : M \to N$ in $\mathcal{M}(H)_A^C$, its kernel and co-kernel are right C-comodules and right A-modules. Hence $\mathcal{M}(H)_A^C$ is a co-complete abelian category with exact direct limits which has a generator, i.e., is a Grothendieck category.

Proof. We have to find a generator for $\mathcal{M}(H)_A^C$. For this choose a generator B in $\mathcal{M}^C$ and consider any M in $\mathcal{M}(H)_A^C$. There exists an epimorphism of right C–comodules $B^{(I)} \to M$, for some index set I, yielding an epimorphism $B^{(I)} \otimes A \to M \otimes A$ in $\mathcal{M}(H)_A^C$. Since $B^{(I)} \otimes A \cong (B \otimes A)^{(I)}$ we get that $M \otimes A$ is $B \otimes A$-generated. Finally, $\nu_M : M \otimes A \to M$ is an epimorphism in $\mathcal{M}(H)_A^C$, whence M is $B \otimes A$-generated showing that $B \otimes A$ is a generator and $\mathcal{M}(H)_A^C$. $\square$

We now assume that C is flat over K. Let P be a right (H, A, C)–Doi-Koppinen module and consider the ring $S = \mathrm{End}_A^C(P)$. The functor

$$\mathrm{Hom}_A^C(P, -) : \mathcal{M}(H)_A^C \longrightarrow \mathcal{M}_S$$

has a left adjoint functor $- \otimes_S P : \mathcal{M}_S \longrightarrow \mathcal{M}(H)_A^C$, where for each right S–module X, the A–module and C–comodule structures on $X \otimes_S P$ are inherited from P. In this situation, $(- \otimes_S P, \mathrm{Hom}_A^C(P, -); S)$ is a right pointed pair of adjoint functors. From Theorem 1.6 we have the following

4.5. Proposition. *Let U be any cogenerator for the Grothendieck category $\mathcal{M}(H)_A^C$ and write $Q = \mathrm{Hom}_A^C(P, U)$. The following statements are equivalent.*

(i) $\mathrm{Hom}_A^C(P, \mathrm{Hom}_A^C(P, M) \otimes_S P) \cong \mathrm{Hom}_A^C(P, M)$, *for every M in $\mathcal{M}(H)_A^C$;*

(ii) $\mathrm{Hom}_A^C(P, N \otimes_S P) \otimes_S P \cong N \otimes_S P$, *for every N in $\mathcal{M}_S$;*

(iii) $\mathrm{Hom}_A^C(P, \ker \delta_M) = 0$, *for every M in $\mathcal{M}(H)_A^C$;*

(iv) $(\mathrm{coker}(\eta_N)) \otimes_S P = 0$, *for every N in $\mathcal{M}_S$;*

(v) $\mathrm{Hom}_A^C(P, X \otimes_S P) \cong X$, *for every X in* $\mathrm{Cop}(Q)$*;*

(vi) $\mathrm{Hom}_A^C(P, Y) \otimes_S P \cong Y$, *for every Y in* $\mathrm{Pres}(P)$*;*

(vii) $\mathrm{Hom}_A^C(P, -) : \mathrm{Pres}(P) \to \mathrm{Cop}(Q)$ *is an equivalence of categories with inverse* $- \otimes_S P$.

Theorem 2.2 applied to the right pointed pair of adjoint functors $(-\otimes_S P, \mathrm{Hom}_A^C(P,-); S)$ yields:

4.6. Theorem. *Let U be any cogenerator for the Grothendieck category $\mathcal{M}(H)_A^C$ and write $Q = \mathrm{Hom}_A^C(P,U)$. If $\mathrm{Hom}_A^C(P,-)$ respects coproducts of P and the exactness of the sequences*

$$0 \to K \to P^{(I)} \to X \to 0,$$

where $K \in \mathrm{Gen}(P)$, then $\mathrm{Hom}_A^C(P,-) : \mathrm{Pres}(P) \to \mathrm{Cog}(Q)$ is an equivalence of categories with inverse $- \otimes_S P$.

4.7. Remarks. From Proposition 3.1 and Theorem 2.6 we obtain a right pointed tilting pair $(-\otimes_S P, \mathrm{Hom}_A^C(P,-); S)$, provided $\mathrm{Gen}(P) = \mathrm{Pres}(P)$ and P is a w-Σ–quasi-projective object of $\mathcal{M}(H)_A^C$. In this case we say that P is a *self-tilting* Doi-Koppinen module. Recall from Proposition 3.2 that P is self-small iff the functor $\mathrm{Hom}_A^C(P,-)$ respects coproducts of P (S is a projective generator of ${}_S\mathcal{M}$). Therefore, P is a self-small and self-tilting Doi-Koppinen module iff the equivalence $\mathrm{Hom}_A^C(P,-) : \mathrm{Gen}(P) \to \mathrm{Cog}(Q)$ hold.

Module case. Putting $C = H = K$, the category $\mathcal{M}(H)_A^C$ is exactly the category $\mathcal{M}_A$ of right A-modules, $\mathrm{Hom}_A^C(P,-) = \mathrm{Hom}_A(P,-)$, and $S = \mathrm{End}_A(P)$.

For the right pointed pair of adjoint functors $(P\otimes_S -, \mathrm{Hom}_A(P,-); S)$ for the categories $\mathcal{M}_A$ and $\mathcal{M}_S$, Proposition 4.5 recovers and extends known characterizations (e.g., Sato [18, Theorem 1.3], [4, Proposition 2.7], [22, 5.5]):

4.8. Proposition. *Let P_A be a right A–module and $Q = \mathrm{Hom}_A(P,U)$ the right S-module, where U is a cogenerator of $\mathcal{M}_A$. Then the following are equivalent:*

(i) $\mathrm{Hom}_A(P, \mathrm{Hom}_A(P,M) \otimes_S P) \cong \mathrm{Hom}_A(P,M)$, *for every M in $\mathcal{M}_A$;*

(ii) $\mathrm{Hom}_A(P, N \otimes_S P) \otimes_S P \cong N \otimes_S P$, *for every N in $\mathcal{M}_S$;*

(iii) $\mathrm{Hom}_A(P, \ker \delta_M) = 0$, *for every M in $\mathcal{M}_A$;*

(iv) $(\mathrm{coker}(\eta_N)) \otimes_S P = 0$, *for every N in $\mathcal{M}_S$;*

(v) $\mathrm{Hom}_A(P, X \otimes_S P) \cong X$, *for every X in* $\mathrm{Cop}(Q)$;

(vi) $\mathrm{Hom}_A(P,Y) \otimes_S P \cong Y$, *for every Y in* $\mathrm{Pres}(P)$;

(vii) $\mathrm{Hom}_A(P,-) : \mathrm{Pres}(P) \to \mathrm{Cop}(Q)$ *is an equivalence of categories with inverse $P\otimes_S -$.*

4.9. Remark. With the above notation, $\mathrm{Hom}_A(P,-) : \mathrm{Gen}(P) \to \mathrm{Cog}(Q)$ is an equivalence if and only if P is a self-small and self-tilting module ($(*)$-module).

4.10. Module coalgebras with group-like elements. In this example we will follow [7]. Let H be a Hopf algebra over a ring K and C a right H-module coalgebra which is flat over K and which has a group-like element $x \in C$. The map

$$\pi_x : H \to C, \quad \pi_x(h) = x \leftharpoonup h,$$

is a right H-module coalgebra map, and any right H-comodule algebra A can be viewed as a right C-comodule via π_x. In particular, A is an object of $\mathcal{M}(H)_A^C$.

We consider the subalgebra A_x of A defined by $A_x = \{a \in A | \sum a_0 \otimes \pi_x(a_1) = a \otimes x\}$. More generally, we define M_x for any $M \in \mathcal{M}(H)_A^C$ by

$$M_x = \{m \in M | \rho_M(m) = m \otimes x\},$$

which is a right A_x-module. The functor

$$(-)_x : \mathcal{M}(H)_A^C \to \mathcal{M}_{A_x}, \quad M \longmapsto M_x,$$

has a left adjoint given by $- \otimes_{A_x} A$. We have a right pointed pair of adjoint functors $(- \otimes_{A_x} A, (-)_x; A_x)$. Theorem 1.6 has the following form:

4.11. Theorem. *Let H be a Hopf algebra over a commutative ring K, A a right H-comodule algebra, and C a right H-module coalgebra, flat over K, with a group-like element x. Assume U to be a cogenerator of $\mathcal{M}(H)_A^C$. Then the following conditions are equivalent.*

(i) $(M_x \otimes_{A_x} A)_x \cong M_x$*, for every M in* $\mathcal{M}(H)_A^C$*;*

(ii) $(V \otimes_{A_x} A)_x \cong V$*, for every V in* $\mathrm{Cop}(U_x)$*;*

(iii) $(V \otimes_{A_x} A)_x \otimes_{A_x} A \cong V \otimes_{A_x} A$*, for every V in* $\mathcal{M}_{A_x}$*;*

(iv) $M_x \otimes_{A_x} A \cong M$*, for every M in* $\mathrm{Pres}(A)$*;*

(v) $\mathrm{coker}(\eta_V) \otimes_{A_x} A = 0$*, for every V in* $\mathcal{M}_{A_x}$*;*

(vi) $(\ker \delta_M)_x = 0$*, for every M in* $\mathcal{M}(H)_A^C$*;*

(vii) $(-)_x : \mathrm{Pres}(A) \to \mathrm{Cop}(U_x)$ *is an equivalence with inverse* $- \otimes_{A_x} A$.

4.12. Remarks. (1) The functor $(-)_x$ commutes with coproducts. Therefore, by Corollary 2.5, if $(-)_x$ is an exact functor then $\mathrm{Pres}(A) \sim \mathcal{M}_{A_x}$.

(2) It follows that $(-)_x : \mathcal{M}(H)_A^C \to \mathcal{M}_{A_x}$ is an equivalence of categories if and only if $(-)_x$ is exact and A is a generator for $\mathcal{M}(H)_A^C$.

(3) By applying Theorem 2.6 to the foregoing situation we get that $\mathrm{Gen}(A) \sim \mathrm{Cog}(U_x)$ if and only if $\mathrm{Gen}(A) = \mathrm{Pres}(A)$ and $(-)_x$ is exact on $\mathrm{Gen}(A)$.

4.13. A **as** H**-extension of the invariants** A^{coH}**.**

For $C = H$ and $x = 1_H$ we have $A^{coH} = A_1 \cong \mathrm{End}_A^H(A)$ and $(-)^{coH} \cong Hom_A^H(A, -)$. So $(- \otimes_{A^{coH}} A, (-)^{coH}; A^{coH})$ is a right pointed pair of adjoint functors for the categories $\mathcal{M}_A^H := \mathcal{M}(H)_A^H$ and $\mathcal{M}_{A^{coH}}$. Now Theorem 4.11 has the form:

Corollary. *Let H be a K-Hopf algebra, flat as K-module and A a right H-comodule algebra. If U is a cogenerator of $\mathcal{M}_A^H$, then the following conditions are equivalent.*

(i) $(M^{coH} \otimes_{A^{coH}} A)^{coH} \cong M^{coH}$*, for every* M *in* $\mathcal{M}_A^H$*;*

(ii) $(V \otimes_{A^{coH}} A)^{coH} \cong V$*, for every* V *in* $\mathrm{Cop}(U^{coH})$*;*

(iii) $(V \otimes_{A^{coH}} A)^{coH} \otimes_{A^{coH}} A \cong V \otimes_{A^{coH}} A$*, for every* V *in* $\mathcal{M}_{A^{coH}}$*;*

(iv) $M^{coH} \otimes_{A^{coH}} A \cong M$*, for every* M *in* $\mathrm{Pres}(A)$*;*

(v) $\mathrm{coker}(\eta_V) \otimes_{A^{coH}} A = 0$*, for every* V *in* $\mathcal{M}_{A^{coH}}$*;*

(vi) $(\ker \delta_M)^{coH} = 0$*, for every* M *in* $\mathcal{M}_A^H$*;*

(vii) $(-)^{coH} : \mathrm{Pres}(A) \to \mathrm{Cop}(U^{coH})$ *is an equivalence with inverse* $- \otimes_{A^{coH}} A$*.*

4.14. Remarks. (1) An algebra extension A/B is called an *H-extension* if A is a H-comodule algebra and B is its *invariant subalgebra* $A_1 = \{a \in A | \rho_A = a \otimes 1\}$. Recall that a *total integral* $\phi : H \to A$ is an H-comodule map with $\phi(1_H) = 1_A$. It is well known that for an H-extension A/B with total integral the canonical map $V \to (V \otimes_B A)_1, v \to v \otimes 1$, is an isomorphism, for every V in $\mathcal{M}_B$. By the above Corollary, the equivalence $\mathrm{Pres}(A) \sim \mathcal{M}_B$ holds.

When A is a left faithfully flat H-Galois extension of the invariants B (i.e. A is a projective generator in $\mathcal{M}_A^H$) we obtain the classical equivalence of Schneider $\mathcal{M}_A^H \sim \mathcal{M}_B$ (see [19, Theorem 1]).

(2) If $C = A = H$ and H is projective as a K-module, then H is a projective generator in $\mathcal{M}_H^H$, $\mathrm{End}_H^H(H) \cong K$, $(-)_1 \cong Hom_H^H(H, -)$, and for every $M \in \mathcal{M}_H^H$,

$$M_1 \otimes_K H \to M, \quad m \otimes h \mapsto mh,$$

is a Hopf module isomorphism and $\mathcal{M}_H^H \sim \mathcal{M}_K$ (Fundamental Theorem for Hopf modules).

5 Graded rings and modules

Let G be a group with neutral element e. For a G-graded ring R, we will denote by $R - gr$ the category of all G-graded unital left R-modules. Our basic reference for graded rings and modules is [13]. Let $x \in G$ and M be a graded left R–module. We denote by $M(x)$ the module M endowed with the new grading given by $M(x)_y = M_{yx}$ for every $y \in G$. This gives the so called x–suspension functor. For $P, Q \in R$-*gr* we consider the G-graded abelian group $\mathrm{HOM}_R(P, Q)$, whose x-th homogeneous component is

$$\mathrm{HOM}_R(P, Q)_x = \{f \in \mathrm{Hom}_R(P, Q) \mid f(P_y) \subseteq Q_{yx}, \text{ for all } y \in G\}.$$

Clearly, $\mathrm{HOM}_R(P, Q)_x = \mathrm{Hom}_{R-gr}(P, Q(x))$. If $Q = P$ then $S = \mathrm{HOM}_R(P, P) = \mathrm{END}_R(P)$ is a G-graded ring and P is a graded (R, S)-bimodule.

For a graded left S-module N, $P \otimes_S N$ is a left R-module that can be graded by

$$(P \otimes_S N)_x := \{\sum_{yz=x} p_y \otimes n_z \mid p_y \in P_y, n_z \in N_z\}.$$

This provides two functors which are graded, i.e. commute with the x-suspension functor, for every $x \in G$,

$$\mathrm{HOM}_R(P, -) : R\text{-}gr \to S\text{-}gr, \text{ and } P \otimes_S - : S\text{-}gr \to R\text{-}gr.$$

It is well known that the functor $P \otimes_S -$ is left adjoint to $\mathrm{HOM}_R(P, -)$. Recall that $\bigoplus_{x\in G} S(x)$ is a generator of S-gr.

Let M be a graded left R–module. The functor $\mathrm{HOM}_R(P, -)$ preserves coproducts of copies of M if and only if $\mathrm{Hom}_{R-gr}(P, -)$ preserves coproducts of $M(x)$ for every $x \in G$. In particular, $\mathrm{HOM}_R(P, -)$ preserves coproducts of $\bigoplus_{x\in G} P(x)$ if and only if $\mathrm{Hom}_{R-gr}(P, -)$ does. In this case we say that P is *gr-self-small*.

5.1. Lemma. *If $P \in R$-gr is gr-self-small then $(P \otimes_S -, \mathrm{HOM}_R(P, -); \bigoplus_{x\in G} S(x))$ is a right pointed pair of adjoint functors.*

Proof. We prove that $\bigoplus_{x\in G} S(x)$ is $\mathrm{HOM}_R(P, -)$-adstatic. Since $\mathrm{HOM}_R(P, -)$ preserves coproducts of $\bigoplus_{x\in G} P(x)$ and the two functors are graded and the functor $P \otimes_S -$ commutes with direct sums, we have

$$\begin{aligned} \mathrm{HOM}_R(P, P \otimes_S (\textstyle\bigoplus_{x\in G} S(x))) &\cong \mathrm{HOM}_R(P, \textstyle\bigoplus_{x\in G}(P \otimes_S S(x)) \\ &\cong \mathrm{HOM}_R(P, \textstyle\bigoplus_{x\in G}(P \otimes_S S)(x)) \\ &\cong \mathrm{HOM}_R(P, \textstyle\bigoplus_{x\in G} P(x)) \\ &\cong \textstyle\bigoplus_{x\in G} \mathrm{HOM}_R(P, P(x)) \cong \textstyle\bigoplus_{x\in G} S(x). \end{aligned}$$

□

By Proposition 3.1, if $P \in R$-gr is gr-w-$\sum$-quasiprojective, then $\mathrm{HOM}_R(P, -)$ respects exactness of sequences

$$0 \to K \to (\bigoplus_{x\in G} P(x))^{(I)} \to X \to 0,$$

where $K \in \mathrm{Gen}(\bigoplus_{x\in G} P(x))$. Now Theorem 2.2 yields for graded modules:

5.2. Theorem. *If $P \in R$-gr is gr-self-small and gr-w-$\sum$-quasi-projective then*

$$\mathrm{HOM}_R(P, -) : \mathrm{Pres}(\bigoplus_{x\in G} P(x)) \to \mathrm{Cog}(P^*)$$

is an equivalence of categories where $P^* = \mathrm{HOM}_R(P, A)$, for any cogenerator A of R-gr.

5.3. Definition. $P \in R$-gr is said to be *gr-self-tilting* provided P is gr-w-$\sum$-quasi-projective and $\mathrm{Pres}(\bigoplus_{x\in G} P(x)) = \mathrm{Gen}(\bigoplus_{x\in G} P(x))$.

By Theorem 5.2, for a gr-self-small and gr-self-tilting module P, we have an equivalence graded functor

$$\mathrm{HOM}_R(P, -) : \mathrm{Gen}(\bigoplus_{x\in G} P(x)) \to \mathrm{Cog}(P^*).$$

6 Left pointed (tilting) pairs

Recall that the *dual or opposite category* $\mathcal{C}^{op}$ of a category $\mathcal{C}$ is the category whose objects are the objects of $\mathcal{C}$ and $\mathrm{Hom}_{\mathcal{C}^{op}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X)$. Every functor $\mathbf{F} : \mathcal{C} \to \mathcal{D}$ can be considered as a functor $\mathbf{F}^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$ in the most natural way. Let $\mathbf{R} : \mathcal{A} \to \mathcal{B}$ be an additive functor between complete and cocomplete abelian categories which has a left adjoint $\mathbf{L} : \mathcal{B} \to \mathcal{A}$. Clearly, $\mathbf{L}^{op}$ is right adjoint to $\mathbf{R}^{op}$. The 'duality principle' in Category Theory allows to state without proofs a number of results which are dual to that we have proved in sections 1 and 2.

Definition. A *left pointed pair of adjoint functors* $(A; \mathbf{L}, \mathbf{R})$ for the categories $\mathcal{A}$ and $\mathcal{B}$ consists of an additive functor $\mathbf{R} : \mathcal{A} \to \mathcal{B}$ with a left adjoint $\mathbf{L} : \mathcal{B} \to \mathcal{A}$, and a cogenerator A for $\mathcal{A}$ which is $\mathbf{L}$-adstatic, i.e., $A \simeq \mathbf{LR}(A)$. Clearly, $(A; \mathbf{L}, \mathbf{R})$ is a left pointed pair if and only if $(\mathbf{R}^{op}, \mathbf{L}^{op}; A)$ is a right pointed pair of adjoint functors.

6.1. Product preserving functors. Let $\mathbf{L} : \mathcal{B} \to \mathcal{A}$ be an additive functor. Given a direct product Q^I of copies of $Q \in \mathcal{B}$, consider the canonical projection $\pi_i : Q^I \to Q$ for each $i \in I$. The family of morphisms

$$\{\mathbf{L}(\pi_i) : \mathbf{L}(Q^I) \to \mathbf{L}(Q) \mid i \in I\}$$

induces a canonical homomorphism $\Theta : \mathbf{L}(Q^I) \to \mathbf{L}(Q)^I$. Assume $(A; \mathbf{L}, \mathbf{R})$ to be a left pointed pair and put $Q = \mathbf{R}(A)$. By 2.1, if $\mathbf{L}$ respects products of Q then any product of copies of Q is $\mathbf{L}$–static.

Theorem 2.2 can be re-stated as the following dual tilting theorem.

6.2. Theorem. *Let $(A; \mathbf{L}, \mathbf{R})$ be a left pointed pair of adjoint functors for $\mathcal{A}$ and $\mathcal{B}$, and let B be any generator for $\mathcal{B}$. Put $P = \mathbf{L}(B)$, $Q = \mathbf{R}(A)$ and assume the following conditions.*

(1) $\mathbf{L}$ respects products of Q.

(2) The functor $\mathbf{L}$ respects the exactness of the sequences $0 \to X \to Q^I \to C \to 0$, where $C \in \mathrm{Cog}(Q)$.

Then $\mathbf{L} : \mathrm{Cop}(Q) \to \mathrm{Gen}(P)$ is an equivalence of categories with inverse $\mathbf{R}$.

Definition. A left pointed pair $(A; \mathbf{L}, \mathbf{R})$ is said to be a *left pointed tilting pair* provided $\mathrm{Cog}(Q) = \mathrm{Cop}(Q)$ and $\mathbf{L}$ is exact on short exact sequences in $\mathrm{Cog}(Q)$.

As an immediate consequence of Theorem 6.2 we have:

6.3. Proposition. *Let $(A; \mathbf{L}, \mathbf{R})$ be a left pointed tilting pair for $\mathcal{A}$ and $\mathcal{B}$ and assume that $\mathbf{L}$ preserves products of $Q = \mathbf{R}(A)$. Then $\mathbf{L} : \mathrm{Cog}(Q) \to \mathrm{Gen}(P)$ is an equivalence of categories with inverse $\mathbf{R}$, where $P = \mathbf{L}(B)$ for some generator B of $\mathcal{B}$.*

6.4. Corollary. *Let $(A; \mathbf{L}, \mathbf{R})$ be a left pointed pair and assume that $\mathbf{L}$ is exact and respects products of $Q = \mathbf{R}(A)$. Then the functor $\mathbf{L} : \mathrm{Cop}(Q) \to \mathcal{A}$ is an equivalence of categories with inverse $\mathbf{R}$.*

7 Comodules

Let C be a coalgebra over a commutative ring K with unity. If C is flat as a K-module, then the category $\mathcal{M}^C$ of all right C–comodules is a Grothendieck category (see [25]). When C is K–projective, the category $\mathcal{M}^C$ is isomorphic to the category $\sigma[{}_{C^*}C]$ of all left C^*–modules subgenerated by ${}_{C^*}C$ (see [8] and [25]). A right C–comodule Q is said to be *quasi-finite* if the functor $-\otimes Q : {}_R\mathcal{M} \to \mathcal{M}^C$ has a left adjoint. As for the case of coalgebras over fields [21], a quasi-finite right C–comodule Q_C gives rise to a co-hom functor $\mathrm{h}_C(Q,-) : \mathcal{M}^C \to \mathcal{M}^D$ which is the left adjoint to the cotensor functor $-\Box_D Q : \mathcal{M}^D \to \mathcal{M}^C$, where $D = e_C(Q) = h_C(Q,Q)$ is the co-endomorphism coalgebra (see [2]). We keep this notation in the rest of this section.

In general the coalgebra D need not be a cogenerator for $\mathcal{M}^D$. However, this is the case when the ground ring K is quasi-Frobenius. Then $(D_D; \mathrm{h}_C(Q,-), -\Box_D Q)$ is a left pointed pair, and Theorem 1.6 yields

7.1. Theorem. *Assume that the ground ring K is QF and let $P = \mathrm{h}_C(Q,G)$, where G is a generator for $\mathcal{M}^C$. The following conditions are equivalent.*

(i) $\mathrm{h}_C(Q, \mathrm{h}_C(Q,M)\Box_D Q) \cong \mathrm{h}_C(Q,M)$*; for every M in $\mathcal{M}^C$,*

(ii) $\mathrm{h}_C(Q, X\Box_D Q) \cong X$*, for every X in* $\mathrm{Pres}(P)$*;*

(iii) $\mathrm{h}_C(Q, N\Box_D Q)\Box_D Q \cong N\Box_D Q$*, for every N in $\mathcal{M}^D$;*

(iv) $\mathrm{h}_C(Q,Y)\Box_D Q \cong Y$*, for every Y in* $\mathrm{Cop}(Q)$*;*

(v) $\ker \delta_Y \Box_D Q = 0$*, for every Y in $\mathcal{M}^D$;*

(vi) $\mathrm{h}_C(Q, \mathrm{coker}(\eta_X)) = 0$*, for every X in $\mathcal{M}^C$;*

(vii) $\mathrm{h}_C(Q,-) : \mathrm{Cop}(Q) \to \mathrm{Pres}(P)$ *is an equivalence of categories with inverse $-\Box_D Q$.*

From Corollary 1.8 we obtain

7.2. Corollary. *Let Q_C be a quasi-finite comodule. The following are equivalent.*

1. $\delta : \mathrm{h}_C(Q, -\Box_D Q) \to 1_{\mathcal{M}^C}$ *is a natural isomorphism.*
2. *The functor* $\mathrm{h}_C(Q,-) : \mathrm{Cop}(Q) \to \mathcal{M}^D$ *is an equivalence of categories with inverse $-\Box_D Q$.*

Our next aim is to apply Theorem 6.2 to comodule categories.

7.3. Lemma. *Let Q_C be a quasi-finite right C–comodule and let*

$$0 \longrightarrow X \longrightarrow Y \longrightarrow X \longrightarrow 0 \qquad (S)$$

be an exact sequence of right C–comodules. The following conditions are equivalent.

(i) The functor $\mathrm{h}_C(Q,-)$ *respects exactness of (S).*

(ii) For every injective K–module W, the functor $\mathrm{Hom}^C(-, W \otimes Q)$ *respects exactness of (S).*

(iii) The functor $\mathrm{Hom}^C(-, E \otimes Q)$ *respects the exactness of (S), for some injective cogenerator E of ${}_K\mathcal{M}$.*

Proof. A slight modification of the proof of [2, Proposition III.2.7] runs here. □

7.4. Let Q be a quasi-finite right C–comodule. In [2, Definition III.2.8] the comodule Q is said to be an *injector* if $W \otimes Q$ is an injective right C–comodule for every injective K–module W or, equivalently, if the functor $\mathrm{h}_C(Q,-)$ is exact. This suggests the following definition.

Definition. The comodule Q is said to be a *w-injector* if the functor $\mathrm{h}_C(Q,-)$ respects exactness of all the exact sequences of the form

$$0 \longrightarrow X \longrightarrow Q^I \longrightarrow M \longrightarrow 0 \qquad (S')$$

where I is any index set and $M \in \mathrm{Cog}(Q)$. By Lemma 7.3 this is equivalent to require that for every injective K–module W, the functor $\mathrm{Hom}^C(-, W \otimes Q)$ respects the exactness of all the sequences of the form (S').

In case the ground ring K is quasi-Frobenius, Lemma 7.3 implies (take $E = K$) that Q is a w–injector if and only if it is w–Π-*quasi-injective* in the sense of [24]. In this case the right C-comodule Q is said to be *self-cotilting* if Q is w-injector and all Q-cogenerated comodules are Q-presented.

7.5. Definition. A quasi-finite right C–comodule Q is said to be *self-co-small* if Q is $\mathrm{h}_C(Q,-)$–co-small, i.e., if the canonical D–colinear homomorphism $\mathrm{h}_C(Q, Q^I) \to \mathrm{h}_C(Q,Q)^I$ is an isomorphism for every index set I (here, $D = \mathrm{Coend}_C(Q)$).

If Q_C is an injector, then, by [2, Satz III.4.7], the functor $\mathrm{h}_C(Q,-)$ is naturally isomorphic to the functor $-\Box_C \mathrm{h}_C(Q,C)$. In particular, $\mathrm{h}_C(Q,-) : \mathcal{M}^C \to \mathcal{M}^D$ preserves arbitrary direct products. This shows that every quasi-finite injector is self-co-small.

We are now ready to formulate Theorem 6.2 for comodule categories when the ground ring K is quasi-Frobenius.

7.6. Theorem. *Let Q be a quasi-finite self-co-small w–π–quasi-injective right C–comodule and let $P = \mathrm{h}_C(Q,G)$, where G is any generator of $\mathcal{M}^C$. If the ground ring K is quasi-Frobenius, then the functor*

$$\mathrm{h}_C(Q,-) : \mathrm{Cop}(Q) \to \mathrm{Gen}(P)$$

is an equivalence of categories with inverse $-\Box_D Q$.

7.7. Corollary. *Let Q be a quasi-finite injector. If the ground ring K is quasi-Frobenius then the functor*

$$\mathrm{h}_C(Q, -) : \mathrm{Cop}(Q) \to \mathcal{M}^D$$

is an equivalence of categories with inverse $-\Box_D Q$.

Proof. A quasi-finite injector is automatically self-co-small. Apply Corollary 6.4. □

7.8. Remark. (1) If the ground ring K is not assumed to be quasi-Frobenius, then our general theory can be applied whenever we assume that there exists an adstatic cogenerator A for the category $\mathcal{M}^D$. For the moment, we do not know if such cogenerator is always on hand.

(2) For K quasi-Frobenius, if Q_C is a quasi-finite *self-co-small* and *self-cotilting* comodule then the functor

$$\mathrm{h}_C(Q, -) : \mathrm{Cog}(Q) \to \mathrm{Gen}(P)$$

is an equivalence of categories with inverse $-\Box_D Q$.

(3) If the quasi-finite comodule Q_C is an injector cogenerator (K quasi-Frobenius) then the functor $\mathrm{h}_C(Q, -) : \mathcal{M}^C \to \mathcal{M}^D$ is an equivalence [2, Satz IV.1.4]. If K is a field and Q_C is a quasi-finite injective cogenerator then we obtain the equivalence of Takeuchi $\mathrm{h}_C(Q, -) : \mathcal{M}^C \to \mathcal{M}^D$ [21, Theorem 3.5].

7.9. Module coalgebras. In this example, we follow [7]. Let C be a right H-module coalgebra which is flat over K, and A a right H-comodule algebra. Assume that there is an algebra map $\alpha : A \to K$. In [7, (1.9)] an additive functor $(-)^\alpha : \mathcal{M}(H)_A^C \to \mathcal{M}^{C^\alpha}$ is constructed as follows:

Define the map $\theta_\alpha : A \to H$ by $\theta_\alpha(a) = \sum \alpha(a_0)a_{-1}$, which is a right H-comodule algebra map. Moreover C is an objet in $\mathcal{M}(H)_A^C$ via this θ_α. Write $A^+ = ker(\alpha)$. Then $CA^+ = C \leftharpoonup A^+$ is a coideal of C. Define $C^\alpha = C/CA^+$ and $M^\alpha = M/MA^+$, for any M in $\mathcal{M}(H)_A^C$. Then C^α has a unique coalgebra structure such that the projection $p : C \to C^\alpha$ is a coalgebra map. M^α has a unique comodule structure $\rho^\alpha : M^\alpha \to M^\alpha \otimes C^\alpha$.

$(-)^\alpha$ is a functor from $\mathcal{M}(H)_A^C$ to $\mathcal{M}^{C^\alpha}$, the category of right C^α-comodules. This functor has a right adjoint given by the cotensor product $-\Box_{C^\alpha} C : \mathcal{M}^{C^\alpha} \to \mathcal{M}_A^C$. If K is a QF ring, then $(C^\alpha; (-)^\alpha, -\Box_{C^\alpha} C)$ is a left pointed pair. From Theorem 1.6 we obtain the following result.

7.10. Theorem. *Let C be a right H-module coalgebra which is flat over a QF ring K. Let A be a right H-comodule algebra with an algebra map $\alpha : A \to K$. If G is a generator of $\mathcal{M}(H)_A^C$, the following are equivalent.*

(i) $(M^\alpha \Box_{C^\alpha} C)^\alpha \cong M^\alpha$ *for every M in $\mathcal{M}(H)_A^C$,*

(ii) $(X \Box_{C^\alpha} C)^\alpha \cong X$ *for every X in* $\mathrm{Pres}(G^\alpha)$*,*

(iii) $(X\Box_{C^\alpha}C)^\alpha\Box_{C^\alpha}C \cong X\Box_{C^\alpha}C$ *for every* X *in* $\mathcal{M}^{C^\alpha}$*,*

(iv) $M^\alpha\Box_{C^\alpha}C \cong M$ *for every* M *in* $\mathrm{Cop}(C)$*.*

(v) $(-)^\alpha : \mathrm{Cop}(C) \to \mathrm{Pres}(G^\alpha)$ *is an equivalence of categories with inverse* $-\Box_{C^\alpha}C$*.*

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