

## ON UNIFORM BOUNDS OF PRIMENESS IN MATRIX RINGS

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### Abstract

A subset  $\mathcal{S}$  of an associative ring  $R$  is a *uniform insulator* for  $R$  provided  $a\mathcal{S}b \neq 0$  for any nonzero  $a, b \in R$ . The ring  $R$  is called *uniformly strongly prime of bound  $m$*  if  $R$  has uniform insulators and the smallest of those has cardinality  $m$ . Here we compute these bounds for matrix rings over fields and obtain refinements of some results of van den Berg in this context.

More precisely, for a field  $F$  and a positive integer  $k$ , let  $m$  be the bound of the matrix ring  $M_k(F)$ , and let  $n$  be  $\dim_F(\mathcal{V})$ , where  $\mathcal{V}$  is a subspace of  $M_k(F)$  of maximal dimension with respect to not containing rank one matrices. We show that  $m + n = k^2$ . This implies, for example, that  $n = k^2 - k$  if and only if there exists a (nonassociative) division algebra over  $F$  of dimension  $k$ .

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### 1. Introduction

Following Handelman and Lawrence [1, page 211], we call a subset  $\mathcal{S}$  of an associative ring  $\mathcal{R}$  a *uniform insulator* for  $\mathcal{R}$  if  $a\mathcal{S}b \neq 0$  for all  $a, b \in \mathcal{R}$  with  $a \neq 0 \neq b$ . The ring  $\mathcal{R}$  is said to be *uniformly strongly prime* if it contains a finite uniform insulator. For such a ring we set  $m(\mathcal{R}) = \min\{|\mathcal{S}| \mid \mathcal{S} \text{ is a uniform insulator of } \mathcal{R}\}$ , and we say  $\mathcal{R}$  is *uniformly strongly prime of bound  $n$*  provided  $m(\mathcal{R}) = n$ .

In what follows  $F$  is a field and  $M_k(F)$  stands for the algebra of  $k \times k$  matrices over  $F$ , where  $k$  is a positive integer. Note that  $M_k(F)$  is always uniformly strongly prime in view of [2, Theorem 3] (or [3, Theorem 1]). For  $\mathcal{R} = M_k(F)$  we put  $m_k(F) := m(\mathcal{R})$ .

The systematic study of  $m(\mathcal{R})$  was initiated by van den Berg in [2, 3] and we recall the following of his results ([3, Theorems 4, 7, 11]).

THEOREM 1.1.

- (i) Let  $\mathcal{D}$  be a division ring and  $\mathcal{R} = M_k(\mathcal{D})$ . Then  $k \leq m(\mathcal{R}) \leq 2k - 1$ .
- (ii) If  $F$  is an algebraically closed field, then  $m_k(F) = 2k - 1$ .
- (iii) Let  $F$  be a field and assume there exists a nonassociative division  $F$ -algebra of dimension  $k$ , then  $m_k(F) = k$ .

In [3, Remark 2], van den Berg asks if the converse of assertion (iii) holds. In the present paper we obtain a positive answer to this question (see 1.4(iii)). We sharpen the above results by studying connections of the uniform bound of  $M_k(F)$  with (maximal) dimension of certain subspaces of  $M_k(F)$  and  $M_{k^2}(F)$ . We also pose some open questions.

Before stating our results we fix some notation. Given positive integers  $k, \ell$  we denote by  $M_{k,\ell}(F)$  the  $k \times \ell$ -matrices over the field  $F$ .

For  $A = (a_{ij})_{1 \leq i \leq k, 1 \leq j \leq \ell} \in M_{k,\ell}(F)$  and  $B \in M_{\ell,k}(F)$ , we define

$$A \bullet B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1\ell}B \\ a_{21}B & a_{22}B & \cdots & a_{2\ell}B \\ \cdots & \cdots & \cdots & \cdots \\ a_{k1}B & a_{k2}B & \cdots & a_{k\ell}B \end{pmatrix} \in M_{k\ell}(F).$$

If  $\ell = 1$ , then  $A \bullet B = AB$ , and it is known that a matrix  $C \in M_k(F)$  has rank one if and only if there exist nonzero matrices  $A \in M_{k,1}(F)$  and  $B \in M_{1,k}(F)$  such that  $C = AB = A \bullet B$ .

If  $\ell = k$ , it is well known that  $\phi : M_k(F) \otimes_F M_k(F) \rightarrow M_{k^2}(F)$ , the linear extension of the map  $A \otimes B \mapsto A \bullet B$ , is an algebra isomorphism.

With this in mind we introduce the following entities which will be helpful for our purposes:

$$n_k(F) = \max \left\{ \dim_F(\mathcal{V}) \mid \begin{array}{l} \mathcal{V} \text{ is a subspace of } M_k(F) \text{ and} \\ \mathcal{V} \cap \{M_{k,1}(F) \bullet M_{1,k}(F)\} = 0 \end{array} \right\},$$

$$\ell_k(F) = \max \left\{ \dim_F(\mathcal{K}) \mid \begin{array}{l} \mathcal{K} \subseteq M_{k^2}(F) \text{ is a left ideal and} \\ \mathcal{K} \cap \{M_k(F) \bullet M_k(F)\} = 0 \end{array} \right\}.$$

We are now in a position to state the main results of the present paper.

THEOREM 1.2. *Given a field  $F$  and positive integer  $k$ , we have:*

- (i)  $m_k(F) = 2k - 1$ , for all  $k$ , if and only if  $F$  is algebraically closed.
- (ii)  $m_k(F) = k$  if and only if there exists a nonassociative division  $F$ -algebra of dimension  $k$ .

The above result sharpens (ii) and (iii) in Theorem 1.1. We note that the theorem is essentially a corollary to van den Berg's results. The next observations provide relationships between the dimensions under consideration.

**THEOREM 1.3.** *Given a field  $F$  and positive integer  $k$ , we have  $m_k(F) + n_k(F) = k^2$  and  $\ell_k(F) = k^2 \cdot n_k(F)$ .*

We list some immediate implications.

**COROLLARY 1.4.** *Let  $\mathcal{V}$  be a  $k$  dimensional vector space over a field  $F$  and let  $\overline{F}$  be the algebraic closure of  $F$ . Then:*

- (i)  $k^2 - 2k + 1 \leq n_k(F) \leq k^2 - k$ .
- (ii)  $n_k(F) = k^2 - 2k + 1$ , for all  $k$ , if and only if  $F$  is algebraically closed.
- (iii)  $n_k(F) = k^2 - k$  if and only if there exists a nonassociative division  $F$ -algebra of dimension  $k$ .
- (iv) A subspace  $\mathcal{W} \subset M_k(F)$  contains a rank one matrix, provided  $\dim_F(\mathcal{W}) > k^2 - k$ , or  $F = \overline{F}$  and  $\dim_F(\mathcal{W}) > k^2 - 2k + 1$ .
- (v) A subspace  $\mathcal{W} \subset \mathcal{V} \otimes_F \mathcal{V}$  contains a non-zero element of the form  $A \otimes B$  for some  $A, B \in \mathcal{V}$ , provided  $\dim(\mathcal{W}) > k^2 - k$ , or  $F = \overline{F}$  and  $\dim(\mathcal{W}) > k^2 - 2k + 1$ .

**PROOF.** (i) follows at once from Theorem 1.1 and Theorem 1.3. (ii) and (iii) are immediate consequences of Theorem 1.2(ii) together with Theorem 1.3. (iv) follows from (i) and (ii). Clearly  $\mathcal{V} \cong M_{k1}(F)$  and  $\mathcal{V} \cong M_{1k}(F)$  as vector spaces. Next, the linear extension of the map  $A \otimes B \mapsto AB$ ,  $A \in M_{k1}(F)$ ,  $B \in M_{1k}(F)$ , is an isomorphism of vector spaces  $M_{k1}(F) \otimes_F M_{1k}(F) \rightarrow M_k(F)$ . Therefore there exists an isomorphism  $\mathcal{V} \otimes_F \mathcal{V} \rightarrow M_k(F)$  of vector spaces sending vectors of the form  $v \otimes u$  to matrices of rank 1. The result now follows from (iv).  $\square$

## 2. Proof of the main theorems

Given a division ring  $\mathcal{D}$  and a positive integer  $k$ , we denote by  $GL(k; \mathcal{D})$  the group of invertible  $k \times k$  matrices over  $\mathcal{D}$ . We need the following result.

**COROLLARY 2.1** ([3, Corollary 5]). *The following assertions are equivalent for a division ring  $\mathcal{D}$  and a positive integer  $k$ :*

- (i)  $M_k(\mathcal{D})$  is uniformly strongly prime of bound  $k$ .
- (ii)  $GL(k; \mathcal{D}) \cup \{0\}$  contains a  $k$ -dimensional  $\mathcal{D}$ -subspace of  $M_k(\mathcal{D})$ .

Recall that a nonassociative  $F$ -algebra  $\mathcal{D}$  is said to be a *division algebra* provided that for any  $a, b \in \mathcal{D}$  with  $a \neq 0$  both equations  $ax = b$  and  $ya = b$  have unique solutions in  $\mathcal{D}$ . We are now in a position to prove Theorem 1.2.

**PROOF OF THEOREM 1.2.** (i) If  $F$  is algebraically closed, then  $m_k(F) = 2k - 1$  by Theorem 1.1. Conversely, if  $F$  is not algebraically closed, then it has a finite extension  $\mathcal{E}$  of dimension  $k > 1$ . Therefore,  $m_k(F) = k < 2k - 1$  by Theorem 1.1 (iii).

(ii) If there exists a nonassociative division  $F$ -algebra of dimension  $k$ , then  $m_k(F) = k$  by Theorem 1.1(iii). Conversely, assume that  $m_k(F) = k$ . Then Corollary 2.1 yields that  $GL(k; F) \cup \{0\}$  contains a  $k$ -dimensional  $F$ -subspace  $\mathcal{V}$  of  $M_k(F)$ . Considering  $M_k(F)$  as the endomorphism algebra of the vector space  $\mathcal{V}$ , we define a product  $\cdot : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$  by the rule  $AB = A(B)$  for all  $A, B \in \mathcal{V}$ . We claim that  $(\mathcal{V}, \cdot)$  is a nonassociative division algebra over  $F$  of dimension  $k$ . Indeed, let  $A, B \in \mathcal{V}$  with  $A \neq 0$ . Consider the map  $\phi : \mathcal{V} \rightarrow \mathcal{V}$  given by  $\phi(X) = XA = X(A)$ . Clearly  $\phi$  is an endomorphism of the vector space  $\mathcal{V}$ . Since  $\mathcal{V} \setminus \{0\} \subseteq GL(k; F)$  and  $A \neq 0$ ,  $X(A) \neq 0$  for all  $X \in \mathcal{V}$  with  $X \neq 0$ . That is  $\ker(\phi) = 0$  and so  $\phi$  is an automorphism of  $\mathcal{V}$ . In particular, there exists a unique  $Y \in \mathcal{V}$  such that  $YA = B$ . Finally, since  $A \in GL(k; F)$ , there exists a unique  $X \in \mathcal{V}$  with  $AX = A(X) = B$ . Thus  $(\mathcal{V}, \cdot)$  is a nonassociative division algebra and the proof is complete.  $\square$

Let  $\text{tr}_k : M_k(F) \rightarrow F$  be the trace map. Given a subspace  $\mathcal{W} \subseteq M_k(F)$ , we set

$$\mathcal{W}^\perp = \{A \in M_k(F) \mid \text{tr}_k(A\mathcal{W}) = 0\}.$$

Given  $A \in M_{k,\ell}(F)$  and  $B \in M_{\ell,k}(F)$ , one can easily check that

$$(1) \quad \text{tr}_k(AB) = \text{tr}_\ell(BA).$$

**LEMMA 2.2.** *Let  $\mathcal{W} \subseteq M_k(F)$  be a subspace containing no rank one matrices. Then any basis of  $\mathcal{W}^\perp$  is a uniform insulator for  $M_k(F)$ . Conversely, let  $\mathcal{S}$  be a uniform insulator for  $M_k(F)$  and let  $\mathcal{V} = \sum_{A \in \mathcal{S}} FA$ . Then  $\mathcal{V}^\perp$  contains no rank one matrices.*

**PROOF.** It is well known that the map  $(A, B) \mapsto \text{tr}_k(AB)$ ,  $A, B \in M_k(F)$ , is a nondegenerate symmetric bilinear form. Therefore,

$$(2) \quad \dim_F(\mathcal{U}) + \dim_F(\mathcal{U}^\perp) = k^2 \quad \text{and} \quad \{\mathcal{U}^\perp\}^\perp = \mathcal{U}$$

for any subspace  $\mathcal{U} \subseteq M_k(F)$ .

Let  $\mathcal{W}$  be as in the lemma and let  $\mathcal{S}$  be a basis of  $\mathcal{W}^\perp$ . Given  $0 \neq A \in M_{k,1}(F)$  and  $0 \neq B \in M_{1,k}(F)$ ,  $AB \in M_k(F)$  has rank one and so  $AB \notin \mathcal{W} = \{\mathcal{W}^\perp\}^\perp$  forcing  $0 \neq \text{tr}_k(ABX)$  for some  $X \in \mathcal{S}$ . Making use of (1), we conclude that  $BXA = \text{tr}_1(BXA) \neq 0$ . We see that  $B\mathcal{S}A \neq 0$  for all  $0 \neq A \in M_{k,1}(F)$  and  $0 \neq B \in M_{1,k}(F)$ . Now let  $P, Q \in M_k(F)$  be nonzero. Write

$$P = \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_k \end{pmatrix} \quad \text{and} \quad Q = (Q^1, Q^2, \dots, Q^k),$$

where  $P_i \in M_{1,k}(F)$  and  $Q^j \in M_{k,1}(F)$ . Then  $PXQ = (P_i X Q^j)_{i,j=1}^k$  for all  $X \in \mathcal{S}$  and so  $P\mathcal{S}Q \neq 0$ . Therefore  $\mathcal{S}$  is a uniform insulator for  $M_k(F)$ .

Now let  $\mathcal{S}$  and  $\mathcal{V}$  be as in the lemma. Assume to the contrary that  $\mathcal{V}^\perp$  contains a matrix  $C$  of rank one. Write  $C = AB$  where  $A \in M_{k,1}(F)$  and  $B \in M_{1,k}(F)$ . Clearly  $A \neq 0$  and  $B \neq 0$  (otherwise  $C = 0$  would be of rank 0). Since  $AB = C \in \mathcal{V}^\perp$ ,  $BXA = \text{tr}_1(BXA) = \text{tr}_k(ABX) = 0$  for all  $X \in \mathcal{S}$ . Let  $P, Q \in M_k(F)$  be matrices such that the first row of  $P$  is equal to  $B$  and all the other ones are equal to 0, the first column of  $Q$  is equal to  $A$  and all the other ones are equal to 0. Clearly  $P \neq 0 \neq Q$  and  $P\mathcal{S}Q = 0$ , a contradiction. The proof is thereby complete.  $\square$

We denote by  $A \mapsto {}^t A$ ,  $A \in M_k(F)$ , the transpose map of  $M_k(F)$ . Define an action of  $M_k(F) \otimes_F M_k(F)$  on  $M_k(F)$  by the rule

$$UX = \left( \sum_{i=1}^n A_i \otimes B_i \right) X = \sum_{i=1}^n A_i X {}^t B_i$$

whenever  $U = \sum_{i=1}^n A_i \otimes B_i$ . It is well known that  $M_k(F)$  is a simple faithful left module over the algebra  $M_k(F) \otimes_F M_k(F)$  under this action and  $M_k(F) \otimes_F M_k(F)$  is the algebra of all linear transformations of the vector space  $M_k(F)$ .

LEMMA 2.3. *With the above notation we have:*

(i) *If  $\mathcal{S}$  is a finite uniform insulator for  $M_k(F)$  such that the set  $\mathcal{S}$  is linearly independent over  $F$ , then  $\mathcal{K} = \{U \in M_k(F) \otimes_F M_k(F) \mid U\mathcal{S} = 0\}$  is a left ideal in  $M_k(F) \otimes_F M_k(F)$  containing no nonzero elements of the form  $A \otimes B$ ,  $A, B \in M_k(F)$ , and  $\dim_F(\mathcal{K}) = k^2(k^2 - |\mathcal{S}|)$ .*

(ii) *If  $\mathcal{K}'$  is a left ideal of  $M_k(F) \otimes_F M_k(F)$  containing no nonzero elements of the form  $A \otimes B$  and  $\mathcal{S}'$  is a basis of the vector space  $\{X \in M_k(F) \mid \mathcal{K}'X = 0\}$ , then  $\mathcal{S}'$  is a uniform insulator for  $M_k(F)$  and  $\dim_F(\mathcal{K}') = k^2(k^2 - |\mathcal{S}'|)$ .*

PROOF. Let  $\mathcal{S}$  and  $\mathcal{K}$  be as in the lemma. Clearly  $\mathcal{K}$  is a left ideal of the algebra  $M_k(F) \otimes_F M_k(F)$ . Since  $\mathcal{S}$  is a uniform insulator for  $M_k(F)$ ,  $(A \otimes B)\mathcal{S} \neq 0$  for all nonzero  $A, B \in M_k(F)$  and so  $\mathcal{K}$  contains no nonzero elements of the form  $A \otimes B$ . Write  $\mathcal{S} = \{X_1, X_2, \dots, X_m\}$  where  $m = |\mathcal{S}|$ . Define a linear map

$$\psi_{\mathcal{S}} : M_k(F) \otimes_F M_k(F) \rightarrow M_k(F)^m, \quad \psi_{\mathcal{S}}(U) = (UX_1, UX_2, \dots, UX_m)$$

for all  $U \in M_k(F) \otimes_F M_k(F)$ . Clearly  $\psi_{\mathcal{S}}$  is a left  $M_k(F) \otimes_F M_k(F)$ -module map and  $\mathcal{K} = \ker(\psi_{\mathcal{S}})$ . Since  $\{X_1, X_2, \dots, X_m\}$  is linearly independent over  $F$  and  $M_k(F) \otimes_F M_k(F)$  is the algebra of all linear transformations of the vector space  $M_k(F)$ , we conclude that  $\psi_{\mathcal{S}}$  is an epimorphism. Therefore,

$$\begin{aligned} \dim_F(\mathcal{K}) &= \dim_F(\ker(\psi_{\mathcal{S}})) = k^4 - \dim_F(\text{Im}(\psi_{\mathcal{S}})) \\ &= k^4 - k^2|\mathcal{S}| = k^2(k^2 - |\mathcal{S}|). \end{aligned}$$

Further let  $\mathcal{K}'$  and  $\mathcal{S}'$  be as in the lemma. Since  $\mathcal{K}'$  is a proper left ideal of  $M_k(F) \otimes_F M_k(F) \cong M_{k^2}(F)$ , there exists an idempotent  $E \in M_k(F) \otimes_F M_k(F)$  such that  $\mathcal{K}' = (M_k(F) \otimes_F M_k(F))E$  and  $E \neq 1$  where 1 is the identity of the algebra  $M_k(F) \otimes_F M_k(F)$ . Clearly

$$(1 - E)M_k(F) = \{X \in M_k(F) \mid \mathcal{K}'X = 0\}$$

and so  $\mathcal{S}'$  is a basis of the vector space  $(1 - E)M_k(F)$ . Write  $\mathcal{S}' = \{Y_1, \dots, Y_r\}$  where  $r = |\mathcal{S}'|$ . Consider the linear map

$$\psi_{\mathcal{S}'} : M_k(F) \otimes_F M_k(F) \rightarrow M_k(F)^r, \quad U \mapsto (UY_1, UY_2, \dots, UY_r).$$

We claim that  $\ker(\psi_{\mathcal{S}'}) = (M_k(F) \otimes_F M_k(F))E = \mathcal{K}'$ . Indeed, the inclusion  $\ker(\psi_{\mathcal{S}'}) \supseteq \mathcal{K}'$  follows from the definition of  $\psi_{\mathcal{S}'}$ . Next, let  $U \in \ker(\psi_{\mathcal{S}'})$ . Then  $UY_i = 0$  for all  $i = 1, 2, \dots, r$ . Since  $\{Y_1, Y_2, \dots, Y_r\}$  is a basis of  $(1 - E)M_k(F)$ , we conclude that  $[U(1 - E)]M_k(F) = 0$ . Recalling that  $M_k(F)$  is a faithful left  $M_k(F) \otimes_F M_k(F)$ -module, we get that  $U(1 - E) = 0$  forcing  $U = UE$ . That is  $U \in \mathcal{K}'$  and our claim is proved.

Since  $\ker(\psi_{\mathcal{S}'}) = \mathcal{K}'$ , it follows from our assumption on  $K'$  that  $\ker(\psi_{\mathcal{S}'})$  contains no nonzero matrices of the form  $A \otimes B$ ,  $A, B \in M_k(F)$ . That is to say,  $\mathcal{S}'$  is a uniform insulator for  $M_k(F)$ . As above we get

$$\dim_F(\mathcal{K}') = \dim_F(\psi_{\mathcal{S}'}) = k^4 - k^2|\mathcal{S}'| = k^2(k^2 - |\mathcal{S}'|).$$

The proof is thereby complete.  $\square$

PROOF OF THEOREM 1.3. Let  $\mathcal{S}$  be a uniform insulator for  $M_k(F)$  with  $|\mathcal{S}| = m_k(F)$  and let  $\mathcal{V} = \sum_{A \in \mathcal{S}} FA$ . According to Lemma 2.2,  $\mathcal{V}^\perp$  contains no rank one matrices and so (2) yields

$$n_k(F) \geq \dim_F(\mathcal{V}^\perp) = k^2 - \dim_F(\mathcal{V}) = k^2 - m_k(F).$$

That is to say  $m_k(F) + n_k(F) \geq k^2$ . On the other hand, if  $\mathcal{W}$  is a subspace of  $M_k(F)$  of dimension  $n_k(F)$  containing no rank one matrices and  $\mathcal{T}$  is a basis of  $\mathcal{W}^\perp$ , then  $\mathcal{T}$  is a uniform insulator for  $M_k(F)$  by Lemma 2.2 and so

$$m_k(F) \leq |\mathcal{T}| = \dim_F(\mathcal{W}^\perp) = k^2 - \dim_F(\mathcal{W}) = k^2 - n_k(F)$$

forcing  $m_k(F) + n_k(F) \leq k^2$ . Therefore,  $m_k(F) + n_k(F) = k^2$ .

Let  $\mathcal{K}'$  be any left ideal of  $M_k(F) \otimes_F M_k(F)$  containing no nonzero elements of the form  $A \otimes B$ ,  $A, B \in M_k(F)$ . We claim that

$$(3) \quad \dim_F(\mathcal{K}') \leq k^2 \cdot n_k(F).$$

Indeed, let  $\mathcal{S}'$  be a basis of the vector space  $\{X \in M_k(F) \mid \mathcal{X}'X = 0\}$ . According to Lemma 2.3,  $\mathcal{S}'$  is a uniform insulator for  $M_k(F)$  and since  $|\mathcal{S}'| \geq m_k(F)$ ,

$$\dim_F(\mathcal{X}') = k^2(k^2 - |\mathcal{S}'|) \leq k^2(k^2 - m_k(F)) = k^2 n_k(F).$$

Now let  $\mathcal{S}$  be a uniform insulator for  $M_k(F)$  with  $|\mathcal{S}| = m_k(F)$ . It follows at once from the definition of  $m_k(F)$  that  $\mathcal{S}$  is a linearly independent subset of  $M_k(F)$ . Therefore Lemma 2.3 implies that  $\mathcal{X} = \{U \in M_k(F) \otimes_F M_k(F) \mid U\mathcal{S} = 0\}$  is a left ideal of  $M_k(F) \otimes_F M_k(F)$  containing no nonzero elements of the form  $A \otimes B$  and  $\dim_F(\mathcal{X}) = k^2(k^2 - m_k(F)) = k^2 n_k(F)$  by the above result. It now follows from (3) that

$$(4) \quad \max\{\dim_F(\mathcal{X}')\} = k^2 n_k(F),$$

where  $\mathcal{X}'$  is a left ideal of  $M_k(F) \otimes_F M_k(F)$  containing no nonzero elements of the form  $A \otimes B$ .

Since  $M_k(F) \otimes_F M_k(F)$  is isomorphic to  $M_{k^2}(F)$  under  $\phi : A \otimes B \mapsto A \bullet B$  (see Section 1), we conclude from (4) that  $\ell_k(F) = k^2 \cdot n_k(F)$ . The proof is complete.  $\square$

REMARK 2.4. We conclude our discussion of the uniform bounds of primeness by considering the following implications for a field  $F$  and a positive integer  $k$ .

(i) If  $\mathcal{S}$  is a uniform insulator for  $M_k(F)$  and  $\mathcal{V} = \sum_{A \in \mathcal{S}} FA$ , then  $\mathcal{V}$  contains a uniform insulator  $\mathcal{S}'$  for  $M_k(F)$  with  $|\mathcal{S}'| = m_k(F)$ .

(ii) If  $\mathcal{W}$  is a subspace of  $M_k(F)$  maximal with respect to the property  $\mathcal{W} \cap \{M_{k,1}(F) \bullet M_{1,k}(F)\} = 0$ , then  $\dim_F(\mathcal{W}) = n_k(F)$ .

(iii) If  $\mathcal{X}$  is a left ideal of  $M_{k^2}(F)$  maximal with respect to the property  $\mathcal{X} \cap \{M_k(F) \bullet M_k(F)\} = 0$ , then  $\dim_F(\mathcal{X}) = \ell_k(F)$ .

We cannot prove any of these but we show that they are equivalent:

PROOF. Suppose that (i) is satisfied. We prove (ii). Let  $\mathcal{W}$  be as in (ii). According to Lemma 2.2 any basis of  $\mathcal{W}^\perp$  is a uniform insulator for  $M_k(F)$ . It now follows from our assumption that  $\mathcal{W}^\perp$  contains a uniform insulator  $\mathcal{S}'$  for  $M_k(F)$  with  $|\mathcal{S}'| = m_k(F)$ . Set  $\mathcal{V} = \sum_{A \in \mathcal{S}'} FA$  and note that  $\dim_F(\mathcal{V}) = m_k(F)$  because the set  $\mathcal{S}'$  is linearly independent. Next, the inclusion  $\mathcal{V} \subseteq \mathcal{W}^\perp$  together with (2) yield that  $\mathcal{V}^\perp \supseteq (\mathcal{W}^\perp)^\perp = \mathcal{W}$ . By Lemma 2.2  $\mathcal{V}^\perp$  contains no rank 1 matrices and so the maximality of  $\mathcal{W}$  implies that  $\mathcal{V}^\perp = \mathcal{W}$ . Therefore  $\mathcal{V} = (\mathcal{V}^\perp)^\perp = \mathcal{W}^\perp$  and so  $\dim_F(\mathcal{W}^\perp) = \dim_F(\mathcal{V}) = m_k(F)$ . Recalling that  $\dim_F(\mathcal{W}) = k^2 - \dim_F(\mathcal{W}^\perp) = k^2 - m_k(F)$ , we conclude that  $\dim_F(\mathcal{W}) = n_k(F)$  by Theorem 1.3.

Now assume that (ii) is fulfilled and show that (i) is true. Let  $\mathcal{S}$  and  $\mathcal{V}$  be as in (i). Then  $\mathcal{V}^\perp$  contains no rank 1 matrices by Lemma 2.2. Let  $\mathcal{W}$  be a subspace of  $M_k(F)$

containing  $\mathcal{V}^\perp$  and maximal with respect to the property  $\mathcal{W} \cap \{M_{1k}(F) \bullet M_{1k}(F)\} = 0$ . By our assumption  $\dim_F(\mathcal{W}) = n_k(F)$  and so (2) together with Theorem 1.3 imply that  $\mathcal{V} = (\mathcal{V}^\perp)^\perp \supseteq \mathcal{W}^\perp$  and  $\dim_F(\mathcal{W}^\perp) = k^2 - n_k(F) = m_k(F)$ . Let  $\mathcal{S}'$  be a basis of  $\mathcal{W}^\perp$ . Then  $\mathcal{S}'$  is a uniform insulator for  $M_k(F)$  by Lemma 2.2. Clearly  $|\mathcal{S}'| = m_k(F)$  and  $\mathcal{S}' \subseteq \mathcal{V}$ .

Finally, making use of Lemma 2.3 the proof of the equivalence of statements (i) and (iii) is similar to that of (i) and (ii).  $\square$

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