On the category of comodules over corings

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Abstract

It is well known that the category $\mathcal{M}^{\mathcal{C}}$ of right comodules over an A-coring $\mathcal{C}$, A an associative ring, is a subcategory of the category of left modules ${}_{^*\mathcal{C}}\mathcal{M}$ over the dual ring ${}^*\mathcal{C}$. The main purpose of this note is to show that $\mathcal{M}^{\mathcal{C}}$ is a full subcatgeory in ${}_{^*\mathcal{C}}\mathcal{M}$ if and only if $\mathcal{C}$ is locally projective as a left A-module.

1 Introduction

For any coassociative coalgebra C over a commutative ring R, the convolution product turns the dual module $C^* = \mathrm{Hom}_R(C, R)$ into an associative R-algebra. The category $\mathcal{M}^C$ of right comodules is an additive subcategory of the category ${}_{C^*}\mathcal{M}$ of left C^*-modules. $\mathcal{M}^C$ is an abelian (in fact a Grothendieck) category if and only if C is flat as an R-module. Moreover, $\mathcal{M}^C$ coincides with ${}_{C^*}\mathcal{M}$ if and only if C is finitely generated and projective as an R-module (e.g. [11, Corollary 33]).

In case C is projective as an R-module, $\mathcal{M}^C$ is a full subcategory of ${}_{C^*}\mathcal{M}$ and coincides with $\sigma[{}_{C^*}C]$, the category of submodules of C-generated C^*-modules (e.g. [9, 3.15, 4.3]). It was well understood from examples that projectivity of C as an R-module was not necessary to achieve $\mathcal{M}^C = \sigma[{}_{C^*}C]$ and that the equality holds provided C satisfies the *α-condition*, i.e., the canonical maps $N \otimes_R C \to \mathrm{Hom}_{\mathbb{Z}}(C^*, N)$ are injective for all R-modules N (e.g. [1, Satz 2.2.13], [2, Section 2], [10, 3.2]). It will follow from our results that this condition is in fact equivalent to $\mathcal{M}^C = \sigma[{}_{C^*}C]$ and also to C being *locally projective* as an R-module.

We do investigate the questions and results mentioned above in the more general case of comodules over any *A-coring*, A an associative ring, and it will turn out that the above observations remain valid almost literally in this extended setting.

2 Some module theory

Let A be any associative ring with unit and denote $(-)^* = \mathrm{Hom}_A(-, A)$. We write ${}_A\mathcal{M}$ ($\mathcal{M}_A$) for the category of unital left (right) A-modules. I (or I_N) will denote the identity map (of the module N).

2.1. Canonical maps. For any left A-module K, consider the maps

$$\begin{array}{rcccccc} \tilde{\varphi}_K : & K & \xrightarrow{\varphi_K} & K^{**} & \xrightarrow{i} & A^{K^*}, \\ & k & \mapsto & [f \mapsto f(k)] & \mapsto & (f(k))_{f \in K^*}. \end{array}$$

For any right A-module N define the maps

$$\begin{array}{rlll} \alpha_{N,K} : & N \otimes_A K & \to \mathrm{Hom}_{\mathbb{Z}}(K^*, N), & n \otimes k \ \mapsto \ [f \mapsto nf(k)], \\ \psi_N : & N \otimes_A A^{K^*} & \to N^{K^*}, & n \otimes (a_f)_{f \in K^*} \mapsto (na_f)_{f \in K^*}. \end{array}$$

By the identification $\mathrm{Map}(K^*, N) = N^{K^*}$ we have the commutative diagram

$$\begin{tikzcd} & N \otimes_A K \arrow[r, "I \otimes \tilde{\varphi}_K"] \arrow[d, "\alpha_{N,K}"'] & N \otimes_A A^{K^*} \arrow[d, "\psi_N"] \\ 0 \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(K^*, N) \arrow[r] & N^{K^*}, \end{tikzcd} \qquad \begin{tikzcd} n \otimes k \arrow[r] \arrow[d] & n \otimes (f(k))_{f \in K^*} \arrow[d] \\ {[f \mapsto nf(k)]} \arrow[r] & (nf(k))_{f \in K^*}. \end{tikzcd}$$

2.2. Injectivity of $\alpha_{N,K}$. *We stick to the notation above.*

(1) The following are equivalent:

(a) $\alpha_{N,K}$ is injective;

(b) for $u \in N \otimes_A K$, $(I \otimes f)(u) = 0$ for all $f \in K^*$, implies $u = 0$.

(2) The following are equivalent:

(a) For every finitely presented right A-module N, $\alpha_{N,K}$ is injective;

(b) $\tilde{\varphi}_K : K \to A^{K^*}$ is a pure monomorphism.

Proof. (1) Let $u = \sum_{i=1}^r n_i \otimes k_i \in N \otimes_A K$. Then $(I \otimes f)(u) = \sum_{i=1}^r n_i f(k_i) = 0$, for all $f \in K^*$ if and only if $u \in \mathrm{Ke}\, \alpha_{N,K}$.

(2) For N finitely presented, ψ_N is injective (bijective) and so $\alpha_{N,K}$ is injective if and only if $I_N \otimes \tilde{\varphi}_K$ is injective. Injectivity of $I_N \otimes \tilde{\varphi}_K$ for all finitely presented N characterizes $\tilde{\varphi}_K$ as a pure monomorphism (e.g., [8, 34.5]). $\square$

We say that K satisfies the *α-condition* provided $\alpha_{N,K}$ is injective for all right A-modules N. Such modules are named *universally torsionless (UTL)* in [4] and we recall some of their characterizations.

The module K is called *locally projective* (see [12]) if, for any diagram of left A-modules with exact lines

$$\begin{array}{ccccc} 0 & \longrightarrow & F & \xrightarrow{i} & K \\ & & & & \downarrow{\scriptstyle g} \\ & & L & \xrightarrow{f} & N & \longrightarrow & 0, \end{array}$$

where F is finitely generated, there exists $h : K \to L$ such that $g \circ i = f \circ h \circ i$.

Clearly every projective module is locally projective. From Garfinkel [4, Theorem 3.2] and Huisgen-Zimmermann [12, Theorem 2.1] we have the following characterizations of these modules which are also studied in Ohm-Bush [5] (as *trace modules*), and in Raynaud-Gruson [6] (as *modules plats et strictement de Mittag-Leffler*).

2.3. Locally projective modules. *For the left A-module K, the following are equivalent:*

(a) K is locally projective;

(b) K is a pure submodule of a locally projective module;

(c) $\alpha_{N,K}$ is injective, for any right A-module N;

(d) $\alpha_{N,K}$ is injective, for any cyclic right A-module N;

(e) for each $m \in K$, we have $m \in K^*(m)K$;

(f) for each finitely generated submodule $i : F \to K$, there exists $n \in I\!N$ and maps $\beta : R^n \to K$, $\gamma : K \to R^n$ with $\beta \circ \gamma \circ i = i$.

Recall the following observations. Notice that for a right noetherian ring A, every product of copies of A is locally projective as left A-module (e.g. [12, Corollary 4.3]).

2.4. Corollary. *Let K be a left A-module.*

(1) Every locally projective module is flat and a pure submodule of some product A^Λ, Λ some set.

(2) If K is finitely generated, or A is left perfect, then K is locally projective if and only if K is projective.

(3) *For a right noetherian ring A, the following are equivalent:*

   (a) *K is locally projective;*

   (b) *K is a pure submodule of a product A^Λ, Λ some set.*

The following facts from general category theory will be helpful (e.g., [7]). In any category $\mathcal{A}$, a morphism $f : A \to B$ is called a *monomorphism* if for any morphisms $g, h : C \to A$ the identity $f \circ g = f \circ h$ implies $g = h$.

In an additive category $\mathcal{A}$ a morphism $\gamma : K \to A$ is called a *kernel* of $f : A \to B$ provided $f \circ \gamma = 0$ and, for every $g : C \to A$ with $f \circ g = 0$, there is exactly one $h : C \to K$ such that $g = \gamma \circ h$.

Recall the following well-known (and easliy proved) observations.

2.5. Monomorphisms. *Let $\mathcal{A}$ be any catgeory and $f : A \to B$ a morphism in $\mathcal{A}$. The following are equivalent:*

(a) *f is a monomorphism;*

(b) *the map* $\mathrm{Mor}(L, f) : \mathrm{Mor}(L, A) \to \mathrm{Mor}(L, B)$, $g \mapsto f \circ g$, *is injective, for any $L \in \mathcal{A}$.*

If $\mathcal{A}$ is additive and has kernels, then (a)-(b) are equivalent to:

(c) *for the kernel $\gamma : K \to A$ of f, $K = 0$.*

The basic properties of adjoint functors will be helpful.

2.6. Adjoint functors. *Let $\mathcal{A}$ and $\mathcal{B}$ be any categories. Assume a functor $F : \mathcal{A} \to \mathcal{B}$ is right adjoint to a functor $G : \mathcal{B} \to \mathcal{A}$, i.e.,*

$$\mathrm{Mor}_{\mathcal{B}}(Y, F(X)) \simeq \mathrm{Mor}_{\mathcal{A}}(G(Y), X)), \ \textit{for any } X \in \mathcal{A},\ Y \in \mathcal{B}.$$

Then

(1) *F preserves monomorphisms and products,*

(2) *G preserves epimorphisms and coproducts.*

For the study of comodules the following type of module categories is of particular interest.

2.7. The category $\sigma[K]$. For any left A-module K we denote by $\sigma[K]$ the full subcategory of ${}_A\mathcal{M}$ whose objects are submodules of K-generated modules. This is the smallest full Grothendieck subcategory of ${}_A\mathcal{M}$ containing K (see [8]).

$\sigma[K]$ coincides with ${}_A\mathcal{M}$ if and only if A embeds into some (finite) coproduct of copies of K. This happens, for example, when K is a faithful A-module which is finitely generated as a module over its endomorphism ring (see [8, 15.4]).

The *trace functor* $\mathcal{T}^K : {}_A\mathcal{M} \to \sigma[K]$, which sends any $X \in {}_A\mathcal{M}$ to

$$\mathcal{T}^K(X) := \sum\{f(N) \mid N \in \sigma[K],\ f \in \mathrm{Hom}_A(N, X)\},$$

is right adjoint to the inclusion functor $\sigma[K] \to {}_A\mathcal{M}$ (e.g., [8, 45.11]). Hence, by 2.6, for any family $\{N_\lambda\}_\Lambda$ of modules in $\sigma[K]$, the product in $\sigma[K]$ is

$$\prod\nolimits_\Lambda^K N_\lambda = \mathcal{T}^K(\prod\nolimits_\Lambda N_\lambda),$$

where the unadorned $\prod$ denotes the usual (cartesian) product of A-modules.

It also follows from 2.6 that for $\{N_\lambda\}_\Lambda$ in $\sigma[K]$ the coproduct in $\sigma[K]$ and the coproduct in ${}_A\mathcal{M}$ coincide.

3 Corings and comodules

As before, let A be any associative ring with unit.

3.1. Corings and their duals. An A-*coring* is an (A, A)-bimodule $\mathcal{C}$ with (A, A)-bimodule maps (*comultiplication* and *counit*)

$$\underline{\Delta} : \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}, \quad \underline{\varepsilon} : \mathcal{C} \to A,$$

satisfying the identities

$$(I \otimes \underline{\Delta}) \circ \underline{\Delta} = (\underline{\Delta} \otimes I) \circ \underline{\Delta}, \quad (I \otimes \underline{\varepsilon}) \circ \underline{\Delta} = I = (\underline{\varepsilon} \otimes I) \circ \underline{\Delta}.$$

For elementwise description of these maps we adopt the Σ-*notation*, writing for $c \in \mathcal{C}$,

$$\underline{\Delta}(c) = \sum c_{\underline{1}} \otimes c_{\underline{2}}.$$

Then coassociativity of $\underline{\Delta}$ is written as

$$\sum \underline{\Delta}(c_{\underline{1}}) \otimes c_{\underline{2}} = \sum c_{\underline{1}\,\underline{1}} \otimes c_{\underline{1}\,\underline{2}} \otimes c_{\underline{2}} = \sum c_{\underline{1}} \otimes c_{\underline{2}\,\underline{1}} \otimes c_{\underline{2}\,\underline{2}} = \sum c_{\underline{1}} \otimes \underline{\Delta}(c_{\underline{2}}),$$

and the conditions on the counit are

$$\sum \underline{\varepsilon}(c_{\underline{1}}) c_{\underline{2}} = c = \sum c_{\underline{1}} \underline{\varepsilon}(c_{\underline{2}}).$$

Of course, when A is commutative and $ac = ca$ for all $a \in A$, $c \in \mathcal{C}$, the coring $\mathcal{C}$ is just a *coalgebra* in the usual sense.

For any A-coring $\mathcal{C}$, the maps $\mathcal{C} \to A$ may be right A-linear or left A-linear and we denote these by

$$\mathcal{C}^* := \mathrm{Hom}_{-A}(\mathcal{C}, A), \quad {}^*\mathcal{C} := \mathrm{Hom}_{A-}(\mathcal{C}, A),$$

and for bilinear maps we have $\mathrm{Hom}_{AA}(\mathcal{C}, A) = {}^*\mathcal{C} \cap \mathcal{C}^*$.

Both $\mathcal{C}^*$ and ${}^*\mathcal{C}$ can be turned to associative rings with unit $\underline{\varepsilon}$ by the (convolution) products

(1) for $f, g \in \mathcal{C}^*$, and $c \in \mathcal{C}$ put $f *^r g\,(c) = \sum g(f(c_{\underline{1}})c_{\underline{2}})$,

(2) for $f, g \in {}^*\mathcal{C}$, and $c \in \mathcal{C}$ put $f *^l g\,(c) = \sum f(c_{\underline{1}}g(c_{\underline{2}}))$.

Notice that for $f, g \in {}^*\mathcal{C} \cap \mathcal{C}^*$ this yields

$$f * g\,(c) = \sum f(c_{\underline{1}})g(c_{\underline{2}}),$$

a formula which is well known from coalgebras.

It is easily verified that the maps

$$\iota_l : A \to {}^*\mathcal{C},\ a \mapsto [c \mapsto \underline{\varepsilon}(c)a], \text{ and } \quad \iota_r : A \to \mathcal{C}^*,\ a \mapsto [c \mapsto a\underline{\varepsilon}(c)],$$

are ring anti-morphisms and hence we may consider left ${}^*\mathcal{C}$-modules as right A-modules and right $\mathcal{C}^*$-modules as left A-modules.

3.2. Right comodules. Let $\mathcal{C}$ be an A-coring and M a right A-module. An A-linear map $\varrho_M : M \to M \otimes_A \mathcal{C}$ is called a *coaction* on M, and it is said to be *counital* and *coassociative* provided

$$(I \otimes \underline{\varepsilon}) \circ \varrho_M = I, \text{ and } (I \otimes \underline{\Delta}) \circ \varrho_M = (\varrho_M \otimes I) \circ \varrho_M.$$

A *right $\mathcal{C}$-comodule* is a right A-module with a counital coassociative coaction.

A *morphism* of right $\mathcal{C}$-comodules $f : M \to N$ is an A-linear map such that

$$\varrho_N \circ f = (f \otimes I) \circ \varrho_M .$$

We denote the set of comodule morphisms between M and N by $\mathrm{Hom}^{\mathcal{C}}(M, N)$. It is easy to show that this is an abelian group and hence the category $\mathcal{M}^{\mathcal{C}}$, formed by right $\mathcal{C}$-comodules and comodule morphisms, is additive.

For any right A-module X, the tensor product $X \otimes_A \mathcal{C}$ is a right $\mathcal{C}$-comodule by

$$I \otimes \underline{\Delta} : X \otimes_A \mathcal{C} \to X \otimes_A \mathcal{C} \otimes_A \mathcal{C},$$

and for any A-morphism $f : X \to Y$, the map

$$f \otimes I : X \otimes_A \mathcal{C} \to Y \otimes_A \mathcal{C}$$

is a comodule morphism.

3.3. The category $\mathcal{M}^{\mathcal{C}}$. *Let $\mathcal{C}$ be an A-coring.*

(1) The category $\mathcal{M}^{\mathcal{C}}$ has direct sums and cokernels.
It has kernels provided $\mathcal{C}$ is flat as a left A-module.

(2) For the functor $- \otimes_A \mathcal{C} : \mathcal{M}_A \to \mathcal{M}^{\mathcal{C}}$ we have natural isomorphisms

$$\mathrm{Hom}^{\mathcal{C}}(M, X \otimes_A \mathcal{C}) \to \mathrm{Hom}_A(M, X), \quad f \mapsto (I \otimes \underline{\varepsilon}) \circ f,$$

for $M \in \mathcal{M}^{\mathcal{C}}$, $X \in \mathcal{M}_A$, with inverse map $h \mapsto (h \otimes I) \circ \varrho_M$, i.e., the functor $- \otimes_A \mathcal{C} : \mathcal{M}_A \to \mathcal{M}^{\mathcal{C}}$ is right adjoint to the forgetful functor $\mathcal{M}^{\mathcal{C}} \to \mathcal{M}_A$ and hence it preserves monomorphisms and products.

(3) For the right comodule endomorphisms we have $\mathrm{End}^{\mathcal{C}}(\mathcal{C}) \simeq \mathcal{C}^*$.

(4) $\mathcal{C}$ is a subgenerator in $\mathcal{M}^{\mathcal{C}}$.

Proof. (1) Consider a family $\{M_\lambda\}_\Lambda$ of right $\mathcal{C}$-comodules. It is easy to prove that the direct sum $\bigoplus_\Lambda M_\lambda$ in $\mathcal{M}_A$ is a right $\mathcal{C}$-comodule and has the universal property of a coproduct in $\mathcal{M}^{\mathcal{C}}$.

For any morphism $f : M \to N$ of right $\mathcal{C}$-comodules, the cokernel of f in $\mathcal{M}_A$ has a comodule structure and hence is a cokernel in $\mathcal{M}^{\mathcal{C}}$. If $\mathcal{C}$ is flat as a left A-module, similar arguments hold for the kernel.

(2) The proof of the corresponding assertion for coalgebras applies (e.g., [9, 3.12]) and then refer to 2.6. Note that the adjointness, for example, was also observed in [3, Lemma 3.1].

(3) The group isomorphism $\mathrm{End}^{\mathcal{C}}(\mathcal{C}) \simeq \mathcal{C}^*$ follows from (2) by putting $M = \mathcal{C}$ and $X = A$. This is a ring isomorphism when writing the morphisms on the right.

(4) For any $M \in \mathcal{M}^{\mathcal{C}}$, there is an epimorphism $A^{(\Lambda)} \to M$ in $\mathcal{M}_A$. Tensoring with $\mathcal{C}$ yields an epimorphism $A^{(\Lambda)} \otimes_A \mathcal{C} \to M \otimes_A \mathcal{C}$ in $\mathcal{M}^{\mathcal{C}}$. As easily checked the structure map $\varrho_M : M \to M \otimes_A \mathcal{C}$ is a morphism in $\mathcal{M}^{\mathcal{C}}$ and hence M is a subobject of a $\mathcal{C}$-generated comodule. $\square$

3.4. $\mathcal{M}^{\mathcal{C}}$ as Grothendieck category.

For an A-coring $\mathcal{C}$ the following are equivalent:

(a) *$\mathcal{C}$ is a flat left A-module;*

(b) *every monomorphism in $\mathcal{M}^{\mathcal{C}}$ is injective;*

(c) *the forgetful functor $\mathcal{M}^{\mathcal{C}} \to \mathcal{M}_A$ respects monomorphisms.*

If these conditions are satisfied, $\mathcal{M}^{\mathcal{C}}$ is a Grothendieck category.

Proof. $(a) \Rightarrow (b) \Leftrightarrow (c)$ are obvious.

$(c) \Rightarrow (a)$ For any monomorphism $f : N \to L$ in $\mathcal{M}_A$, the map $f \otimes I : N \otimes_A \mathcal{C} \to L \otimes_A \mathcal{C}$ is a monomorphism in $\mathcal{M}^{\mathcal{C}}$ (by 3.3(2)) and hence injective by assumption. This shows that $- \otimes_A \mathcal{C} : \mathcal{M}_A \to \mathbb{Z}$-Mod is exact and hence $\mathcal{C}$ is a flat left A-module.

Now assume that (a)-(c) are satified. Then $\mathcal{M}^{\mathcal{C}}$ is abelian and cocomplete. Since $\mathcal{C}$ is a subgenerator it is routine to show that the subcomodules of $\mathcal{C}^n$, $n \in \mathbb{N}$, form a generating set for $\mathcal{M}^{\mathcal{C}}$. Hence $\mathcal{M}^{\mathcal{C}}$ is a Grothendieck category. □

Every right $\mathcal{C}$-comodule M allows a left ${}^*\mathcal{C}$-module structure by

$$\rightharpoonup : {}^*\mathcal{C} \otimes_{\mathbb{Z}} M \to M, \quad f \otimes m \mapsto (I \otimes f) \circ \varrho_M(m).$$

With this structure any comodule morphisms $M \to N$ is ${}^*\mathcal{C}$-linear, i.e.

$$\mathrm{Hom}^{\mathcal{C}}(M, N) \subset \mathrm{Hom}_{{}^*\mathcal{C}}(M, N),$$

and hence $\mathcal{M}^{\mathcal{C}}$ is a subcategory of ${}_{{}^*\mathcal{C}}\mathcal{M}$. As shown in [3, Lemma 4.3], $\mathcal{M}^{\mathcal{C}}$ can be identified with ${}_{{}^*\mathcal{C}}\mathcal{M}$ provided $\mathcal{C}$ is finitely generated and projective as left A-module.

Notice that in any case $\mathcal{C}$ is a faithful ${}^*\mathcal{C}$-module since $f \rightharpoonup c = 0$ for all $c \in \mathcal{C}$ implies $f(c) = \underline{\varepsilon}(f \rightharpoonup c) = 0$ and hence $f = 0$.

The question arises when, more generally, $\mathcal{M}^{\mathcal{C}}$ is a full subcategory of ${}_{{}^*\mathcal{C}}\mathcal{M}$, i.e., when $\mathrm{Hom}^{\mathcal{C}}(M, N) = \mathrm{Hom}_{{}^*\mathcal{C}}(M, N)$, for any $M, N \in \mathcal{M}^{\mathcal{C}}$. The answer is given in our main theorem:

3.5. $\mathcal{M}^{\mathcal{C}}$ as full subcategory of ${}_{{}^*\mathcal{C}}\mathcal{M}$

For the A-coring $\mathcal{C}$, the following are equivalent:

(a) *$\mathcal{M}^{\mathcal{C}} = \sigma[{}_{{}^*\mathcal{C}}\mathcal{C}]$;*

(b) *$\mathcal{M}^{\mathcal{C}}$ is a full subcategory of ${}_{{}^*\mathcal{C}}\mathcal{M}$;*

(c) for all $M, N \in \mathcal{M}^{\mathcal{C}}$, $\mathrm{Hom}^{\mathcal{C}}(M,N) = \mathrm{Hom}_{{}^*\mathcal{C}}(M,N)$*;*

(d) $\mathcal{C}$ *satifies the* α*-condition as left* A*-module;*

(e) every ${}^*\mathcal{C}$*-submodule of* $\mathcal{C}^n$, $n \in \mathbb{N}$*, is a subcomodule of* $\mathcal{C}^n$*;*

(f) $\mathcal{C}$ *is locally projective as left* A*-module.*

If these conditions are satisfied we have, for any family $\{M_\lambda\}_\Lambda$ *of right* A*-modules,*

$$(\prod\nolimits_\Lambda M_\lambda) \otimes_A \mathcal{C} \;\simeq\; \prod\nolimits_\Lambda^{\mathcal{C}} (M_\lambda \otimes_A \mathcal{C}) \subset \prod\nolimits_\Lambda (M_\lambda \otimes_A \mathcal{C}).$$

Proof. The implications $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Rightarrow (e)$ are obvious.

$(a) \Rightarrow (d)$ By 3.4 ${}_A\mathcal{C}$ is flat. For any $N \in \mathcal{M}_A$ we prove the injectivity of the map

$$\alpha : N \otimes_A \mathcal{C} \to \mathrm{Hom}_{\mathbb{Z}}({}^*\mathcal{C}, N), \quad n \otimes c \mapsto [f \mapsto nf(c)].$$

Considering $\mathrm{Hom}_{\mathbb{Z}}({}^*\mathcal{C}, N)$ and the right $\mathcal{C}$-comodule $N \otimes_A \mathcal{C}$ as left ${}^*\mathcal{C}$-modules in the canonical way, we observe that α is ${}^*\mathcal{C}$-linear. So for any right $\mathcal{C}$-comodule L we have the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{{}^*\mathcal{C}}(L, N \otimes_A \mathcal{C}) & \xrightarrow{\mathrm{Hom}(L,\alpha)} & \mathrm{Hom}_{{}^*\mathcal{C}}(L, \mathrm{Hom}_{\mathbb{Z}}({}^*\mathcal{C}, N)) \\
\downarrow \simeq & & \downarrow \simeq \\
\mathrm{Hom}_A(L, N) & \xrightarrow{\quad i \quad} & \mathrm{Hom}_{\mathbb{Z}}(L, N),
\end{array}$$

where the first vertical isomorphism is obtained by assumption and 3.3,

$$\mathrm{Hom}_{{}^*\mathcal{C}}(L, N \otimes_A \mathcal{C}) = \mathrm{Hom}^{\mathcal{C}}(L, N \otimes_A \mathcal{C}) \simeq \mathrm{Hom}_A(L, N),$$

and the second one by canonical isomorphisms

$$\mathrm{Hom}_{{}^*\mathcal{C}}(L, \mathrm{Hom}_{\mathbb{Z}}({}^*\mathcal{C}, N)) \simeq \mathrm{Hom}_{\mathbb{Z}}({}^*\mathcal{C} \otimes_{{}^*\mathcal{C}} L, N) \simeq \mathrm{Hom}_{\mathbb{Z}}(L, N).$$

This shows that $\mathrm{Hom}(L, \alpha)$ is injective and so (by 2.5) the corestriction of α is a monomorphism in $\mathcal{M}^{\mathcal{C}}$. Since ${}_A\mathcal{C}$ is flat this implies that α is injective (by 3.4).

$(e) \Rightarrow (a)$ First we show that every finitely generated module $N \in \sigma[{}_{{}^*\mathcal{C}}\mathcal{C}]$ is a $\mathcal{C}$-comodule. There exists some ${}^*\mathcal{C}$-submodule $X \subset \mathcal{C}^n$, $n \in \mathbb{N}$, and an epimorphism $h : X \to N$. By assumption X and the kernel of h are comodules and hence N is a comodule.

Now for any $L \in \sigma[{}_{{}^*\mathcal{C}}\mathcal{C}]$ the finitely generated submodules are comodules and hence L is a comodule.

For any ${}^*\mathcal{C}$-morphism in $\sigma[{}_{{}^*\mathcal{C}}\mathcal{C}]$, the kernel is a ${}^*\mathcal{C}$-submodule and hence a comodule. As easily verified this implies that monomorphisms and epimorphisms in $\sigma[{}_{{}^*\mathcal{C}}\mathcal{C}]$ are comodule morphisms and hence this is true for all morphisms in $\sigma[{}_{{}^*\mathcal{C}}\mathcal{C}]$.

$(d) \Leftrightarrow (f)$ follows by 2.3.

$(d) \Rightarrow (e)$ We show that for right $\mathcal{C}$-comodules M, any ${}^*\mathcal{C}$-submodule N is a subcomodule. For this consider the map

$$\rho_N : N \to \mathrm{Hom}_A({}^*\mathcal{C}, N),\ n \mapsto [f \mapsto f \rightharpoonup n].$$

With the inclusion $i : N \to M$, we have the commutative diagram with exact lines

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "i"] & M \arrow[r, "p"] \arrow[d, "\varrho_M"] & M/N \arrow[r] & 0 \\
0 \arrow[r] & N \otimes_A \mathcal{C} \arrow[r, "i\otimes I"] \arrow[d, "\alpha_{N,\mathcal{C}}"] & M \otimes_A \mathcal{C} \arrow[r, "p\otimes I"] \arrow[d, "\alpha_{M,\mathcal{C}}"] & M/N \otimes_A \mathcal{C} \arrow[r] \arrow[d, "\alpha_{M/N,\mathcal{C}}"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_A({}^*\mathcal{C}, N) \arrow[r, "\mathrm{Hom}({}^*\mathcal{C},i)"] & \mathrm{Hom}_A({}^*\mathcal{C}, M) \arrow[r] & \mathrm{Hom}_A({}^*\mathcal{C}, M/N) & ,
\end{tikzcd}$$

where all the α's are injective and $\mathrm{Hom}({}^*\mathcal{C}, i) \circ \rho_N = \alpha_{M,\mathcal{C}} \circ \varrho_M \circ i$. This implies $(p \otimes I) \circ \varrho_M \circ i = 0$, and by the kernel property, $\varrho_M \circ i$ factors through $N \to N \otimes_A \mathcal{C}$ thus yielding a $\mathcal{C}$-coaction on N.

The final assertion follows by 2.6 and the characterization of products in $\sigma[{}_{{}^*\mathcal{C}}\mathcal{C}]$ (see 2.7). □

As a corollary we can show when all ${}^*\mathcal{C}$-modules are $\mathcal{C}$-comodules. This includes the reverse conclusion of [3, Lemma 4.3] and extends [11, Lemma 33].

3.6. $\mathcal{M}^{\mathcal{C}} = {}_{{}^*\mathcal{C}}\mathcal{M}$**.**

For any A-coring $\mathcal{C}$, the following are equivalent:

(a) $\mathcal{M}^{\mathcal{C}} = {}_{{}^*\mathcal{C}}\mathcal{M}$*;*

(b) the functor $-\otimes_A \mathcal{C} : \mathcal{M}_A \to {}_{*\mathcal{C}}\mathcal{M}$ *has a left adjoint;*

(c) ${}_A\mathcal{C}$ *is finitely generated and projective;*

(d) ${}_A\mathcal{C}$ *is locally projective and* $\mathcal{C}$ *is finitely generated as right* $\mathcal{C}^*$*-module.*

Proof. $(a) \Rightarrow (b)$ and $(c) \Rightarrow (d)$ are obvious.

$(b) \Rightarrow (c)$ By 2.6, $-\otimes_A\mathcal{C}$ preserves monomorphisms (injective morphisms) and hence ${}_A\mathcal{C}$ is flat. Moreover we obtain, for any family $\{M_\lambda\}_\Lambda$ in $\mathcal{M}_A$, the isomorphism

$$(\prod_\Lambda M_\lambda) \otimes_A \mathcal{C} \simeq \prod_\Lambda (M_\lambda \otimes_A \mathcal{C}),$$

which implies that ${}_A\mathcal{C}$ is finitely presented (e.g., [8, 12.9]) and hence projective.

$(d) \Rightarrow (a)$ Recall that $\mathcal{C}^*$ is the endomorphism ring of the faithful module ${}_{*\mathcal{C}}\mathcal{C}$. Hence $\mathcal{C}_{\mathcal{C}^*}$ finitely generated implies $\mathcal{M}^\mathcal{C} = \sigma[{}_{*\mathcal{C}}\mathcal{C}] = {}_{*\mathcal{C}}\mathcal{M}$ (see 2.7). $\square$

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