ROBERT WISBAUER

On module classes closed under extensions

Classes of modules with given properties (like being closed under direct sums and factor modules) play an important part in various aspects of Module Theory. Most of these classes can be determined by a single module. In this report we consider some of these classes and we relate their properties to properties of the defining module. Our aim is to expose the relationship between various parts of Module Theory studied independently so far. Some of the results and the proofs are taken from the dissertation of Berning [7].

1 Classes related to a given module

Let R be an associative ring with unit and R–Mod the category of unital left R-modules. M will always denote a left R-module. For basic notions see [28, 39].

A full subcategory $\mathcal{C}$ of R–Mod which is closed under submodules, factor modules and direct sums is called a *closed subcategory.*

Given an R-module M we define the class of *M-generated modules*,

$$\mathrm{Gen}\,(M) = \{N \in R\text{–}\mathsf{Mod} \mid \text{ there exists an epimorphism } M^{(\Lambda)} \to N,\ \Lambda \text{ any set}\},$$

and the class of *M-cogenerated modules*,

$$\mathrm{Cog}\,(M) = \{N \in R\text{–}\mathsf{Mod} \mid \text{ there exists a monomorphism } N \to M^{\Lambda},\ \Lambda \text{ any set}\}.$$

Related to these are two *preradicals (trace and reject)* in R–Mod defined by

$$\mathrm{Tr}(M,N) = \sum \{\mathrm{Im}\ f \mid f \in \mathrm{Hom}(M,N)\} \text{ and}$$
$$\mathrm{Re}(N,M) = \bigcap \{\mathrm{Ke}\ f \mid f \in \mathrm{Hom}(N,M)\},\ \text{ for } N \in R\text{–}\mathsf{Mod}.$$

Obviously $\mathrm{Gen}\,(M)$ is a prototype of a module class closed under direct sums and factor modules whereas $\mathrm{Cog}\,(M)$ is closed under direct products and submodules.

Considered as full subcategories of R–Mod, $\mathrm{Gen}\,(M)$ is a cocomplete category with cokernels and $\mathrm{Cog}\,(M)$ is a complete category with kernels.

1.1 The category $\sigma[M]$

To study properties of the module M we form the full subcategory of R–Mod,

$$\sigma[M] = \{N \in R\text{–}\mathsf{Mod} \mid N \text{ is a submodule of an } M\text{-generated module}\}.$$

$\sigma[M]$ is a Grothendieck category and allows a homological characterization of properties of the module M. By construction, $\sigma[M]$ is the smallest closed subcategory of R–Mod containing the module M. It is easy to see that every closed subcategory in R–Mod is of the form $\sigma[M]$ for a suitable module M.

Moreover, $\sigma[M]$ has a generator, for example,

$$G := \bigoplus\{K \subset M^{(I\!N)} \mid K \text{ cyclic }\}.$$

An R-module N for which $\sigma[N] = \sigma[M]$ is called a *subgenerator* of $\sigma[M]$.

In particular, M is a subgenerator in R–Mod if $\sigma[M] = R$–Mod. Such modules are also called *cofaithful.* Over a left artinian ring R every faithful R-module is cofaithful.

For the investigation of the lattice of all closed subcategories of $\sigma[M]$ we refer to [34, 35].

Since $\sigma[M]$ is a Grothendieck category every $N \in \sigma[M]$ has an *injective hull in* $\sigma[M]$, also called the *M-injective hull* of N, which is usually denoted by $\widehat{N}$. It is isomorphic to the trace of M in the injective hull $I(N)$ of N in R–Mod, i.e. we may identify $\widehat{N} = \mathrm{Tr}(M, I(N))$. It is well-known that $\widehat{N}$ is a maximal essential extension of N in $\sigma[M]$ (e.g. [39, § 17]).

The internal properties of Gen (M) and $\sigma[M]$ are determined by 'internal' properties of M, i.e. properties related to M itself, like M-projectivity or M-injectivity. For example, independent of the ring R, for a semisimple module M, all modules in $\sigma[M]$ are M-injective and M-projective. For an extensive account of the study of such interdependencies we refer to [39] and [13]. Here we will be interested in the relationship of the classes defined to surrounding subcategories in R–Mod.

1.2 Definitions. Let M be an R-module. For any two classes of modules $\mathtt{C}$ and $\mathtt{D}$ in $\sigma[M]$ denote by $\mathtt{E}_M(\mathtt{D}, \mathtt{C})$ the class of R-modules N for which there is an exact sequence

$$0 \to C \to N \to D \to 0$$

in $\sigma[M]$, where $C \in \mathtt{C}$ and $D \in \mathtt{D}$.

$\mathtt{C}$ is said to be *closed under extensions in* $\sigma[M]$ if $\mathtt{C} = \mathtt{E}_M(\mathtt{C}, \mathtt{C})$.

In case $\sigma[M] = R$–Mod we put $\mathtt{E}(\mathtt{D}, \mathtt{C}) := \mathtt{E}_M(\mathtt{D}, \mathtt{C})$.

These operations yield a 'product' of ideals (see [15]) related to the given classes based on the following observation:

1.3 Proposition. *Let* $\mathtt{C}$ *and* $\mathtt{D}$ *be subclasses of* R–Mod *which are closed under isomorphisms. For any left ideal* I *of* R*,* $R/I \in \mathtt{E}(\mathtt{C},\mathtt{D})$ *if and only if there exists a left ideal* $J \supset I$ *of* R *such that* $J/I \in \mathtt{C}$ *and* $R/J \in \mathtt{D}$*.*

Proof: Assume $R/I \in \mathtt{E}(\mathtt{C},\mathtt{D})$. Then there exists an exact sequence

$$0 \longrightarrow C \xrightarrow{f} R/I \longrightarrow D \longrightarrow 0$$

where $C \in \mathtt{C}$ and $D \in \mathtt{D}$. For the unique left ideal $J \supset I$ of R with $\mathsf{Im}\, f = J/I$, $J/I \simeq C \in \mathtt{C}$ and $R/J \simeq D \in \mathtt{D}$.

The converse implication is trivial. $\square$

Properties of $\mathtt{C}$ and $\mathtt{D}$ are transferred to $\mathtt{E}(\mathtt{D},\mathtt{C})$:

1.4 Properties of $\mathtt{E}(\mathtt{D},\mathtt{C})$. *Let* $\mathtt{C}$ *and* $\mathtt{D}$ *be subclasses of* $\sigma[M]$*.*

(i) *If* $\mathtt{C}$ *and* $\mathtt{D}$ *are closed under submodules (factor modules, direct sums) then* $\mathtt{E}_M(\mathtt{D},\mathtt{C})$ *is also closed under submodules (resp. factor modules, direct sums).*

(ii) *If* $\mathtt{C}$ *and* $\mathtt{D}$ *are (finitely) closed subcategories of* R–Mod *then* $\mathtt{E}_M(\mathtt{D},\mathtt{C})$ *is also a (finitely) closed subcategory of* $\sigma[M]$*.*

Proof: Let $0 \to C \to N \to D \to 0$ be an exact sequence in $\sigma[M]$ with $C \in \mathtt{C}$ and $D \in \mathtt{D}$. If $0 \to L \to N$ is exact we obtain (as a pullback $(\star)$) the following commutative exact diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P & \longrightarrow & L & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \downarrow & \star & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C & \longrightarrow & N & \longrightarrow & D & \longrightarrow & 0.
\end{array}$$

If $\mathtt{C}$ and $\mathtt{D}$ are closed under submodules we have $P \in \mathtt{C}$ and $Q \in \mathtt{D}$ and hence $L \in \mathtt{E}_M(\mathtt{D},\mathtt{C})$.

The other assertions are proved similarly. $\square$

2 Torsion theory in $\sigma[M]$

We recall some notions from torsion theory. These techniques are familiar from R–Mod but it is well-known that they also apply to Grothendieck categories. For basic facts we refer to [10] or [28].

In [38] some of this notions were applied for $\sigma[M]$. In [4], [5] and [6] torsion theory in $\sigma[M]$ was used to define strongly and properly semiprime modules. Also the investigations in [17], [18], [40] and [41] on generalized composition series may be considered as a part of torsion theory in $\sigma[M]$.

2.1 Definitions. A class $\mathcal{T}$ of modules in $\sigma[M]$ is called a

pretorsion class if $\mathcal{T}$ is closed under direct sums and factor modules;

hereditary pretorsion class if $\mathcal{T}$ is closed under direct sums, factor and submodules;

torsion class if $\mathcal{T}$ is closed under direct sums, factors and extensions in $\sigma[M]$;

hereditary torsion class if $\mathcal{T}$ is closed under direct sums, factors, submodules and extensions in $\sigma[M]$;

stable class if $\mathcal{T}$ is closed under essential extensions in $\sigma[M]$;

TTF class if $\mathcal{T}$ is closed under direct products in $\sigma[M]$, factor modules, submodules and extensions in $\sigma[M]$.

A pair $(\mathtt{T}, \mathtt{F})$ of subclasses of $\sigma[M]$ is called a *torsion theory in* $\sigma[M]$ if they satisfy

(i) $\mathtt{T} = \{T \in \sigma[M] \mid \text{ for all } F \in \mathtt{F}, \operatorname{Hom}_R(T, F) = 0\}$,

(ii) $\mathtt{F} = \{F \in \sigma[M] \mid \text{ for all } T \in \mathtt{T}, \operatorname{Hom}_R(T, F) = 0\}$.

In this case $\mathtt{T}$ is a torsion class and $\mathtt{F}$ is closed under submodules, direct products and extensions. $(\mathtt{T}, \mathtt{F})$ is called *hereditary* if $\mathtt{T}$ is closed under submodules, *cohereditary* if $\mathtt{F}$ is closed under factor modules, *stable* if $\mathtt{T}$ is closed under essential extensions in $\sigma[M]$.

Any class of modules in $\sigma[M]$ can be extended to a torsion class (see [10, 28]):

2.2 Generating and cogenerating torsion theories

For any subclass $\mathtt{C}$ *of* $\sigma[M]$ *we define*

$$\begin{aligned} \mathtt{T}(\mathtt{C}) &:= \{T \in \sigma[M] \mid \textit{ for all } C \in \mathtt{C}, \operatorname{Hom}_R(T, C) = 0\}, \\ \mathtt{F}(\mathtt{C}) &:= \{F \in \sigma[M] \mid \textit{ for all } C \in \mathtt{C}, \operatorname{Hom}_R(C, F) = 0\}. \end{aligned}$$

Then $\mathtt{C} \subset \mathtt{T}(\mathtt{F}(\mathtt{C}))$ *and* $\mathtt{C} \subset \mathtt{F}(\mathtt{T}(\mathtt{C}))$. *Moreover,*

(i) *the following assertions are equivalent:*

  (a) $(\mathtt{C}, \mathtt{F}(\mathtt{C}))$ *is a torsion theory,*

  (b) $\mathtt{C} = \mathtt{T}(\mathtt{F}(\mathtt{C}))$,

  (c) $\mathtt{C}$ *is a torsion class;*

(ii) *the following assertions are equivalent:*

  (a) $(\mathtt{T}(\mathtt{C}), \mathtt{C})$ *is a torsion theory,*

  (b) $\mathtt{C} = \mathtt{F}(\mathtt{T}(\mathtt{C}))$,

  (c) $\mathtt{C}$ *is closed under submodules, direct products and extensions;*

(iii) *if* $(\mathtt{C}, \mathtt{F}(\mathtt{C}))$ *is a torsion theory the following are equivalent:*

  (a) $(\mathtt{C}, \mathtt{F}(\mathtt{C}))$ *is hereditary,*

  (b) $\mathtt{F}(\mathtt{C})$ *is closed under injective hulls,*

  (c) *there exists an* M*-injective* $Q \in \sigma[M]$ *with* $\mathtt{F}(\mathtt{C}) = \mathrm{Cog}\,(Q)$*;*

(iv) *the following assertions are equivalent:*

  (a) $(\mathtt{C}, \mathtt{F}(\mathtt{C}))$ *and* $(\mathtt{T}(\mathtt{C}), \mathtt{C})$ *are torsion theories,*

  (b) $\mathtt{C}$ *is a TTF class.*

For $M = R$, TTF classes can be described by the trace ideal ([10, II.3.1], [28, VI.8]):

2.3 TTF classes in R–Mod. *For a class* $\mathtt{C}$ *in* R–Mod *the following are equivalent:*

(a) $(\mathtt{C}, \mathtt{F}(\mathtt{C}))$ *and* $(\mathtt{T}(\mathtt{C}), \mathtt{C})$ *are torsion theories;*

(b) $\mathtt{C}$ *is a TTF class;*

(c) *there exists an idempotent ideal* I *in* R *such that*

$$\begin{array}{rcl} \mathtt{C} & = & \{C \in R\text{–}\mathsf{Mod} \mid IC = 0\} \; (= R/I\text{–}\mathsf{Mod}), \\ \mathtt{T}(\mathtt{C}) & = & \{T \in R\text{–}\mathsf{Mod} \mid IT = T\}, \\ \mathtt{F}(\mathtt{C}) & = & \{F \in R\text{–}\mathsf{Mod} \mid \mathsf{Hom}_R(R/I, F) = 0\}. \end{array}$$

3 $\mathrm{Gen}\,(U)$ closed under extensions

We investigate under which conditions the classes introduced are closed under extensions. For this we need some more definitions.

3.1 Definitions. Let P be an R-module and $p : N \to L$ an epimorphism in R–Mod. P is called *pseudo-projective with respect to* p if for all non-zero $f \in \mathsf{Hom}_R(P, L)$ there are $s \in \mathrm{End}(P)$ and $g : P \to N$ satisfying $gp = sf \neq 0$, i.e. the exact diagram

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow{\scriptstyle f} & & \\ N & \xrightarrow{p} & L & \longrightarrow & 0 \end{array}$$

can be non-trivially extended to the commutative diagram

$$\begin{array}{ccccc} P & \xrightarrow{s} & P & & \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \\ N & \xrightarrow{p} & L & \longrightarrow & 0. \end{array}$$

Adapting a notation from [12], P is called *im-projective with respect to* p if the above conditions are satisfied with s an epimorphism.

As usual we say P is *projective with respect to* p if the above conditions are satisfied with s an isomorphism (or $s = id_P$).

It is easy to verify that P is pseudo-projective with respect to p if and only if $\mathsf{Hom}_R(P, N)p$ is essential in $\mathsf{Hom}_R(P, L)$ as an $\mathrm{End}(P)$-submodule.

We will be interested in modules which are pseudo-projective with respect to different classes of epimorphisms.

3.2 Definitions. Let M be an R-module and $U, P \in \sigma[M]$. P is called

U-pseudo-projective in $\sigma[M]$ if P is pseudo-projective with respect to all epimorphisms $p : N \to L$ in $\sigma[M]$ with $\mathrm{Ke}\, p \in \mathrm{Gen}\,(U)$;

self-pseudo-projective in $\sigma[M]$ if it is *P-pseudo-projective* in $\sigma[M]$;

pseudo-projective in $\sigma[M]$ if it is pseudo-projective with respect to all epis in $\sigma[M]$;

im-projective in $\sigma[M]$ if it is im-projective with respect to all epis in $\sigma[M]$.

Instead of *pseudo-projective in* R–Mod we will just say *pseudo-projective* for short.

Two R-modules U and M are called *trace equivalent* if $\mathrm{Gen}\,(U) = \mathrm{Gen}\,(M)$. In R–Mod we have the following relationship (compare [12, 1.1]):

3.3 Proposition. *For an R-module U the following are equivalent:*

(a) *U is pseudo-projective in R–*Mod*;*

(b) *for some set Λ, $U^{(\Lambda)}$ is im-projective;*

(c) *U is trace equivalent to an im-projective R-module;*

(d) *there exists a free (projective) R-module P and $f \in \mathrm{End}(P)$ with $\mathrm{Im}\ f = \mathrm{Im}\ f^2$, such that U and $\mathrm{Im}\ f$ are trace equivalent;*

(e) *there exists a free (projective) R-modul P and $f, h \in \mathrm{End}(P)$ with $f = hf^2$, such that U and $\mathrm{Im}\ f$ are trace equivalent.*

With the notions introduced we are now able to describe the case

3.4 Self-pseudo-projective modules in $\sigma[M]$
For an R-module M and $U \in \sigma[M]$, the following are equivalent:

(a) *U is self-pseudo-projective in $\sigma[M]$;*

(b) $\mathrm{Gen}\,(U)$ *is closed under extensions in $\sigma[M]$;*

(c) *every generator in* $\mathrm{Gen}\,(U)$ *is U-pseudo-projective in $\sigma[M]$;*

(d) *every $N \in \sigma[M]$ with an exact sequence $U^{(\Lambda)} \to N \to U^{(\Lambda)} \to 0$ belongs to* $\mathrm{Gen}\,(U)$.

Proof: $(a) \Rightarrow (b)$ Let $0 \to K \to L \to N \to 0$ be an exact sequence in $\sigma[M]$ with $K, N \in \mathrm{Gen}\,(U)$. We have to show that $L \in \mathrm{Gen}\,(U)$. Suppose $\mathrm{Tr}(U, L) \neq L$. Then $K \subset \mathrm{Tr}(U, L)$ and $N' := L/\,\mathrm{Tr}(U, L)$ is a factor module of N and hence is U-generated. For any non-zero $f : U \to N'$ we obtain (by (b)) a commutative diagram

$$\begin{array}{ccccccccc} & & & & U & \xrightarrow{s} & U & \to & 0 \\ & & & & \downarrow g & & \downarrow f & & \\ 0 & \to & \mathrm{Tr}(U, L) & \to & L & \xrightarrow{p} & N' & \to & 0, \end{array}$$

where $gp \neq 0$. This contradicts $(U)g \subset \mathrm{Tr}(U, L)$.

$(b) \Rightarrow (a)$ Let $p : N \to L$ be an epimorphism in $\sigma[M]$ with $\mathrm{Ke}\ p \in \mathrm{Gen}\,(U)$. For any non-zero $f \in \mathrm{Hom}_R(U, L)$ we form a pullback to obtain the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & K & \to & P & \xrightarrow{s'} & U & \to & 0 \\ & & \| & & \downarrow g' & & \downarrow f & & \\ 0 & \to & K & \to & N & \xrightarrow{p} & L & \to & 0. \end{array}$$

Since $g'p = s'f \neq 0$ and $P \in \mathrm{Gen}\,(U)$ (by (a)) there exists $h : U \to P$ satisfying $hg'p = hs'f \neq 0$. This shows that U is U-pseudo-projective in $\sigma[M]$.

The remainig implications are standard (compare 4.2). □

For $M = R$, 3.4 yields the following characterizations of modules which include [36, Proposition 2.1] (see [7, 8.4]):

3.5 Self-pseudo-projective modules

For an R-module U the following are equivalent:

(a) *U is self-pseudo-projective;*

(b) $\mathrm{Gen}\,(U)$ *is closed under extensions;*

(c) *every generator in* $\mathrm{Gen}\,(U)$ *is self-pseudo-projective;*

(d) *every $N \in R$–Mod with an exact sequence $U^{(\Lambda)} \to N \to U^{(\Lambda)} \to 0$ belongs to* $\mathrm{Gen}\,(U)$.

The above notions are closely related to ext-projective modules considered in representation theory of algebras. Adapting a definition from Auslander-Smalø [2] we say:

3.6 Definition. Let M be an R-module and $U, P \in \sigma[M]$. P is called *U-ext-projective* in $\sigma[M]$ if P is projective with respect to all epimorphisms $p : N \to L$ in $\sigma[M]$ with $\mathrm{Ke}\, p \in \mathrm{Gen}\,(U)$, i.e. any exact diagram

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow_f & & \\ N & \xrightarrow{p} & L & \longrightarrow & 0 \end{array}$$

can be extended commutatively by some $P \to N$ provided $\mathrm{Ke}\, p \in \mathrm{Gen}\,(U)$ and $N \in \sigma[M]$.

U is called *self-ext-projective* if it is U-ext-projective in R–Mod. Obviously such a module is self-pseudo-projective.

3.7 Properties. *Let M be an R-module and $U, P \in \sigma[M]$.*

(1) *Direct sums and direct summands of U-ext-projective modules are again U-ext-projective (in $\sigma[M]$).*

(2) *A direct sum of copies of a self-ext-projective module is again self-ext-projective (in* $\sigma[M]$*).*

(3) P *is* U*-ext-projective if and only if every exact sequence*

$$0 \to K \to X \xrightarrow{p} P \to 0$$

in $\sigma[M]$ *with* $\mathrm{Ke}\, p \in \mathrm{Gen}\,(U)$ *splits, i.e.* $\mathrm{Ext}^1_R(P, K) = 0$ *(in* $\sigma[M]$*) for every* $K \in \mathrm{Gen}\,(U)$*.*

Proof: (1) is easily seen by standard arguments.

(2) Assume U is self-ext-projective. Then $\mathrm{Gen}\,(U) = \mathrm{Gen}\,(U^{(\Lambda)})$ and the assertion follows from (1).

(3) This follows from the pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & X & \longrightarrow & P & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0. \end{array}$$

□

As mentioned before, any self-ext-projective module is self-pseudo-projective. The inverse implication does not hold in general: Consider a self-pseudo-projective module U with a U-generated submodule $K \subset U$. Obviously $U \oplus U/K$ is a generator in $\mathrm{Gen}\,(U)$ and hence it is self-pseudo-projective by 3.5(c). If K is not a direct summand in U, then $U \oplus U/K$ is certainly not self-ext-projective.

However, under suitable conditions we may find a self-ext-projective generator in $\mathrm{Gen}\,(U)$ (compare Assem [1]).

Definition. A generator $X \in \mathrm{Gen}\,(U)$ is called *minimal* if for any decomposition $X = X' \oplus X''$, $\mathrm{Tr}(X', X'') \neq X''$.

For example, if U is finitely generated and a direct sum of modules with local endomorphism rings, then there exists a minmal U-generator.

3.8 Proposition. *Let* $U \in \sigma[M]$ *be finitely generated and minimal (in the above sense) and* $T = \mathrm{End}(U)$*. Assume that* T *is right perfect or* U_T *is finitely generated and* T *is semiperfect. Then the following assertions are equivalent:*

(a) $\mathrm{Gen}\,(U)$ *is closed under extensions in* $\sigma[M]$*;*

(b) U *is self-pseudo-projective in* $\sigma[M]$*;*

(c) *U is self-ext-projective in $\sigma[M]$.*

Proof: It remains to show that $(a) \Rightarrow (c)$. We adopt an argument in Assem [1, 1.3].

Assume (a). We have to show that $\text{Ext}^1_R(U, K) = 0$ for all $K \in \text{Gen}(U)$.

Let $U = U_0 \oplus U_1$ with $S := \text{End}(U_0)$ local. By minimality of U, $\text{Tr}(U_1, U_0) \neq U_0$. Suppose that $\text{Ext}^1_R(U_0, K) \neq 0$ for some $K \in \text{Gen}(U)$. Consider a non-split extension

$$0 \to K \to E \xrightarrow{f} U_0 \to 0.$$

By (a), E is U-generated and hence there is an epimorphism

$$U_1^k \oplus U_0^k = U^k \xrightarrow{h} E \xrightarrow{f} U_0.$$

Since f is not a retraction, none of the $hf|_{U_0} \in S$ is an isomorphism and hence they all belong to Jac S. By our assumptions, $\text{Rad}_S(U_0)$ is a small S-submodule of U_0. Hence we conclude

$$U_0 = (U_1^k)hfS \subset \text{Tr}(U_1, U_0),$$

a contradiction to the minimality of U. □

Next we characterize pseudo-projective modules in $\sigma[M]$. The implications are more or less straightforward.

3.9 Pseudo-projective modules in $\sigma[M]$

For $U \in \sigma[M]$ the following assertions are equivalent:

(a) *U is pseudo-projective in $\sigma[M]$;*

(b) *for $N \in \sigma[M]$ and $\text{Tr}(U, N) \subset L \subset N$ with N/L cocyclic, $\text{Tr}(U, N/L) = 0$;*

(c) *for any $N \in \sigma[M]$ and $K \subset N$, $\text{Tr}(U, N/K) = (\text{Tr}(U, N) + K)/K$;*

(d) *$(\text{Gen}(U), \mathtt{F}(\text{Gen}(U))$ is a cohereditary torsion theory in $\sigma[M]$.*

Pseudo-projective modules U in R–Mod can be characterized by their trace ideals $\text{Tr}(U, R)$. For this we recall the following facts:

3.10 Lemma. *Let I be a left ideal in R and N an R-module. Then $IN \in \text{Gen}(I)$, hence $IN \subset \text{Tr}(I, N)$. If I is idempotent then $IN = \text{Tr}(I, N)$.*

The assertions in 3.9 can be extended by properties of the trace ideal. Some of the resulting characterizations (and some others) may be found in [36, 2.3] and [9, 2.1,3.5].

3.11 Pseudo-projective modules in R–Mod *(see [7, Satz 8.7])*
For an R-module U and $T := \mathrm{Tr}(U,R)$ the following are equivalent:

(a) *U is pseudo-projective;*

(b) *for any free R-module N and $K \subset N$, $\mathrm{Tr}(U, N/K) = (\mathrm{Tr}(U,N)+K)/K$;*

(c) *$(\mathrm{Gen}\,(U), \mathtt{F}(\mathrm{Gen}\,(U))$ is a cohereditary torsion theory;*

(d) *there is a free (projective) R-module P and an epimorphism $p_0 : P \to U$ with $P = \mathrm{Ke}\ p_0 + \mathrm{Tr}(U,P)$;*

(e) *there exists a free (projective) R-module P and an epimorphism $p_0 : P \to U$ and a homomorphism $q_0 : U^{(\Delta)} \to P$ such that $q_0 p_0 : U^{(\Delta)} \to U$ is epic;*

(f) *for every R-module L, $\mathrm{Tr}(U,L) = TL$;*

(g) *$U = TU$;*

(h) $\mathtt{F}(\mathrm{Gen}\,(U)) = R/T$-Mod.

If these conditions hold then $T = T^2$ and $\mathrm{Gen}\,(U) = \mathrm{Gen}\,(T)$.

4 $\sigma[U]$ closed under extensions

$\sigma[U]$ is closed under extensions in $\sigma[M]$ if and only if it is a hereditary torsion class in $\sigma[M]$. To study this case recall that there always exists a generator G in $\sigma[U]$. Then $\mathrm{Gen}\,(G) = \sigma[U]$ and we can refer to 3.4. Note that this is not quite satisfying because we cannot see the property under consideration directly from the module M.

4.1 $\boldsymbol{\sigma[U]}$ closed under extensions in $\sigma[M]$
For $U \in \sigma[M]$ the following assertions are equivalent:

(a) *$\sigma[U]$ is closed under extensions in $\sigma[M]$;*

(b) *there exists a generator G in $\sigma[U]$ which is G-pseudo-projective in $\sigma[M]$;*

(c) *every generator in $\sigma[U]$ is self-pseudo-projective in $\sigma[M]$;*

(d) *$\sigma[U]$ contains all cyclic modules from $\mathtt{E}_M(\sigma[U], \sigma[U])$;*

(e) *every $N \in \sigma[M]$ which allows an exact sequence $U^{(\Lambda)} \to N \to U^{(\Lambda)}$ (for some set Λ) belongs to $\sigma[U]$.*

Proof: By the preceding remark the equivalence of (a), (b) and (c) follows from 3.5. $(a)\Rightarrow(d)\Rightarrow(e)$ are obvious.

$(e)\Rightarrow(a)$ Let $0\to K\to L\to N\to 0$ be an exact sequence in R–Mod, $K,N\in\sigma[U]$. For a suitable set Λ, we can find an epimorphism $g: U^{(\Lambda)}\to X$ and embeddings $K\hookrightarrow X$ and $N\hookrightarrow X$. For $V:=(N)g^{-1}\subset U^{(\Lambda)}$ we have $(V)g=N$. A pushout $(\star)$ yields the commutative exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & N & \longrightarrow & 0\\
 & & \downarrow & \star & \downarrow & & \| & & \\
0 & \longrightarrow & X & \longrightarrow & P & \longrightarrow & N & \longrightarrow & 0.
\end{array}$$

By a pullback $(\star)$ we obtain the commutative exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & Q & \longrightarrow & V & \longrightarrow & 0\\
 & & \| & & \downarrow & \star & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & P & \longrightarrow & N & \longrightarrow & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & . &
\end{array}$$

Combining these yields an exact sequence $U^{(\Lambda)}\longrightarrow Q\longrightarrow U^{(\Lambda)}$.
By (c), $Q\in\sigma[U]$ and hence $P\in\sigma[U]$ and $L\in\sigma[U]$. □

Putting $U=R$ the above theorem has the following form:

4.2 $\sigma[U]$ closed under extensions in R–Mod

For an R-module U the following assertions are equivalent:

(a) *$\sigma[U]$ is closed under extensions;*

(b) *there exists a self-pseudo-projective generator in $\sigma[U]$;*

(c) *every generator in $\sigma[U]$ is self-pseudo-projective;*

(d) *$\sigma[U]$ contains all cyclic modules from $\mathtt{E}(\sigma[U],\sigma[U])$;*

(e) *every R-module N which allows an exact sequence $U^{(\Lambda)}\to N\to U^{(\Lambda)}$ (for some set Λ) belongs to $\sigma[U]$.*

In view of 4.1 it is natural to consider the case when $\sigma[U]$ has a generator which is pseudo-projective in $\sigma[U]$. This case is essentially described by 3.9.

When $\sigma[U]$ has a pseudo-projective generator in R–Mod we can again refer to properties of the trace ideal. For this several techniques developed in different situations are helpful. We recall some facts and definitions.

4.3 Flat factor rings

For a right ideal J of R the following are equivalent:

(a) *R/J is a flat right R-module;*

(b) *the exact sequence $0 \to J \to R \to R/J \to 0$ is pure in* Mod-*R;*

(c) *for every left ideal I of R, $JI = J \cap I$.*

If J is a (two-sided) ideal in R, then the following are also equivalent to (a)-(c):

(d) *every injective left R/J-module is R-injective;*

(e) *R/J*-Mod *contains an R-injective cogenerator.*

In Tominaga [29] modules over rings T without identity were considered. He calls a left T-module N *s-unital* if $u \in Tu$ for every $u \in N$. We refer to [37] for an account on the relationship of this notion to $\sigma[U]$.

For an ideal $T \subset R$, every R-module is a T-module and we have:

4.4 s-unital T-modules

Let T be an ideal in R. For any R-module N the following are equivalent:

(a) *N is an s-unital T-module;*

(b) *for every submodule $_RL \subset {}_RN$, $L = TL$;*

(c) *for any $k \in \mathbb{N}$ and $n_1, \ldots, n_k \in N$, there exists $t \in T$ with $n_i = tn_i$ for $i = 1, \ldots, k$;*

(d) *for any set Λ, $N^{(\Lambda)}$ is an s-unital T-module.*

With straightforward arguments we obtain (see 4.3):

4.5 s-unital rings

For an ideal T in R the following are equivalent:

(a) *T is left s-unital;*

(b) *for every left ideal I of R, $TI = T \cap I$;*

(c) *for every R-module ${}_RL \subset {}_RN$, $TL = TN \cap L$;*

(d) *R/T is a flat right R-module.*

We are now ready for our next theorem:

4.6 $\sigma[U]$ with a pseudo-projective generator *(see [7, Satz 9.8])*

Let U be an R-module, $G := \bigoplus\{K \subset U^{(\mathbb{N})} \mid K$ cyclic $\}$ and $T := \mathrm{Tr}(\sigma[U], R) = \mathrm{Tr}(G, R)$. The following assertions are equivalent:

(a) *G is pseudo-projective;*

(b) *every generator in $\sigma[U]$ is pseudo-projective;*

(c) *$\sigma[U]$ is closed under extensions and $\mathtt{F}(\sigma[U])$ is closed under factor modules;*

(d) *$(\sigma[U], \mathtt{F}(\sigma[U]))$ is a cohereditary torsion theory;*

(e) *G is an s-unital T-module;*

(f) *every module in $\sigma[U]$ is an s-unital T-module;*

(g) *U is an s-unital T-module;*

(h) *for any $k \in \mathbb{N}$ and $g_1, \ldots, g_k \in G$, $R = T + \bigcap_{i=1}^k \mathsf{Ann}_R(g_i)$;*

(i) *for every $g \in G$, $R = T + \mathsf{Ann}_R(g)$;*

(j) *for every $N \in \sigma[U]$ the canonical morphismus $\varphi_N : T \otimes_R N \to N$ is a bijection;*

(k) *$U = TU$ and $(R/T)_R$ is U-flat.*

Proof: The equivalences (a)-(e) follow from 3.11.

$(e) \Rightarrow (h)$ For $k \in \mathbb{N}$ and $g_1, \ldots, g_k \in G$, consider the monomorphism

$$\gamma : R/\bigcap_{i=1}^k \mathsf{Ann}_R(g_i) \longrightarrow G^k, \quad \overline{r} \longmapsto (rg_i)_{1 \le i \le k}.$$

By (e), $T \cdot \mathsf{Im}\,\gamma = \mathsf{Im}\,\gamma$. From $\mathsf{Im}\,\gamma \simeq R/\bigcap_{i=1}^k \mathsf{Ann}_R(g_i)$ we conclude

$$T \cdot R/\bigcap_{i=1}^n \mathsf{Ann}_R(g_i) = R/\bigcap_{i=1}^n \mathsf{Ann}_R(g_i),$$

i.e. $R = T + \bigcap_{i=1}^n \mathsf{Ann}_R(g_i)$.

$(f) \Rightarrow (g)$ and $(h) \Rightarrow (i)$ are trivial.

$(g) \Rightarrow (e)$ By (g) and 4.4, G is a submodule of the s-unital T-module $U^{(\Lambda)}$ and hence is s-unital.

$(i) \Rightarrow (e)$ For $g \in G$ there exist (by (i)) elements $x \in \mathsf{Ann}_R(g)$, $t \in T$ with $1_R = x + t$. Hence $g = (x+t)g = tg \in Tg$, i.e. G is an s-unital T-module.

$(e) \Rightarrow (j)$ For $N \in \sigma[U]$, $N = TN$ (by (e)), i.e. φ_N is surjective. To prove injectivity of φ_N consider $\sum_{i=1}^k t_i \otimes n_i \in \mathsf{Kern}\,\varphi_N$ (where $t_i \in T$, $n_i \in N$), i.e. $\sum_{i=1}^k t_i n_i = 0$. Since ${}_RT$ is an s-unital T-module there exist $t \in T$ with $t_i = tt_i$ for all $i = 1, \ldots, k$ (by 4.4). From this we obtain

$$\sum_{i=1}^k t_i \otimes n_i = \sum_{i=1}^k tt_i \otimes n_i = t \otimes \sum_{i=1}^k t_i n_i = 0.$$

Hence φ_N is injective.

$(j) \Rightarrow (e)$ Consider a submodule $K \subset G$ and form the commutative exact diagram

$$\begin{array}{ccccccccc} & & T \otimes_R K & \longrightarrow & T \otimes_R G & \longrightarrow & T \otimes_R G/K & \longrightarrow & 0 \\ & & \downarrow\varphi_K & & \downarrow\varphi_G & & \downarrow\varphi_{G/K} & & \\ 0 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & G/K & \longrightarrow & 0. \end{array}$$

By (j), φ_G and $\varphi_{G/K}$ are isomorphisms. Then φ_K is epic and $K = \mathrm{Im}\ \varphi_K = TK$, i.e. G is s-unital.

$(a) \Rightarrow (k)$ $U = TU$ is clear by $(a) \Rightarrow (h)$. Since G is pseudo-projective, $T = T^2$ (by 3.11). By 4.7, $(R/T)_R$ is flat and hence, in particular, U-flat.

$(k) \Rightarrow (g)$ For any submodule $N \subset U$ consider the commutative diagram with canonical isomorphisms

$$\begin{array}{ccccc} 0 & \longrightarrow & R/T \otimes_R N & \longrightarrow & R/T \otimes_R U \\ & & \downarrow\simeq & & \downarrow\simeq \\ 0 & \longrightarrow & N/TN & \longrightarrow & U/TU\,. \end{array}$$

Since $(R/T)_R$ is U-flat the rows are exact. From $U/TU = 0$ follows $N/TN = 0$, i.e. $N = TN$ and hence U is an s-unital T-module. $\square$

Some of the characterizations in 4.7 were obtained in [42, 2.4], [22, Th.1.9] and [21, Th.2.2]. Combining the preceding results with 3.11 we obtain the following observations about idempotent ideals part of which were proved in [25, 1.2].

4.7 Corollary. *For an idempotent ideal T of R the following are equivalent:*

(a) $\mathsf{Gen}\,_R T$ *is closed under submodules;*

(b) *for any R-modules* $_RL \subset {_RN}$, $TL = TN \cap L$*;*

(c) R/T *is a flat right R-module;*

(d) T *is a left s-unital ring;*

(e) *for every* $t \in T$, $R = T + \mathsf{Ann}_R(t)$*;*

(f) *for any* $k \in I\!N$ *and* $t_1, \ldots, t_k \in T$, $R = T + \bigcap_{i=1}^k \mathsf{Ann}_R(t_i)$*;*

(g) *every injective left* R/T*-module is R-injective;*

(h) R/T-Mod *contains an R-injective cogenerator.*

5 Stable subcategories

Recall that a class $\mathcal{T} \subset \sigma[M]$ is called *stable in* $\sigma[M]$ if $\mathcal{T}$ is closed under essential extensions in $\sigma[M]$.

5.1 Stable subcategories in $\sigma[M]$ *(see [7, Satz 5.6])*
For $U \in \sigma[M]$ *the following assertions are equivalent:*

(a) $\sigma[U]$ *is stable in* $\sigma[M]$*;*

(b) $\sigma[U]$ *is closed under M-injective hulls;*

(c) *every U-injective module in* $\sigma[U]$ *is M-injective;*

(d) *for any* $N \in \sigma[M]$, $\mathrm{Tr}(\sigma[U], N)$ *is essentially closed in* N*;*

(e) *for any M-injective module* $N \in \sigma[M]$, $\mathrm{Tr}(U, N)$ *is a direct summand in* N*;*

(f) *for any U-injective module* $N \in \sigma[M]$, $\mathrm{Tr}(U, N)$ *is M-injective.*

Proof: The assertions follow from basic properties of M-injectivity. □

Notice that for a stable subcategory $\sigma[U] \subset \sigma[M]$, U-injective modules in $\sigma[M]$ need not be M-injective. In fact we have (see [7, Satz 5.8]):

5.2 Proposition. *For $U \in \sigma[M]$ the following are equivalent:*

(a) *Every U-injective module in $\sigma[M]$ is M-injective;*

(b) *every U-injective module in $\sigma[U]$ and every module $X \in \sigma[M]$ for which $\mathrm{Tr}(\sigma[U], X) = 0$, is M-injective.*

Proof: $(a) \Rightarrow (b)$ If $\mathrm{Tr}(\sigma[U], X) = 0$ then X is trivially U-injective.

$(b) \Rightarrow (a)$ Consider a U-injective $X \in \sigma[M]$. Then $Y := \mathrm{Tr}(\sigma[U], X) \subset X$ is also U-injective, hence M-injective (by (b)) and therefore a direct summand in X. For ${}_RZ \subset {}_RX$ with $X = Y \oplus Z$, $\mathrm{Tr}(\sigma[U], Z) = 0$ and hence Z - and X - are M-injective. $\square$

5.3 Proposition. *Let $U \in \sigma[M]$ and assume that $\sigma[U]$ is stable in $\sigma[M]$. Then $\sigma[U]$ is closed under extensions in $\sigma[M]$.*

Proof: Let $0 \to K \to L \to N \to 0$ be an exact sequence in $\sigma[M]$, where $K, N \in \sigma[U]$ and $L \in \sigma[M]$. By assumption the M-injective hull $\widehat{K}$ of K belongs to $\sigma[U]$. A pushout $(\star)$ yields the exact commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow & \star & \downarrow & & \| & & \\ 0 & \longrightarrow & \widehat{K} & \longrightarrow & Q & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

The lower row splits. Hence $Q \simeq \widehat{K} \oplus N \in \sigma[U]$ and $L \in \sigma[U]$. $\square$

6 $\mathrm{Cog}\,(U)$ closed under extensions

The fact that $\mathrm{Cog}\,(U)$ is closed under extensions is related to injectivity conditions for the module U which are dual to the projectivity conditions considered above.

6.1 Definitions. Two R-module N and L are called

reject equivalent if $\mathrm{Cog}\,(N) = \mathrm{Cog}\,(L)$;

subisomorphic if there are momorphisms $N \to L$ and $L \to N$.

We observe the following relationship between these notions ([7, A.2]):

6.2 Proposition. *For two R-modules N, L the following are equivalent:*

(a) *N and L are reject equivalent;*

(b) *there is a set Λ such that N^Λ and L^Λ are subisomorphic.*

The notions dual to those of 3.1 are as follows:

6.3 Definitions. Let Q be an R-module and $h : L \to N$ a monomorphism in R–Mod. Q is called *pseudo-injective with respect to h* if for any non-zero $f : L \to Q$ there exist $s \in \mathrm{End}(Q)$ and $g : N \to Q$ with $fs = hg \neq 0$, i.e. the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \xrightarrow{h} & N \\ & & \downarrow_f & & \\ & & Q & & \end{array}$$

can be extended non-trivially to a commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \xrightarrow{h} & N \\ & & \downarrow_f & & \downarrow_g \\ & & Q & \xrightarrow{s} & Q; \end{array}$$

Obviously Q is pseudo-injective with respect to h if and only if $h\mathsf{Hom}_R(N,Q)$ is an essential $\mathrm{End}(Q)$-submodule of $\mathsf{Hom}_R(L,Q)$.

Q is called *kern-injective with respect to h* if the above conditions can be satisfied with s monic.

Q is *injective with respect to h* if the above conditions can be satisfied with s an isomorphism (or $s = id_Q$).

We will apply these notions with respect to different classes of monomorphisms.

6.4 Definitions. Let M be an R-module and $U, Q \in \sigma[M]$. Q is called

U-pseudo-injective in $\sigma[M]$ if Q is pseudo-injective with respect to all monos $h : L \to N$ in $\sigma[M]$ with Coke $h \in \mathrm{Cog}\,(U)$;

self-pseudo-injective in $\sigma[M]$ if it is pseudo-injective with respect to all monos $h : L \to N$ in $\sigma[M]$ with Coke $h \in \mathrm{Cog}\,(Q)$;

pseudo-injective in $\sigma[M]$ if Q is pseudo-injective with respect to all monos in $\sigma[M]$;

ker-injective in $\sigma[M]$ if it is ker-injective with respect to all monos in $\sigma[M]$.

Dual to 3.4 we have (see also [36, 2.2]):

6.5 Self-pseudo-injective modules in $\sigma[M]$

For $U \in \sigma[M]$ the following assertions are equivalent:

(a) *U is self-pseudo-injective in $\sigma[M]$;*

(b) *if $h : L \to N$ is a mono in $\sigma[M]$ with* Coke $h \in \text{Cog}\,(U)$ *and $h\mathsf{Hom}_R(N,U) = 0$, then* $\text{Hom}(L,U) = 0$*;*

(c) $\text{Cog}\,(U)$ *is closed under extensions;*

(d) $(\mathtt{T}(U), \text{Cog}\,(U))$ *is a torsion theory;*

(e) *every $N \in \sigma[M]$ with an exact sequence $0 \to U^\Lambda \to N \to U^\Lambda$ belongs to* $\text{Cog}\,(U)$.

The next result is dual to 3.9. Compare also [36, 2.4], [10, IV.7.1], [31] and [19].

6.6 Pseudo-injective modules in $\sigma[M]$

For $U \in \sigma[M]$ the following assertions are equivalent:

(a) *U is pseudo-injective in $\sigma[M]$;*

(b) *for all monos $h : L \to N$ in $\sigma[M]$ with $h\mathsf{Hom}_R(N,U) = 0$, $\mathsf{Hom}_R(L,U) = 0$;*

(c) $\text{Cog}\,(U)$ *is closed under extensions,* $\mathtt{T}(U)$ *is closed under submodules;*

(d) $(\mathtt{T}(U), \text{Cog}\,(U))$ *is a hereditary torsion theory;*

(e) *for every monomorphism $h : L \to N$ in $\sigma[M]$ with $L \in \text{Cog}\,(U)$ there exists some $g : N \to U^\Lambda$ such that $hg : L \to U^\Lambda$ is monic;*

(f) $\widehat{U} \in \text{Cog}\,(U)$.

By (f), M is pseudo-injective in $\sigma[M]$ if and only if M cogenerates its M-injective hull. Such modules are also called *(self) QF-3' modules*.

As a special case of the above result we have (compare [11]):

6.7 Ker-injective modules in $\sigma[M]$

For $U \in \sigma[M]$ the following assertions are equivalent:

(a) *U is ker-injective;*

(b) *for all monos $h : L \to N$ and all $f : L \to U$ in $\sigma[M]$ there exists $g : N \to U$ with* Ke $hg \subset$ Ke f*;*

(c) *for all monos $h : U \to N$ in $\sigma[M]$ there exists $g : N \to U$ such that $hg : U \to U$ is monic;*

(d) *U and $\widehat{U}$ are subisomorphic;*

(e) *U is subisomorphic to an M-injective module.*

Between pseudo-injective and ker-injective modules we have the connection:

6.8 Proposition. *For $U \in \sigma[M]$ the following assertions are equivalent:*

(a) *U is pseudo-injective in $\sigma[M]$;*

(b) *there is a set Λ such that U^Λ is ker-injective;*

(c) *U is reject equivalent to a ker-injective module in $\sigma[M]$.*

7 Linear topologies

Any linear topology $\mathcal{T}$ on ${}_RR$ is characterized by the $\mathcal{T}$-discrete modules which form a closed subcategory of R–Mod. As noticed above this category is of type $\sigma[M]$ and hence every linear topology on ${}_RR$ is of the following type:

7.1 The M-adic topology. For any R-module M, the set

$$\{\text{Ke } f \mid f : R \to N,\ N \in \sigma[M]\}$$

forms a filter of left ideals of R which is a *linear topology* of R, called the *M-adic topology* of R. A basis of the neighbourhoods of 0 is given by

$$T_M := \{\text{Ke } f \mid k \in I\!N,\ f \in \mathsf{Hom}_R(R, M^k)\}.$$

Since $\sigma[M]$ is uniquely determined by its cyclic modules there is a bijective correspondence between the subcategories $\sigma[M]$ of R–Mod and the left linear topologies on R. So every linear topology on R is an M-adic topology for some R-module M.

Asking which properties of an M-adic topology correspond to which property of $\sigma[M]$ we note as a first example:

7.2 Jansian topologies

An M-adic topology on R is called a *Jansian topology* if $\sigma[M]$ is closed under direct products in R–Mod. It is easy to verify that this is equivalent to the fact that $\sigma[M] = R/I$–Mod, where $I = Ann_R(M)$.

Linear topologies for which the discrete modules are closed under extensions deserve special attention:

7.3 Gabriel topologies
An M-adic topology is called a *Gabriel topology* if $\sigma[M]$ is closed under extensions.

We can apply 4.2 to study these topologies. The property that $\sigma[M]$ is closed under extensions can be described by the fact that the filter of left ideals corresponding to $\sigma[M]$ is closed under the product described in 5.3 (see [15], [33]).

7.4 Stable topologies
An M-adic topology is called a *stable topology* if $\sigma[M]$ is closed under injective hulls.

Properties of such topologies can be derived from 5.1. We know from 5.3 that stable topologies are in particular Gabriel topologies.

An interesting interplay between the M-adic topology and $\sigma[M]$ is the following observation:

7.5 Closed ideals
Let Q be a cogenerator in $\sigma[M]$ and $I \subset R$ a left ideal. Then the following are equivalent:

(a) *I is closed in the M-adic topology;*

(b) *R/I is cogenerated by Q, i.e. $R/I \subset Q^{\Lambda}$ (product in R–Mod).*

The above result is useful for our next investigations.

7.6 Definition. Two left linear topologies on R are called *equivalent* if they have the same closed left ideals in R.

It follows from 7.5 that an M-adic topology is equivalent to an N-adic topology if and only if the cogenerators in $\sigma[M]$ and $\sigma[N]$ cogenerate the same cyclic R-modules.

7.7 The Leptin topology
The coarset topology which is equivalent to a given linear topology $\mathcal{T}$ on ${}_RR$ is called the *Leptin topology for $\mathcal{T}$*. The set

$$\mathcal{B}_* := \{ {}_RI \subset {}_RR \mid I \in \mathcal{T} \text{ and } R/I \text{ finitely cogenerated } \}$$

is a basis for the open left ideals of the Leptin topology for $\mathcal{T}$.

For the M-adic topology we have the interesting fact (compare [23, L.5,L.6]):

Let $\{E_\lambda\}_\Lambda$ *be a minimal representing set of the simple modules in* $\sigma[M]$ *and* $K := \bigoplus_\Lambda \widehat{E_\lambda}$ *(minimal cogenerator in* $\sigma[M]$*).*

Then the K*-adic topology is the Leptin topology for the* M*-adic topology on* R*.*

7.8 The Warner topology

The finest topology which is equivalent to a given linear topology $\mathcal{T}$ on ${}_RR$ is called the *Warner topology for* $\mathcal{T}$. The set

$$\mathcal{B}^* := \{{}_RI \subset {}_RR \mid \text{ all left ideals } J \text{ of } R \text{ with } I \subset J \text{ are closed}\}$$

is a basis for the open left ideals of the Leptin topology for $\mathcal{T}$.

The existence of the Warner topology was shown in [30] by using a duality argument. The main problem is to show that $\mathcal{B}^*$ is closed under finite intersections.

It is an open question to find an R-module U (somehow related to $\sigma[M]$) such that the U-adic topology is the Warner topology for the M-adic topology.

References

[1] I. Assem, *Tilting Theory - An Introduction*, Topics in Algebra, Banach Center Publ. 26, PWN Warsaw (1990), 127-180.

[2] M. Auslander, S.O. Smalø, *Almost split sequences in subcategories*, J. Algebra 69 (1981), 426-454, Addendum J. Algebra 71 (1981), 592-594.

[3] J. A. Beachy, *Cotorsion radicals and projective modules* Bull. Austral. Math. Soc. 5 (1971), 241-253.

[4] K.I. Beidar, R. Wisbauer, *Strongly semiprime modules and rings*, Usp. Mat. Nauk 48 (1993), 161-162.

[5] K.I. Beidar, R. Wisbauer, *Strongly and properly semiprime rings and modules*, Ring Theory, Proc. Ohio State-Denison Conf. 1992, World Sci. Pub. Singapore (1993), 58-94.

[6] K.I. Beidar, R. Wisbauer, *Properly semiprime self-pp-modules*, Comm. Algebra 23 (1995), 841-861.

[7] J. Berning, *Beziehungen zwischen links-linearen Toplogien und Modulkategorien*, Dissertation, Universität Düsseldorf (1994).

[8] L. Bican, *Corational extensions and pseudoprojective modules*, Acta Math. Acad. Sci. Hungaria 28 (1976), 5-11.

[9] L. Bican, P. Jambor, T. Kepka, P. Němec, *Pseudoprojective modules*, Math. Slovaca 29 (1979), 106-115.

[10] L. Bican, T. Kepka, P. Němec, *Rings, modules, and preradicals*, New York : Marcel Dekker, 1982.

[11] G. F. Birkenmeier, *Modules which are subisomorphic to injective modules*, J. Pure Appl. Algebra 13 (1978), 169-177.

[12] G. F. Birkenmeier, *Modules which are epi-equivalent to projective modules*, Acta Univ. Carolinae-Math. Phys. 24 (1983), 9-16.

[13] N.V. Dung, D.V. Huynh, P.F. Smith, R. Wisbauer, *Extending Modules*, Pitman Research Notes in Math. 313, London (1994).

[14] V. K. Goel, *A study of cotorsion theory*, Ring Theory : Proc. Ohio Univ. Conference, Marcel Dekker, New York (1977), 113-132.

[15] J. S. Golan, *Linear topologies on a ring : an overview*, Pitman, New York (1987)

[16] E. Gregorio, *Classical Morita equivalence for linear topologies; appl. to quasi-dualities*, Comm. Algebra 18 (1990), 1137-1146.

[17] T. Izawa, *Composition series relative to a module*, J. Pure Appl. Algebra 35 (1985), 15-33.

[18] T. Izawa, *A generalization of composition series*, J. Algebra 118 (1988), 388-407.

[19] A.I. Kashu, *When is the radical associated with a module a torsion*, Math. Notes 16 (1974), 608-612.

[20] H. Komatsu, *The category of s-unital Modules*, Math. J. Okayama Univ. 28 (1986), 65-91.

[21] You Lin, *Cotorsion theories*, Northeastern Math. J. 8 (1992), 110-120.

[22] R. J. McMaster, *Cotorsion theories and colocalization*, Can. J. Math. 27 (1975), 618-628.

[23] C. Menini, *Linearly compact rings and strongly quasi-injective modules*, Rend. Sem. Mat. Univ. Padova 65 (1980), 251-262.

[24] R. W. Miller *Endomorphism rings of finitely generated projective modules*, Pac. J. Math. 47 (1973), 199-220.

[25] R. W. Miller, *TTF classes and quasi-generators*, Pac. J. Math. 51 (1974), 499-507.

[26] V. S. Ramamurthi, *Weakly regular rings*, Canad. Math. Bull. 16 (1973), 317-321.

[27] E. A. Rutter,jr., *Torsion theories over semiperfect rings*, Proc. Amer. Math. Soc. 34 (1972), 389-395.

[28] Bo Stenström, *Rings of quotients*, Springer, Berlin (1975)

[29] H. Tominaga, *On s-unital rings*, Math. J. Okayama Univ. 18 (1976), 117-134.

[30] A. Tonolo, *On the existence of a finest equivalent linear topology*, Comm. Algebra 20 (1992), 437-455.

[31] G. M. Tsukerman, *Pseudo-injective modules and self-pseudo-injective rings*, Math. Notes 7 (1970), 220-226.

[32] J. E. Viola-Prioli, *When is every kernel functor idempotent?*, Canad. J. Math. 27 (1975), 545-554.

[33] A. M. Viola-Prioli, J. E. Viola-Prioli, *Rings whose kernel functors are linearly ordered*, Pac. J. Math. 132 (1988), 21-34.

[34] A. M. de Viola-Prioli, J. E. Viola-Prioli, R. Wisbauer, *Module categories with linearly ordered closed categories*, Comm. Algebra 22 (1994), 3613-3627.

[35] A. M. de Viola-Prioli, J. E. Viola-Prioli, R. Wisbauer, *A description of closed subcategories of $\sigma[M]$*, Comm. Algebra 23 (1995), 4173-4188.

[36] T. Wakamatsu, *Pseudo-projectives and pseudo-injectives in Abelian categories*, Math. Rep. Toyama Univ. 2 (1979), 133-142.

[37] R. Wisbauer, *Co-semisimple modules and nonassociative V-rings*, Comm. Algebra 5 (1977), 1193-1209.

[38] R. Wisbauer, *Localization of modules and the central closure of rings*, Comm. Algebra 9 (1981), 1455-1493.

[39] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Reading (1991)

[40] Xu Bangteng, *The lattice of P-cotorsionless submodules of a module*, J. Algebra 150 (1992), 28-34.

[41] Xu Bangteng, *On composition series relative to a module*, Comm. Algebra 20 (1992), 1279-1297.

[42] B. Zimmermann-Huisgen, *Endomorphism rings of self-generators*, Pac. J. Math. 61 (1975), 567-602.

Address:

Robert Wisbauer
Mathematical Institute of the University
40225 Düsseldorf, Germany
e-mail: wisbauer@math.uni-duesseldorf.de