Dual coalgebras of algebras over commutative rings

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Abstract

In the study of algebraic groups the representative functions related to monoid algebras over fields provide an important tool which also yields the finite dual coalgebra of any algebra over a field. The purpose of this note is to transfer this basic construction to monoid algebras over commutative rings R. As an application we obtain a bialgebra (Hopf algebra) structure on the finite dual of the polynomial ring $R[x]$ over a noetherian ring R. Moreover we give a sufficient condition for the finite dual of any R-algebra A to become a coalgebra. In particular this condition is satisfied provided R is noetherian and hereditary.

Introduction

Let k be a field and consider a group G. The commutative Hopf algebra $\mathcal{R}_k(G)$ of all k-valued representative functions over G plays a prominent role in the finite dimensional representation theory of G (e.g., [4]). From the point of view of the algebraic theory of Hopf algebras, $\mathcal{R}_k(G)$ can be considered as the dual Hopf algebra $k[G]^\circ$ of the group algebra $k[G]$. In fact, the construction of the dual coalgebra A° of any k-algebra can be performed by means of the k-valued representative functions of a monoid (see [1, Ch. 2]). This leads to the dual Hopf algebra H° of any Hopf algebra H over the field k. In this paper we study to which extent these basic constructions are possible for (Hopf) algebras over commutative rings.

The definition of comultiplication over the finite dual A° of a k-algebra uses the exactness of the tensor product bifunctor $-\otimes_k -$, and the existence of a basis in any vector space (cf. [1, Ch. 2]). Considering algebras A over a commutative ring R, technical difficulties arise at the very beginning because of the lack of these properties. Of course, if A is finitely generated and projective over R then A° is just $A^* = \mathrm{hom}_R(A, R)$, which is known to be an R-coalgebra, without any condition on R. The construction of the coalgebra structure

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of the finite dual A° for algebras over fields (see [8, Ch. VI]) is extended to (non finitely generated) projective R-algebras in Cao-Yu - Nichols [2], provided R is a Dedekind domain.

Here we choose a different approach to this problem (similar to [1]) by first considering representative functions of a monoid G to R (Section 1). For the following results we need R to be a noetherian ring.

Considering any R-algebra A as a multiplicative monoid, we define the finite dual A° and - assuming that $A^\circ \subset R^A$ is a pure R-submodule - we obtain that A° is a coalgebra (Theorem 2.8). This implies that A° is a coalgebra, provided R is a noetherian hereditary ring, e.g., a Dedekind domain (Proposition 2.11, Corollary 2.14).

Under the condition that the finite dual $R[G]^\circ$ of the monoid R-algebra $R[G]$ is R-pure in $R[G]^*$, we prove that $R[G]^\circ$ is a bialgebra isomorphic to the bialgebra of all R-valued representative functions (Theorem 2.13). In Section 3 we show in particular that for the polynomial algebra $R[x]$, the finite dual $R[x]^\circ$ is a bialgebra. Another possibility for a coproduct on $R[x]$ is to consider x as a primitive element. This makes $R[x]$ a Hopf R-algebra and moreover yields a Hopf R-algebra structure on $R[x]^\circ$.

In Section 4 we show that $R[x]^\circ$ can be identified with the linearly recursive sequences over R which can be given an R-bialgebra structure and a Hopf R-algebra structure corresponding to those on $R[x]^\circ$ (introduced in Section 3).

1 Representative functions

Let R be a noetherian commutative ring. For any set S, consider the R-algebra R^S of all maps $f : S \to R$. The symbol $\otimes$ always denotes the tensor product over R.

1.1. Canonical map. Let S, T be sets. For R-submodules $X \subseteq R^S$ and $Y \subseteq R^T$, define the canonical map

$$\pi : X \otimes Y \to R^{S\times T},$$

given on generators by $\pi(f \otimes g)(s,t) = f(s)g(t)$, for every $f \otimes g \in X \otimes Y$ and $(s,t) \in S \times T$. Observe that the canonical map depends on the R-submodules X and Y but we will always denote it with the same letter π. The map

$$\pi : R^S \otimes R^T \to R^{S\times T}$$

is a homomorphism of R-algebras. Since R is noetherian, R^S and R^T are flat R-modules.

1.2. Proposition. *The canonical map $\pi : R^S \otimes R^T \to R^{S\times T}$ is injective.*

Proof. Let M be a f.g. submodule of R^S. Since R is noetherian we have

$$M \otimes R^T \simeq M^T \subset (R^S)^T \simeq R^{S\times T},$$

and this monomorphism is just the restriction of π to $M \otimes R^T$. Going to the direct limit, we see that π is injective. $\square$

1.3. Monoid algebras. Let G be a monoid with neutral element e and denote by $R[G]$ the associated monoid algebra. The algebra R^G is endowed with a structure of an $R[G]$-bimodule as follows: For $x, y \in G$ and $f \in R^G$ consider $xfy \in R^G$, defined by $xfy(z) = f(yzx)$ for all $z \in G$. These left and right G-actions are extended in a unique way to make R^G an $R[G]$-bimodule.

1.4. Maps between R^G and $R^{G\times G}$. Let $m : G \times G \to G$ be the multiplication map of the monoid G. It induces an R-module homomorphism

$$m^\bullet : R^G \to R^{G\times G} \text{ given by } m^\bullet(f)(x,y) = f(xy), \text{ for every } f \in R^G, x, y \in G.$$

Consider the maps $I \times m, m \times I : G \times G \times G \to G \times G$, where I denotes the identity map on G. The associativity of m implies

$$(I \times m)^\bullet \circ m^\bullet = (m \times I)^\bullet \circ m^\bullet.$$

On the other hand, defining $\alpha, \beta : R^{G\times G} \to R^G$ by $\alpha(h)(x) = h(x,e)$, and $\beta(h)(x) = h(e,x)$ for every $h \in R^{G\times G}$, $x \in G$, we have

$$\alpha \circ m^\bullet = I = \beta \circ m^\bullet.$$

We are now ready to study R-valued representative functions on the monoid G.

1.5. Finiteness conditions. Let B be an $R[G]$-subbimodule of R^G. Define

$$\begin{aligned} {}^{\mathsf{f}}B &= \{b \in B \mid R[G]b \text{ is f.g. as an } R\text{-module}\}, \quad \textit{and} \\ B^{\mathsf{f}} &= \{b \in B \mid bR[G] \text{ is f.g. as an } R\text{-module}\}. \end{aligned}$$

It is easy to prove that ${}^{\mathsf{f}}B$ is a left and B^{f} is a right $R[G]$-submodule of B.

The following is the technical result that supports all our constructions.

1.6. Proposition. *For an element $f \in R^G$ the following are equivalent:*

(i)	$f \in {}^{\mathsf{f}}B$;	*(i)′*	$f \in B^{\mathsf{f}}$;
(ii)	$m^\bullet(f) \in \pi(B \otimes R^G)$;	*(ii)′*	$m^\bullet(f) \in \pi(R^G \otimes B)$;
(iii)	$m^\bullet(f) \in \pi({}^{\mathsf{f}}B \otimes R^G)$;	*(iii)′*	$m^\bullet(f) \in \pi(R^G \otimes B^{\mathsf{f}})$;
(iv)	$m^\bullet(f) \in \pi({}^{\mathsf{f}}B \otimes R^G) \cap \pi(R^G \otimes B^{\mathsf{f}})$;		
(v)	*$f \in B$ and $R[G]fR[G]$ is finitely generated as an R-module.*		

As a consequence $B^{\mathsf{f}} = {}^{\mathsf{f}}B$ and B^{f} is an $R[G]$-subbimodule of B.

Proof. $(i) \Rightarrow (iii)$. We have that $R[G]f$ is a finitely generated R-submodule of ${}^{\mathsf{f}}B$. Hence, there are $b_1, \dots, b_n \in {}^{\mathsf{f}}B$ such that $R[G]f = \sum_{i=1}^n Rb_i$. For each $y \in G$, choose $f_1(y), \dots, f_n(y) \in R$ such that $yf = \sum_{i=1}^n f_i(y)b_i$. Now, for $x, y \in G$,

$$m^\bullet(x,y) = f(xy) = (yf)(x) = \sum_{i=1}^n f_i(y)b_i(x) = \pi(\sum_{i=1}^n b_i \otimes f_i)(x,y)$$

Thus, $m^\bullet(f) \in \pi({}^{\mathsf{f}}B \otimes R^G)$.

$(iii) \Rightarrow (ii)$. This is evident since ${}^{\mathsf{f}}B \subseteq B$.

$(ii) \Rightarrow (i)'$. First we prove $f \in B$. In fact, for $x \in G$, we have

$$f(x) = f(xe) = m^\bullet(x,e) = \sum_{i=1}^{n} b_i(x) f_i(e),$$

where $m^\bullet(f) = \pi(\sum_{i=1}^n b_i \otimes f_i)$, $b_i \in B$ and $f_i \in R^G$. Hence, $f = \sum_{i=1}^n f_i(e) b_i \in B$. Now, for $y, x \in G$,

$$(fy)(x) = f(yx) = m^\bullet(f)(y,x) = \sum_{i=1}^{n} b_i(y) f_i(x) = \left(\sum_{i=1}^{n} b_i(y) f_i \right)(x).$$

Therefore, $fy \in Rf_1 + \cdots + Rf_n$. Since R is noetherian, $fR[G]$ is a finitely generated R-module.

$(i)' \Rightarrow (iii)' \Rightarrow (ii)' \Rightarrow (i)$ follow by symmetry.

$(iii), (iii)' \Leftrightarrow (iv)$ and $(v) \Rightarrow (i)$ are clear.

$(i) \Rightarrow (v)$ For any $x, y \in G$, we will prove that xfy is contained in a fixed finitely generated R-module. Let $b_1, \ldots, b_n \in B$ be such that $R[G]f = \sum_{i=1}^n Rb_i$. Now, $R[G]b_i \subseteq R[G]f$, whence $R[G]b_i$ is finitely generated, and thus $b_i \in {}^{\mathsf{f}}B$. We have already proved that $(i) \Rightarrow (i)'$, so $b_i \in B^{\mathsf{f}}$. This means that $b_i R[G]$ is finitely generated as an R-module and, therefore, $xfy \in \sum_{i=1}^n b_i R[G]$ which is a finitely generated R-module that does not depend on $x, y \in G$. □

1.7. Representative functions. Consider the particular case $B = R^G$. Given a function $f \in R^G$, it follows from Proposition 1.6 that $R[G]f$ is finitely generated as an R-module if and only if $fR[G]$ is. In this case, f will be said to be an *R-valued representative function* on the monoid G. The set $\mathcal{R}_R(G)$ of all representative functions on G is an $R[G]$-subbimodule of R^G, since $\mathcal{R}_R(G) = (R^G)^{\mathsf{f}}$. Moreover we deduce from Proposition 1.6:

1.8. Corollary. *The following conditions are equivalent for $f \in R^G$:*

(i) $f \in \mathcal{R}_R(G)$,

(ii) $m^\bullet(f) \in \pi(R^G \otimes R^G)$,

(iii) $m^\bullet(f) \in \pi(\mathcal{R}_R(G) \otimes R^G)$,

(iii)' $m^\bullet(f) \in \pi(R^G \otimes \mathcal{R}_R(G))$,

(iv) $m^\bullet(f) \in \pi(\mathcal{R}_R(G) \otimes R^G) \cap \pi(R^G \otimes \mathcal{R}_R(G))$.

Recall that an R-submodule $W \subset V$ is *pure* if, for each R-module X, the canonical map $W \otimes X \to V \otimes X$ is injective.

1.9. Comultiplication on B^{f}. Assume for an $R[G]$-subbimodule $B \subset R^G$, B^{f} is a pure R-submodule of R^G. So, by Proposition 1.6, for any $b \in B^{\mathsf{f}}$ and $\pi : R^G \otimes R^G \to R^{G\times G}$,

$$m^\bullet(b) \in \pi(R^G \otimes B^{\mathsf{f}}) \cap \pi(B^{\mathsf{f}} \otimes R^G) = \pi((R^G \otimes B^{\mathsf{f}}) \cap (B^{\mathsf{f}} \otimes R^G)) = \pi(B^{\mathsf{f}} \otimes B^{\mathsf{f}}).$$

Therefore we have the R-linear map

$$\Delta := B^{\mathsf{f}} \xrightarrow{m^\bullet} \pi(B^{\mathsf{f}} \otimes B^{\mathsf{f}}) \xrightarrow{\pi^{-1}} B^{\mathsf{f}} \otimes B^{\mathsf{f}}$$

We show that this comultiplication on B^{f} is co-associative. For this consider the diagram

$$\begin{tikzcd}
R^G \arrow[rrr, "m^\bullet"] \arrow[ddd, "m^\bullet"'] & & & R^{G\times G} \arrow[ddd, "(I\times m)^\bullet"] \\
& B^{\mathsf{f}} \arrow[r, "\Delta"] \arrow[d, "\Delta"] \arrow[ul] & B^{\mathsf{f}} \otimes B^{\mathsf{f}} \arrow[d, "I\otimes\Delta"] \arrow[ur] & \\
& B^{\mathsf{f}} \otimes B^{\mathsf{f}} \arrow[r, "\Delta\otimes I"] \arrow[dl] & B^{\mathsf{f}} \otimes B^{\mathsf{f}} \otimes B^{\mathsf{f}} \arrow[dr] & \\
R^{G\times G} \arrow[rrr, "(m\times I)^\bullet"] & & & R^{G\times G\times G}.
\end{tikzcd}$$

The external rectangle is commutative because m is an associative map. Also, all the trapezia, whose non parallel edges are the obvious canonical maps, are commutative. Since B^{f} is pure in R^G, the canonical map $B^{\mathsf{f}} \otimes B^{\mathsf{f}} \otimes B^{\mathsf{f}} \to R^{G\times G\times G}$ is injective. Therefore, the internal square is commutative.

1.10. Counit. With the neutral element e of the monoid G, we define

$$\epsilon : R^G \to R,\ h \mapsto h(e),$$

and use the same letter to denote the restriction

$$\epsilon : B^{\mathsf{f}} \to R,\ h \mapsto h(e).$$

It is easy to prove that ϵ is a counit for the comultiplication $\Delta : B^{\mathsf{f}} \to B^{\mathsf{f}} \otimes B^{\mathsf{f}}$. So we have shown:

1.11. Theorem. *If B^{f} is pure as an R-submodule of R^G, then $(B^{\mathsf{f}}, \Delta, \epsilon)$ is a co-associative co-unitary R-coalgebra.*

1.12. Corollary. *If $\mathcal{R}_R(G)$ is pure in R^G as an R-module, then $(\mathcal{R}_R(G), \Delta, \epsilon)$ is a co-associative co-unitary R-coalgebra. Moreover, with this coalgebra structure, $\mathcal{R}_R(G)$ is an R-bialgebra when considered as a subalgebra of R^G.*

Proof. Notice that if $B = R^G$, then $B^{\mathsf{f}} = \mathcal{R}_R(G)$, whence it is an R-coalgebra. Now a straightforward computation shows that $\mathcal{R}_R(G)$ is a subalgebra of R^G and that Δ is an algebra map. $\square$

2 Dual coalgebra

Let (A, m_A, u_A) be an associative unitary R-algebra. Throughout this section we will assume R to be noetherian. Denote by A^* the R-submodule of R^A consisting of all R-linear maps, i.e., $A^* = \hom_R(A, R)$. For any R-map $f : M \to A$, where M is an R-module, put $f^\circ := f^*|_{A^\circ} : A^\circ \to M^*$.

2.1. The monoid ring $R[A]$. Considering A as a multiplicative monoid we have the monoid ring $R[A]$. The $R[A]$-bimodule structure on R^A is given as follows. For $(r_a)_{a\in A} \in R[A]$, $f \in R^A$ and $b \in A$ we put

$$((r_a)_{a\in A} \cdot f)(b) = \sum_{a\in A} r_a f(b\ a) \quad \text{and} \quad (f \cdot (r_a)_{a\in A})(b) = \sum_{a\in A} r_a f(ab).$$

It can be easily checked that A^* is an $R[A]$-subbimodule of R^A. On the other hand, A^* has the structure of an A-bimodule which coincides on elements of A with the previous one, that is, for $a, b \in A$ and $f \in A^*$, we have

$$(af)(b) = f(ba) \quad \text{and} \quad (fa)(b) = f(ab).$$

2.2. Lemma. *Let $X \subseteq A^*$ be an R-submodule. Then X is a left (resp. right) $R[A]$-submodule of A^* if and only if X is a left (resp. right) A-submodule of A^*. In particular, the $R[A]$-subbimodules and the A-subbimodules of A^* coincide.*

Proof. Straightforward. $\square$

2.3. Finite dual of A. Adopting the notation used for algebras over fields we define

$$A^\circ = \{f \in A^* \mid Af \text{ is f.g. as an } R\text{-module}\}.$$

Observe that, for $f \in A^*$, $Af = R[A]f$ and $fA = fR[A]$. It follows from Proposition 1.6 and Lemma 2.2 that $A^\circ = (A^*)^{\mathsf{f}}$ and, hence, A° is an A-subbimodule of A^*. Moreover, we have the equality

$$A^\circ = \{f \in A^* \mid fA \text{ is f.g. as an } R\text{-module}\}.$$

The following proposition is a particular case of Proposition 1.6.

2.4. Proposition. *For $f \in R^A$, the following statements are equivalent:*

(i)	$f \in A^\circ$;		
(ii)	$m^\bullet(f) \in \pi(A^* \otimes R^A)$;	*(ii)′*	$m^\bullet(f) \in \pi(R^A \otimes A^*)$;
(iii)	$m^\bullet(f) \in \pi(A^\circ \otimes R^A)$;	*(iii)′*	$m^\bullet(f) \in \pi(R^A \otimes A^\circ)$;
(iv)	$m^\bullet(f) \in \pi(A^\circ \otimes R^A) \cap \pi(R^A \otimes A^\circ)$;		
(v)	*$f \in A^*$ and AfA is finitely generated as an R-module.*		

2.5. Cofinite submodules. An R-submodule X of A is called *R-cofinite* if A/X is a finitely generated R-module. With this notion we recover the characterization of the finite dual of algebras over fields.

2.6. Proposition. *The following statements are equivalent for $f \in A^*$:*

(i) $f \in A^\circ$;

(ii) $\operatorname{Ker} f$ *contains an R-cofinite ideal of A;*

(iii) $\operatorname{Ker} f$ *contains an R-cofinite left ideal of A;*

(iv) $\operatorname{Ker} f$ *contains an R-cofinite right ideal of A.*

Proof. $(i) \Rightarrow (ii)$ If $f \in A^\circ$ then, by Proposition 2.4, AfA is finitely generated as an R-module. Let $f_1, \ldots, f_n$ be a set of generators and consider $I = \bigcap_{i=1}^n \operatorname{Ker} f_i$. It is clear that I is an R-cofinite R-submodule of A. Moreover, $I \subseteq \operatorname{Ker} f$. Let us see that I is an ideal of A. For $a, b \in A$ and $c \in I$, we have $f_i(acb) = (bf_ia)(c)$. Since $bf_ia \in AfA = \sum_{i=1}^n Rf_i$, it follows that $(bf_ia)(c) = 0$, whence $acb \in I$.

$(ii) \Rightarrow (iii)$ This is evident.

$(iii) \Rightarrow (i)$ Let I be an R-cofinite left ideal of A contained in $\operatorname{Ker} f$. Define $\varphi : Af \to \operatorname{hom}_R(A/I, R)$ by $\varphi(af)(b + I) = (af)(b) = f(ba)$. Easy computations show that φ is a well-defined injective homomorphism of R-modules. Since R is noetherian and A/I is R-finitely generated, it follows that $\operatorname{hom}_R(A/I, R)$ is a finitely generated R-module. But Af is isomorphic to an R-submodule of $\operatorname{hom}_R(A/I, R)$, which implies that Af is finitely generated as an R-module. By definition, $f \in A^\circ$.

Finally, the equivalence between (iv) and (i) follows by symmetry. □

2.7. Dual coalgebra. The constructions in 1.9 and 1.10 can be reinterpreted here as follows. Assume that A° is pure in R^A as an R-module. For every $f \in A^\circ$ there is a unique $\sum_{i=1}^n f_i \otimes g_i \in A^\circ \otimes A^\circ$ such that

$$f(ab) = \sum_{i=1}^n f_i(a) g_i(b),$$

for every $a, b \in A$. Then the map

$$m_A^\circ : A^\circ \to A^\circ \otimes A^\circ, \quad f \mapsto \sum_{i=1}^n f_i \otimes g_i\,,$$

is a well-defined co-associative R-linear comultiplication over A°. Moreover, the restriction

$$u_A^\circ : A^\circ \to R, \quad f \mapsto f(1_A),$$

defines a counit.

2.8. Theorem. *Let R be noetherian and assume A° to be pure in R^A as an R-module. Then*

(i) $(A^\circ, m_A^\circ, u_A^\circ)$ *is a co-associative co-unitary R-coalgebra.*

(ii) Assume $(A, m_A, u_A, \Delta_A, \epsilon_A)$ *is an R-bialgebra. Then* $(A^\circ, \Delta_A^\circ, \epsilon_A^\circ, m_A^\circ, u_A^\circ)$ *is an R-bialgebra. Moreover if A is a Hopf R-algebra with antipode S_A, then A° is a Hopf R-algebra with antipode S_A°.*

Proof. (i) By 2.7 this is a specialization of Theorem 1.11.

(ii) The proof is similar to the argument in [5, 9.1.3] due to the fact that over noetherian rings, submodules of finitely generated modules are again finitely generated. □

2.9. Let $R[A]$ be the free R-module with basis A and consider the surjective homomorphism of R-modules $R[A] \to A$ sending every $a \in A$ to itself. Moreover, by the universal property of free modules, we have an isomorphism of R-modules $R[A]^* \cong R^A$, given by restriction to the basis A of linear maps defined over $R[A]$. Notice that the composition $A^* \to R[A]^* \cong R^A$ is just the inclusion $A^* \subseteq R^A$.

2.10. There are various kinds of conditions which imply that A° is an R-pure submodule of R^A. For example, if A is a left semisimple algebra (left artinian with zero Jacobson radical) then A° is a direct summand of R^A as a left A-module and so, in particular, it is an R-pure submodule. One might also ask which conditions on the ring R imply this property. Clearly for every field or semisimple ring this is the case. Proposition 2.11 describes two more situations where A° is an R-pure submodule of R^A.

The ring R is *hereditary* if every ideal is projective. A noetherian ring R is hereditary if and only if every submodule of an R-cogenerated module is flat (e.g., [9, 39.13]).

2.11. Proposition. *Let A be an algebra over a noetherian ring R.*

(i) Assume R to be hereditary. Then A° is pure in R^A.

(ii) Assume A to be projective as an R-module. Then A° is a pure R-submodule of A^* if and only if A° is pure in R^A.

Proof. (i) The conditions on R imply that for every epimorphism of R-modules $X \to Y$, the monomorphism $Y^* \to X^*$ is pure. So in particular the monomorphism $A^* \to R[A]^*$ given in 2.9 is pure, i.e., A^* is a pure submodule of R^A.
It remains to prove that A° is a pure submodule of A^*. For each ideal I of A, define

$$I^\perp = \{f \in A^* \mid I \subseteq \mathrm{Ker}(f)\},$$

which is an R-submodule of A^*. By Proposition 2.6, $A^\circ = \bigcup I^\perp$, where I ranges over all cofinite ideals of A. On the other hand, $I^\perp$ is the image of the pure monomorphism $0 \to (A/I)^* \to A^*$, associated to the canonical projection $A \to A/I \to 0$ and so it is a pure submodule of A^*. Hence A° is a direct union of pure submodules of A^* and therefore it is pure in A^*.

(ii) Since A is projective as an R-module, the epimorphism $R[A] \to A$ given in 2.9 splits and so $A^* \to R[A]^* \cong R^A$ is a splitting monomorphism. Hence the assertion is evident. $\square$

Remark. In case R is a noetherian hereditary ring, Proposition 2.11 (i) combined with Theorem 2.8 sharpen the main result in [2] where the coalgebra structure of A° is defined over Dedekind domains R.

2.12. Let G be any monoid, R a noetherian commutative ring, and $R[G]$ the monoid algebra. The image of $R[G]^\circ$ under the isomorphism $R[G]^* \simeq R^G$ given in 2.9 is precisely the algebra of all representative functions $\mathcal{R}_R(G)$. In fact, $R[G]^\circ$ is a subalgebra of $R[G]^*$ and, thus, we have an isomorphism of R-algebras $R[G]^\circ \cong \mathcal{R}_R(G)$. Moreover, $\mathcal{R}_R(G)$ is R-pure in R^G if and only if $R[G]^\circ$ is R-pure in $R[G]^*$, so that we obtain the following result from Corollary 1.12 and Theorem 2.8:

2.13. Theorem. *Let G be any monoid, R noetherian, and $R[G]$ the monoid algebra. Assume that $R[G]^\circ$ is R-pure in $R[G]^*$. Then $R[G]^\circ$ and $\mathcal{R}_R(G)$ are isomorphic R-bialgebras.*

2.14. Corollary. *Let R be noetherian and hereditary. Then for any monoid G, $R[G]^\circ$ and $\mathcal{R}_R(G)$ are isomorphic R-bialgebras.*

2.15. Corollary. *Let G be a group and assume R to be noetherian. Then the bialgebra $R[G]$ is a Hopf algebra with antipode*

$$S : R[G] \to R[G],\ g \mapsto g^{-1},\ \textit{for } g \in G.$$

Assume the conditions of Theorem 2.13 (resp. Corollary 2.14) hold. Then

$$S^\circ : R[G]^\circ \to R[G]^\circ,\ \alpha \mapsto [g \mapsto \alpha(g^{-1}),\ \textit{for } g \in G]$$

yields an antipode for the bialgebra $R[G]^\circ$, which becomes a Hopf R-algebra.

3 The bialgebra $R[x]^\circ$

As an application of Theorem 2.13 consider the case $G = \mathbb{N} = \{0, 1, 2, ...\}$ and any commutative ring R. Using the fact that the monoid algebra $R[\mathbb{N}]$ and the polynomial algebra $R[x]$ are isomorphic, we show that - for R noetherian - $R[x]^\circ \simeq R[\mathbb{N}]^\circ$ is a pure R-submodule of $R[x]^*$, and so has a structure of an R-bialgebra.

An ideal $I \subset R[x]$ is called *monic* if it contains a polynomial with leading coefficient 1. The following properties of such ideals will be of importance.

3.1. Proposition. *1. Let $I \subset R[x]$ be a monic ideal. Then $R[x]/I$ is f.g. as an R-module.*

2. *Assume R is noetherian and $R[x]/I$ is f.g. as an R-module. Then I is a monic ideal in $R[x]$.*

Proof. (1) Let $g(x) = x^k + a_{k-1}x^{k-1} + \ldots + a_0 \in I$, and $h(x) \in R[x]$. Then there exist $q(x), r(x) \in R[x]$, such that

$$h(x) = q(x)g(x) + r(x), \text{ where } \deg(r(x)) < k.$$

So $\{1 + I, x + I, \ldots, x^{k-1} + I\}$ is a finite generating set of $R[x]/I$ over R.

(2) Assume $R[x]/I$ is f.g. as an R-module. Consider the chain

$$R\{1+I\} \subset \sum_{i=0}^{1} R\{x^i + I\} \subset \sum_{i=0}^{2} R\{x^i + I\} \subset \cdots$$

of R-submodules of $R[x]/I$. Since R is noetherian and $R[x]/I$ is f.g., $R[x]/I$ is also noetherian as an R-module. Hence there exists a positive integer n with

$$\sum_{i=0}^{n-1} R\{x^i + I\} = \sum_{i=0}^{n} R\{x^i + I\},$$

and so there are $r_0, r_1, \ldots, r_{n-1} \in R$, such that

$$x^n - r_{n-1}x^{n-1} - \ldots - r_1 x - r_0 \in I,$$

showing that I is a monic ideal in $R[x]$. $\square$

3.2. The coalgebra $\mathbf{R[x]^\circ}$. *Let R be noetherian. Then $R[x]^\circ$ is a co-associative co-unitary R-coalgebra with coproduct*

$$\Delta : R[x]^\circ \to R[x]^\circ \otimes R[x]^\circ, \ \alpha \mapsto [x^i \otimes x^j \mapsto \alpha(x^{i+j}), \ i, j \ \geq \ 0],$$

and counit

$$\varepsilon : R[x]^\circ \to R, \ \alpha \mapsto \alpha(1).$$

Proof. By Theorem 2.13 it suffices to show that $R[x]^\circ$ is a pure R-submodule of $R[x]^*$. As a consequence of Proposition 3.1, $R[x]^\circ$ may be identified with the direct union of $(R[x]/(g(x)))^*$ for all monic polynomials $g(x)$. Note that for a monic polynomial $g(x)$, the canonical R–linear map $R[x] \to R[x]/(g(x))$ splits because $R[x]/(g(x))$ is a free R–module and so $(R[x]/(g(x)))^*$ is a direct summand of $R[x]^*$. Hence

$$R[x]^\circ = \bigcup (R[x]/(g(x)))^*$$

is pure in $R[x]^*$.

The coproduct on $R[x]^\circ$ is induced by the usual product of polynomials in $R[x]$, $m : x^i \otimes x^j \mapsto x^{i+j}$, and the counit is induced by the unit of the algebra $R[x]$, $1 \mapsto x^0$ and so we have the formulas given above. $\square$

3.3. The algebra structures on $\mathbf{R[x]^\circ}$. Algebra structures on $R[x]^\circ$ are induced by coalgebra structures on $R[x]$. We consider two of these, given by the following coproducts and counits:

$$\Delta_1 : R[x] \to R[x] \otimes R[x],\ x^i \mapsto x^i \otimes x^i, \qquad \varepsilon_1 : R[x] \to R,\ x^i \mapsto 1,\ i \geq 0;$$

$$\Delta_2 : R[x] \to R[x] \otimes R[x],\ x^i \mapsto \textstyle\sum_{j=0}^{i} \binom{i}{j}\ x^j \otimes x^{i-j}, \quad \varepsilon_2 : R[x] \to R,\ x^i \mapsto \delta_{i,0},\ i \geq 0.$$

For $\alpha, \beta \in R[x]^\circ$, Δ_1 and ε_1 induce the product and the unit

$$\begin{aligned} &(\alpha \cdot \beta)(x^i) = (\alpha \otimes \beta)\ \Delta_1(x^i) = \alpha(x^i)\beta(x^i),\ i \geq 0, \\ &u_1 : R \to R[x]^\circ,\ 1 \mapsto [x^i \mapsto 1,\ i \geq 0]. \end{aligned}$$

whereas Δ_2 and ε_2 induce the product and the unit

$$\begin{aligned} &(\alpha * \beta)(x^i) = (\alpha \otimes \beta)\Delta_2(x^i) = \sum_{j=0}^{i} \tbinom{i}{j}\alpha(x^j)\beta(x^{i-j}), i \geq 0 \\ &u_2 : R \to R[x]^\circ, 1 \mapsto [x^i \mapsto \delta_{i,0},\ i \geq 0]. \end{aligned}$$

It is easy to see that the coproduct Δ_1 and the counit ε_1 are compatible with the usual algebra structure of $R[x]$ giving $R[x]$ an R-bialgebra structure. The same holds for Δ_2 and ε_2. In case R is a field it was shown in [3] that these are in fact the only possible R-bialgebra structures on $R[x]$.

Note that the R-bialgebra $(R[x], m, u, \Delta_1, \varepsilon_1)$ cannot be given a Hopf R-algebra structure because x is a group like element and in a Hopf algebra such elements have to be invertible.

3.4. A bialgebra structure on $\mathbf{R[x]^\circ}$. Let R be noetherian. As proved in 3.2, $R[x]^\circ$ is R-pure in $R[x]^*$ and $(R[x]^\circ, \Delta, \varepsilon)$ is an R-coalgebra. Combining this coalgebra structure with the algebra structure defined by "$\cdot$" and u_1 (3.3), and applying Theorem 2.13, it follows that $(R[x]^\circ, \cdot, u_1, \Delta, \varepsilon)$ is an R-bialgebra which is dual to the R-bialgebra $(R[x], m, u, \Delta_1, \varepsilon_1)$.

3.5. A Hopf algebra structure on $R[x]^\circ$. For the R-bialgebra $(R[x], m, u, \Delta_2, \varepsilon_2)$,

$$S : R[x] \to R[x],\ x^i \mapsto (-1)^i\ x^i,\ i \geq 0,$$

is an antipode and $(R[x], m, u, \Delta_2, \varepsilon_2, S)$ is a Hopf R-algebra.

Let R be noetherian. It is easy to see that Δ and ε are R-algebra morphisms for $(R[x]^\circ, *, u_2)$ and so $(R[x]^\circ, *, u_2, \Delta, \varepsilon)$ is an R-bialgebra. Simple computations show that the R-linear map (induced by S)

$$S^\circ : R[x]^\circ \to R[x]^\circ,\ \alpha \mapsto [x^i \mapsto (-1)^i\ \alpha(x^i),\ i \geq 0],$$

is an antipode and so $(R[x]^\circ, *, u_2, \Delta, \varepsilon, S^\circ)$ is a Hopf R-algebra which is dual to the Hopf R-algebra $(R[x], m, u, \Delta_2, \varepsilon_2, S)$.

4 Linearly recursive sequences:

In this section we identify $R[x]^\circ$ with the R-module $\mathbb{L}$ of linearly recursive sequences over R. Applying our previous results we show that $\mathbb{L}$ allows an R-bialgebra structure and a Hopf R-algebra structure corresponding to those on $R[x]^\circ$ (given in Section 3). In case R is a field this is explained in Peterson-Taft [7] and Chin-Goldman [3]. For details on linearly recursive sequences we refer to [6].

4.1. Definition. Let $\mathcal{S} = \{\mu : \mathbb{N} \to R\}$ be the set of all sequences over R. $\mathcal{S}$ has a structure of an $R[x]$-module as follows:
For $h(x) = a_0 + a_1x + ... + a_nx^n \in R[x]$ and $w \in \mathcal{S}$, define

$$h(x) \cdot w = \mu \in \mathcal{S}, \text{ where } \mu(i) = \sum_{j=0}^{n} a_j w(i+j) \text{ for all } i \in \mathbb{N}.$$

The set of *linearly recursive sequences* (abbreviated l.r.s.) over R is defined as

$$\mathbb{L} = \{w \in \mathcal{S} \,|\, g(x) \cdot w = 0 \text{ fore some monic polynomial } g(x) \in R[x]\}.$$

Notice that for $g(x) = a_0 + a_1x + \ldots a_{k-1}x^{k-1} + x^k$ and $w \in \mathcal{S}$, this condition means

$$w(i+k) = -\sum_{j=0}^{k-1} a_j w(i+j), \quad \text{for all} \quad i \geq 0.$$

Putting $n = i + k$ we have

$$w(n) = -(a_{k-1}w(n-1) + ... + a_0w(n-k)),$$

which is usual the definition of linearly recursive sequences.

We call $(w(0), \ldots, w(k-1))$ the *initial vector* of w and $g(x)$ a *characteristic polynomial* of w.

For $g(x) \in R[x]$ and $\mu \in \mathcal{S}$, we define the annihilators

$$\mathrm{An}_{\mathcal{S}}(g(x)) = \{\mu \in \mathcal{S} \,|\, g(x) \cdot \mu = 0\}, \ \mathrm{An}_{R[x]}(\mu) = \{g(x) \in R[x] \,|\, g(x) \cdot \mu = 0\}.$$

Clearly $\mathrm{An}_{\mathcal{S}}(g(x))$ is an $R[x]$-submodule of $\mathcal{S}$, $\mathrm{An}_{R[x]}(\mu)$ is an ideal of $R[x]$ and $\mu \in \mathbb{L}$ if and only if $\mathrm{An}_{R[x]}(\mu)$ is a monic ideal in $R[x]$.

4.2. Lemma. (Compare [6, 2.2]) *Let $g(x) = x^k + a_{k-1}x^{k-1} + ... + a_1x + a_0 \in R[x]$. Then $An_{\mathcal{S}}(g(x))$ is a free R-submodule of $\mathbb{L}$ with basis $\{e_0^g, ..., e_{k-1}^g\}$ given by*

$$e_i^g(j) = \begin{cases} 1, & \textit{for } j = i, \\ 0, & \textit{for } j \neq i, \quad j = 0, \ldots, k-1. \end{cases}$$

4.3. Canonical isomorphisms. There is an R-module isomorphism

$$\Phi : R[x]^* \to \mathcal{S}, \;\; \varphi \mapsto \mu \in \mathcal{S}, \;\text{ where } \mu(i) = \varphi(x^i), \;\text{ for all }\; i \geq 0.$$

If R is noetherian, then its restriction to $R[x]^\circ$ yields an isomorphism

$$\Phi' : R[x]^\circ \to \mathbb{L}.$$

Proof. It is easy to verify that Φ is an isomorphism. For the second assertion we first show $\Phi'(R[x]^\circ) \subset \mathbb{L}$. Assume $\varphi \in R[x]^\circ$. By Proposition 2.6 there exists a cofinite ideal $I \subset R[x]$, such that $\varphi(I) = 0$. Since R is noetherian, I is a monic ideal (by Proposition 3.1) containing some monic polynomial $g(x) = a_0 + a_1 x + \cdots + a_{k-1}x^{k-1} + x^k \in R[x]$ with $\varphi(g(x)) = 0$. For this we also have

$$\varphi(x^i\, g(x)) = 0, \;\text{ for all }\; i \geq 0,$$

and so

$$a_0\varphi(x^i) + a_1\varphi(x^{i+1}) + \cdots + a_{k-1}\varphi(x^{i+k-1}) + \varphi(x^{i+k}) = 0.$$

Putting $\mu = \Phi(\varphi)$, this means (with $a_k = 1$)

$$\sum_{j=0}^{k} a_j \mu(i+j) = 0, \;\text{ for all }\; i \geq 0.$$

So we have $g(x) \cdot \mu = 0$, and by definition $\mu \in \mathbb{L}$.

Now let $\mu \in \mathbb{L}$ and $\varphi = \Phi^{-1}(\mu)$. Then there exists some monic polynomial $g(x)$ with $g(x) \cdot \mu = 0$. Assuming $g(x)$ to be of the form given above, this yields

$$0 = \sum_{j=0}^{k} a_j\, \mu(i+j) = \varphi(x^i g(x)), \;\text{for each } i \geq 0,$$

which implies $\varphi((\,g(x))\,) = 0$. Since $(\,g(x))$ is a monic ideal we conclude from Proposition 2.6 and Proposition 3.1 that $\varphi \in R[x]^\circ$. $\square$

From now let R be a noetherian ring. By the isomorphism Φ' the structure of $R[x]^\circ$ transfers to $\mathbb{L}$ in the following way:

4.4. Coporoduct. (Compare [6, 14.16]) $(\mathbb{L}, \Delta, \varepsilon)$ is a coalgebra by

$$\Delta(\mu) = \sum_{i=0}^{k-1} (x^i \cdot \mu) \otimes e_i^g, \qquad \varepsilon(\mu) = \mu(0), \text{ for } \mu \in \mathbb{L},$$

where $g(x)$ is a characteristic polynomial of μ of degree k and $\{e_0^g, ..., e_{k-1}^g\}$ is the basis of $An_{\mathcal{S}}(g(x))$ from Lemma 4.2. Note that this coproduct on $\mathbb{L}$ corresponds to the coproduct on $R[x]^\circ$ given in 3.2.

Besides this we have two products on $\mathbb{L}$:

4.5. The Hadamard product. The algebra $(R[x]^\circ, \cdot, u_1)$ (see 3.3) yields the multiplication

$$\cdot : \mathbb{L} \otimes \mathbb{L} \to \mathbb{L},\ v \otimes w \longmapsto (v \cdot w)(i) = v(i)w(i),\ \ i \geq 0,$$

with identity element $u_1 : R \to \mathbb{L},\ 1 \longmapsto (1, 1, 1, ...)$.

Together with Δ and ε we have a bialgebra structure on $\mathbb{L}$.

4.6. The Hurwitz product. The algebra $(R[x]^\circ, *, u_2)$ (see 3.3) yields the multiplication

$$* : \mathbb{L} \otimes \mathbb{L} \to \mathbb{L},\ v \otimes w \longmapsto (v * w)(i) = \sum_{j=0}^{i} \binom{i}{j} v(j)w(i-j),\ \ i \geq 0,$$

with identity $u_2 : R \to \mathbb{L},\ 1 \longmapsto (1, 0, 0, ...)$.

Together with Δ and ε we have a Hopf R-algebra structure on $\mathbb{L}$, where the antipode is given by $S^\circ : \mathbb{L} \to \mathbb{L}, S^\circ(\mu)(i) = (-1)^i \mu(i)$.

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