Modules with every subgenerated module lifting

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Abstract

It was shown in Dung-Smith [2] that, for a module $M$, every module in $\sigma[M]$ is extending ($CS$ module) if and only if every module in $\sigma[M]$ is a direct sum of indecomposable modules of length $\leq 2$ or - equivalently - every module in $\sigma[M]$ is a direct sum of an $M$-injective module and a semisimple module. Here we characterize these modules by the fact that every module in $\sigma[M]$ is lifting or - equivalently - decompose as a direct sum of a semisimple module and a projective module in $\sigma[M]$. They are also determined by the functor ring of $\sigma[M]$ being a $QF$-2 ring with Jacobson radical square zero.

As a Corollary we obtain a result of Vanaja-Purav [8]: All (left) $R$-modules are lifting if and only if $R$ is a generalized uniserial ring with Jacobson radical square zero.

1 Preliminaries

Let $R$ denote an associative ring with unit, $R\text{-Mod}$ the category of unital left $R$-modules, and $M$ a left $R$-module. We call $M$ locally artinian, noetherian, of finite length if every finitely generated submodule of $M$ has the corresponding property. The notation $K \ll M$ means that $K$ is a small (superfluous) submodule of $M$. 
By $\sigma[M]$ we denote the full subcategory of $R$-$Mod$ whose objects are submodules of $M$-generated modules.

For any $R$-module $N$, $E(N)$ will denote the injective hull of $N$ in $R$-$Mod$. For $N \in \sigma[M]$, $\hat{N}$ is the injective hull of $N$ in $\sigma[M]$. $\hat{N}$ is also called the $M$-injective hull of $N$ and is isomorphic to the trace of $M$ in $E(N)$.

$N \in \sigma[M]$ is injective in $\sigma[M]$ if and only if $N$ is $M$-injective.

1.1 Functor ring. Denote by $\{U_\lambda\}_\Lambda$ a representing set of all finitely generated modules in $\sigma[M]$ and $U = \bigoplus_\Lambda U_\lambda$.

$$T := \hat{\text{End}}_R(U) = \{ f \in \text{End}_R(U) | (U_\lambda)f = 0 \text{ almost everywhere} \}$$

is called the functor ring of $\sigma[M]$. $T$ has no unit but has enough idempotents.

1) $T$ is left perfect if and only if every module in $\sigma[M]$ is a direct sum of finitely generated modules. In this case $M$ is called pure semisimple ([10], 53.4).

2) Assume $M$ is locally of finite length. Then $T$ is semiperfect ([10], 51.7).

3) Assume for every primitive idempotent $e \in T$, $Te$ is finitely cogenerated.

Then $M$ is locally artinian ([10], 52.1).

A ring $T$ with enough idempotents is called semiperfect if simple $T$-modules have projective covers (see [10], 49.10). $T$ is said to be a left (right) $QF$-$2$ ring if it is semiperfect and, for every primitive idempotent $e \in T$, $Te$ (resp. $eT$) has a simple essential socle (e.g., [3], section 4).

1.2 Theorem. For an $R$-module $M$ with functor ring $T$ the following are equivalent:

(a) For some $k \in \mathbb{N}$, every module in $\sigma[M]$ is a direct sum of uniserial modules of length $\leq k$;

(b) $T$ is a left and right $QF$-$2$ ring and $\text{Jac}(T)$ is nilpotent.

Proof: Consider a representing set $\{U_\lambda\}_\Lambda$ of all finitely generated modules in $\sigma[M]$, $U = \bigoplus_\Lambda U_\lambda$ and $T = \hat{\text{End}}_R(U)$.

(a) $\Rightarrow$ (b) By condition (a), $U$ is a direct sum of indecomposable modules of bounded length. Hence, by the Harada-Sai Lemma (e.g., [10], 54.1), $T$ is semiperfect and $\text{Jac}(T)$ is nilpotent.
Since $M$ is locally of finite length we know from [10], 53.5 that $U_T$ is $T$-injective. Now we can use the conclusions $(a) \implies (b) \implies (c)$ of [10], 55.15 to derive that $T$ is left and right $QF$-2.

$(b) \implies (a)$ Assume $T$ is a left and right $QF$-2 ring and $Jac(T)^n = 0$, for some $n \in \mathbb{N}$. Then $M$ is pure semisimple and locally artinian (see 1.1) and hence locally of finite length. With the proof of $(c) \implies (a)$ in [10], 55.15 we see that indecomposable modules in $\sigma[M]$ are uniserial.

It remains to show that for every uniserial $N \in \sigma[M]$, length $N \leq n$. Assume $N$ has a composition series

$$0 \neq N_1 \subset \ldots \subset N_n \subset N_{n+1} = N.$$  

From this we obtain a sequence of $n$ morphisms in $Jac(T)$,

$$N_n \to N \to N/N_1 \to \cdots \to N/N_{n-1},$$

whose product is not zero, contradicting $Jac(T)^n = 0$. \hfill \Box

## 2 Lifting modules

An $R$-module $M$ is called extending or CS module if every submodule is essential in a direct summand of $M$.

$M$ is said to be lifting if every submodule $K \subset M$ lies above a direct summand, i.e., there is a direct summand $X \subset M$ with $X \subset K$ and $K/X \ll M/X$. For characterizations of this condition refer to [10], 41.11 and 41.12.

A family $\{N_\lambda\}_\lambda$ of independent submodules of $M$ is said to be a local direct summand of $M$ if any finite (direct) sum of $N_\lambda$’s is a direct summand in $M$, and we say it is a direct summand if $\bigoplus_\lambda N_\lambda$ is a direct summand in $M$ (see [4], Definition 2.15).

A module is called continuous if it is extending and direct injective. In particular, self-injective modules are continuous.

Recall two results about these modules:

### 2.1 Lemma. Let $M$ be a continuous $R$-module.

1. Assume every local direct summand of $M$ is a direct summand. Then $M$ is a direct sum of indecomposable submodules.
Assume $M$ is lifting. Then local direct summands of $M$ are direct summands.

**Proof:** (1) See [5], Lemma 2.4 or [4], Theorem 2.17.

(2) This is shown in [5], Lemma 2.5. \qed

A ring $R$ is called a left $H$-ring if every injective module in $R$-$Mod$ is lifting. Some of the characterizations of $H$-rings (see [5], Theorem 1) can be extended to modules. For this we need the

**Definition.** A module $K \in \sigma[M]$ is said to be small in $\sigma[M]$ if it is a small submodule in its $M$-injective hull, i.e., $K \ll \widehat{K}$.

2.2 **Theorem** For any $R$-module $M$, the following are equivalent:

(a) Every injective module in $\sigma[M]$ is lifting;

(b) $M$ is locally noetherian and every non-small module in $\sigma[M]$ contains an $M$-injective submodule;

(c) every module in $\sigma[M]$ is a direct sum of an $M$-injective module and a small module.

**Proof:** (a) $\Rightarrow$ (b) By 2.1, every injective module in $\sigma[M]$ is a direct sum of indecomposable submodules. This implies that $M$ is locally noetherian (see [10], 27.5).

Assume $N$ is not small in its $M$-injective hull $\widehat{N}$. Since $\widehat{N}$ is lifting there is a direct summand $X \subset \widehat{N}$ with $X \subset N$ and $N/X \ll \widehat{N}/X$. By assumption, $X$ is not zero.

(b) $\Rightarrow$ (a) Referring to [10], 27.3, apply the proof of Proposition 2.7 in [5].

(a) $\Rightarrow$ (c) Consider $N \in \sigma[M]$ with $M$-injective hull $\widehat{N}$. Since $\widehat{N}$ is lifting, by [10], 41.11, a direct summand $X \subset \widehat{N}$ is contained in $N$ and $N = X + Y$ with $Y \ll \widehat{N}$. This implies that $Y$ is small in $\sigma[M]$.

(c) $\Rightarrow$ (a) With respect to [10], 41.11, this is obvious. \qed

It was pointed out in Osofsky [6], Lemma B (also in the proof (1) $\Rightarrow$ (3) of Vanaja-Purav, Proposition 2.13) that, for a uniserial module $M$ with composition series $0 \neq V \subset U \subset M$, $M \oplus U/V$ is not an extending module. For the same situation we observe:
2.3 Lemma. Assume $M$ is a uniserial module with composition series $0 \neq V \subset U \subset M$. Then the module $M \oplus U/V$ is not lifting.

**Proof:** Assume $M \oplus U/V$ is lifting. Then, by Theorem 1 in [1], $U/V$ is $M$-projective. However, the diagram

$$
\begin{array}{c}
U/V \\
\downarrow \\
M \rightarrow M/V \rightarrow 0
\end{array}
$$

cannot be extended commutatively by any $h : U/V \rightarrow M$, since the image of such a morphism always is contained in $V$. $\square$

The main purpose of this note is to prove:

2.4 Theorem. For any $R$-module $M$ the following are equivalent:

(a) Every module in $\sigma[M]$ is lifting;

(b) every module in $\sigma[M]$ is a direct sum of a semisimple module and a projective module in $\sigma[M]$;

(c) every module in $\sigma[M]$ is a direct sum of modules of length $\leq 2$;

(d) $T$ is a left and right QF-$2$ ring and $\text{Jac}(T)^2 = 0$.

*If this conditions hold, there is a projective generator in $\sigma[M]$ and all indecomposable modules of length $\leq 2$ are $M$-projective.*

**Proof:** $(a) \Rightarrow (d)$ Assume every module in $\sigma[M]$ is lifting. Then by Theorem 2.2, $M$ is locally noetherian. It is easy to see that finitely generated uniform lifting modules are local modules, i.e., their factor modules are indecomposable.

Consider an indecomposable injective module $Q \in \sigma[M]$. Then for any finitely generated submodule $K \subset Q$, $K/\text{Rad}(K)$ is simple and hence $Q$ is uniserial (see [10], 55.1). In particular, every uniform module in $\sigma[M]$ is uniserial of length $\leq 2$ (by Lemma 2.3). So the $M$-injective hull $\widehat{M}$ of $M$ is a direct sum of modules of length $\leq 2$ and hence $\widehat{M}$ (and $M$) is locally of finite length. This implies that every finitely generated module in $\sigma[M]$ is a direct sum of indecomposable modules (of length $\leq 2$).
Denote by \( \{ U_\lambda \}_\Lambda \) a representing set of all finitely generated modules in \( \sigma[M] \) and \( U = \bigoplus_\Lambda U_\lambda \). By the Harada-Sai Lemma, the functor ring \( T := \hat{\text{End}}_R(U) \) has the properties that \( T/Jac(T) \) is semisimple and \( Jac(T) \) is nilpotent.

In particular, \( M \) is pure-semisimple, i.e., every module in \( \sigma[M] \) is a direct sum of finitely generated modules and these are direct sums of uniserial submodules of length \( \leq 2 \). Now the assertion follows from Theorem 1.2.

Since \( T \) is right perfect, there exists a projective generator in \( \sigma[M] \) by [10], 51.13.

Consider an indecomposable module \( N \) of length 2. This is a factor module of a supplemented projective module in \( \sigma[M] \) and hence has a projective cover \( P \) (see [10], 42.1), which again is indecomposable and hence of length \( \leq 2 \). This implies \( P = N \), i.e., \( N \) is \( M \)-projective.

\((c) \iff (d)\) This is clear by Theorem 1.2.

\((c) \Rightarrow (a)\) Consider any module \( N = \bigoplus_\Lambda N_\alpha \) in \( \sigma[M] \), with \( N_\alpha \) uniserial of length \( \leq 2 \). By Theorem 1 in [1], \( N \) is lifting if and only if \( \{ N_\alpha \}_\Lambda \) is locally semi \( T \)-nilpotent and \( N_\alpha \) is almost \( N_\beta \)-projective for any \( \alpha \neq \beta \) in \( \Lambda \).

The first condition is satisfied by the Harada-Sai Lemma (see [10], 54.1). Any \( N_\alpha \) of length 2 is projective in \( \sigma[M] \) (as noted above) and hence is almost \( K \)-projective for any \( K \in \sigma[M] \).

Assume \( N_\alpha \) has length 1 and consider any diagram with exact line

\[
\begin{align*}
N_\alpha & \downarrow^f \\
N_\beta & \xrightarrow{p} L \
\end{align*}
\]

with length \( N_\beta \leq 2 \). If \( p \) is not an isomorphism and \( f \neq 0 \), there exists an epimorphism \( g : N_\beta \to N_\alpha \) with \( p = gf \). From this we see that \( N_\alpha \) is almost \( N_\beta \)-projective and \( N \) is lifting.

\((c) \Rightarrow (b)\) It is clear from the above that modules of length 2 are \( M \)-projective. Recall that finitely generated \( M \)-projective modules are projective in \( \sigma[M] \). From this the assertion is obvious.

\((b) \Rightarrow (c)\) Consider a finitely generated \( N \in \sigma[M] \). Then every factor module of \( N \) is a direct sum of a projective module and a noetherian module and hence \( N \) is noetherian by [7], section 3. This implies that \( M \) is locally noetherian.
Now let $K \in \sigma[M]$ be any indecomposable $M$-injective module. Assume $K$ is not semisimple. Then it is projective in $\sigma[M]$. Since $\text{End}_R(K)$ is local, $K$ is a local module, i.e., every factor module is indecomposable (see [10], 19.7) and hence simple. From this we deduce that $K$ has length $\leq 2$.

Since every $M$-injective module in $\sigma[M]$ is a direct sum of indecomposables the assertion follows. \qed

From Theorem 2.4 together with Theorem 11 in Dung-Smith [2] we obtain a characterization of rings with all modules lifting which extends Proposition 2.13 in Vanaja-Purav [8]:

2.5 Corollary. For any ring $R$ the following are equivalent:

(a) Every left $R$-module is lifting;
(b) every left $R$-module is extending;
(c) every left $R$-module is a direct sum of a semisimple module and a projective module;
(d) every left $R$-module is a direct sum of modules of length $\leq 2$;
(e) $R$ is a generalized uniserial ring with $\text{Jac}(R)^2 = 0$.

It follows from (e) that the conditions (a) – (d) are left right symmetric.

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References


