COLLOQUIUM MATHEMATICUM

VOL. * 200* NO.

FINITE PRESENTATION AND PURITY IN CATEGORIES $\sigma[M]$

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Abstract. For any module M over an associative ring R, let $\sigma[M]$ denote the smallest Grothendieck subcategory of Mod-R containing M. If $\sigma[M]$ is locally finitely presented the notions of purity and pure injectivity are defined in $\sigma[M]$. In this paper the relationship between these notions and the corresponding notions defined in Mod-R is investigated, and the connection between the resulting Ziegler spectra is discussed. An example is given of an M such that $\sigma[M]$ does not contain any non-zero finitely presented objects.

1. Local finite presentation of categories $\sigma[M]$. Given an R-module M, let $\sigma[M]$ denote the category subgenerated by M (see [15]). This is the smallest Grothendieck subcatgory of Mod-R containing M. We say that a category $\mathcal{C}$ is *locally finitely presented*, *lfp*, if it has a set of finitely presented objects such that every object of $\mathcal{C}$ is a direct limit of copies of objects from this set. Recall that the object C of $\mathcal{C}$ is *finitely presented* if the functor $(C, -)$ commutes with direct limits. This can be characterised by the fact that the kernel of any epimorphism $X \to C$ in $\mathcal{C}$ is finitely generated provided that X is finitely generated. Locally finitely presented abelian categories are Grothendieck [2] and they share many properties with module categories. In general $\sigma[M]$ need not be locally finitely presented although it is easy to see (e.g. [1, 1.70]) that it is locally α-presentable for some α (for this notion see [1] for example). Indeed, we shall see that $\sigma[M]$ need not contain any non-zero finitely presented object. In this paper we give a necessary and sufficient condition for a category of the kind $\sigma[M]$ to be locally finitely presented. Our criterion is one which is often easily checkable.

Every locally finitely presented abelian category is a localisation of a functor category (that is, a category of modules over a ring perhaps without unit but with enough local units) at a torsion theory of finite type (see [9, 2.3] for the exact criterion on the torsion theory for the localised category to be of finite type). However, the relation between Mod-R and its full subcategory $\sigma[M]$, even when $\sigma[M]$ is lfp, is usually not of this type: in general $\sigma[M]$ does not sit nicely within Mod-R. Nevertheless we are able,

2000 *Mathematics Subject Classification*: Primary 16D90; Secondary 18E15.

to some extent, to relate purity and pure-injectivity between these categories.

Our interest in this paper is in categories of modules but we remark that there is a general theory of locally α-presentable categories (see [1], [3], [8]) from which some of the results here could (in a more general context) be derived.

If $A \in \sigma[M]$ then A is finitely generated as an object of $\sigma[M]$ iff it is finitely generated as an R-module and so the category $\sigma[M]$ is determined by the finitely generated, hence by the cyclic, modules in it. Therefore $\sigma[M]$ is determined by the filter $\mathcal{F}_M = \{I \leq R_R : R/I \in \sigma[M]\}$ of right ideals of R. Say that $J \in \mathcal{F}_M$ is *$\mathcal{F}_M$-finitely generated*, *$\mathcal{F}_M$-fg* for short, if for all $J' \leq J$ with $J' \in \mathcal{F}_M$ we have J/J' finitely generated.

LEMMA 1.1. *A right ideal $J \in \mathcal{F}_M$ is $\mathcal{F}_M$-finitely generated if and only if whenever $J = \sum_\lambda J_\lambda$ with $J_\lambda \in \mathcal{F}_M$ we have $J = J_{\lambda_1} + \cdots + J_{\lambda_n}$ for some $\lambda_1, \ldots, \lambda_n$.*

Proof. If J is $\mathcal{F}_M$-finitely generated and $J = \sum_\lambda J_\lambda = J_{\lambda_1} + \sum_{\lambda \neq \lambda_1} J_\lambda$ then, since J/J_{λ_1} is finitely generated, there are $\lambda_2, \ldots, \lambda_n$ such that $J/J_{\lambda_1} = \sum_{i=2}^n (J_{\lambda_i} + J_{\lambda_1})/J_{\lambda_1}$ and hence $J = \sum_{i=1}^n J_{\lambda_i}$.

Conversely, if $J \geq J' \in \mathcal{F}_M$ and if J/J' were not finitely generated then there would be $(J_\lambda)_\lambda$ with $J_\lambda \geq J'$, hence $J_\lambda \in \mathcal{F}_M$, and $J/J' = \sum_\lambda J_\lambda/J'$ but with no finite subsum equal to J/J'. Then we would have $J = \sum J_\lambda$ with no finite subsum equal to J. ∎

For a category $\mathcal{C}$ let $\mathcal{C}^{\mathrm{fp}}$ denote the full subcategory of finitely presented objects of $\mathcal{C}$. It is quite common to write mod-R for $(\text{Mod-}R)^{\mathrm{fp}}$.

PROPOSITION 1.2. *Given $\sigma[M]$ and $J \in \mathcal{F}_M$ we have $R/J \in \sigma[M]^{\mathrm{fp}}$ if and only if J is $\mathcal{F}_M$-finitely generated.*

Proof. Suppose that J is $\mathcal{F}_M$-fg. Let $((L_\lambda)_\lambda, (g_{\lambda\mu} : L_\lambda \to L_\mu)_{\lambda \leq \mu})$ be a directed system in $\sigma[M]$ with limit $(L, (g_{\lambda\infty} : L_\lambda \to L)_\lambda)$ and suppose that we have a morphism $f : R/J \to L$. We must show that f factors through some $g_{\lambda\infty}$. Set $a = f(1 + J)$. For each λ and each $b \in L_\lambda$ such that $g_{\lambda\infty}(b) = a$ (if there is such in L_λ) set $I_{\lambda,b} = \operatorname{ann}_R b$. So $I_{\lambda,b} \in \mathcal{F}_M$ and $\operatorname{ann}_R b \leq \operatorname{ann}_R a$. Since $\operatorname{ann}_R a = \sum_{\lambda,b} \operatorname{ann}_R b$, we have $J \leq \sum_{\lambda,b} I_{\lambda,b}$ and hence $J = \sum_{\lambda,b} J \cap I_{\lambda,b}$. Note that $J \cap I_{\lambda,b} \in \mathcal{F}_M$. Therefore, since J is $\mathcal{F}_M$-finitely generated we have $J = \sum_{i=1}^n J \cap I_{\lambda_i,b_i}$ for some λ_i, b_i. For each $i, j \in \{1, \ldots, n\}$ we have $g_{\lambda_i\infty} b_i = g_{\lambda_j\infty} b_j$ so there is $\lambda \geq \lambda_1, \ldots, \lambda_n$ such that $g_{\lambda_i\lambda} b_i = g_{\lambda_j\lambda} b_j = b_0$, say, for all i, j and so $J = J \cap I_{\lambda,b_0}$. Thus $\operatorname{ann}_R b_0 \geq J$ and so f factors through $g_{\lambda\infty}$, as required.

For the converse, let $J \in \mathcal{F}_M$ be such that R/J is finitely presented in $\sigma[M]$. Then for any $I \in \mathcal{F}_M$ where $I \subset J$, the kernel of $R/I \to R/J$ is finitely generated and is equal to J/I, i.e., J is $\mathcal{F}_M$-fg. ∎

Say that $\mathcal{F}_M$ is *cofinally $\mathcal{F}_M$-finitely generated* if for every $I \in \mathcal{F}_M$ there is some $\mathcal{F}_M$-finitely generated $J \in \mathcal{F}_M$ with $J \leq I$.

THEOREM 1.3. *The category $\sigma[M]$ is locally finitely presented if and only if $\mathcal{F}_M$ is cofinally $\mathcal{F}_M$-finitely generated.*

Proof. Suppose first that $\sigma[M]$ is lfp. Let $I \in \mathcal{F}_M$. Then $R/I \in \sigma[M]$ and so there is an epimorphism $\bigoplus_i F_i \to R/I$ with the $F_i \in \sigma[M]^{\mathrm{fp}}$. Since R/I is finitely generated there is even an epimorphism $f : F \to R/I$ with $F \in \sigma[M]^{\mathrm{fp}}$. Let $a_1, \dots, a_n$ be a finite set of generators for F where, without loss of generality, $f(a_1) = 1 + I$. Say $f(a_i) = r_i + I$, $i = 2, \dots, n$. Set $F' = F/\langle a_1 r_i - a_i : i = 2, \dots, n\rangle$ and let $p : F \to F'$ be the projection. Then F' is cyclic and also finitely presented. We have a factorisation of f through p, say $f' : F' \to R/I$ is such that $f'p = f$.

Now, F' is cyclic, isomorphic to R/J with $J = \mathrm{ann}_R\, p(a_1)$ and is finitely presented, so by 1.2, J is $\mathcal{F}_M$-finitely generated. Furthermore, J is contained in I, as required.

For the converse, supposing that $\mathcal{F}_M$ is cofinally $\mathcal{F}_M$-finitely generated, we find that the R/J with J a $\mathcal{F}_M$-fg member of $\mathcal{F}_M$ form a generating (by cofinality of these in $\mathcal{F}_M$) set of finitely presented (by 1.2) objects of $\sigma[M]$, as required. ∎

COROLLARY 1.4. *If $\sigma[M]$ is locally finitely presented then the R/J with J $\mathcal{F}_M$-finitely generated and in $\mathcal{F}_M$ form a generating set of finitely presented objects.*

The condition of 1.3 is often readily checkable and one can recover known conditions for $\sigma[M]$ being lfp quite easily. For example if R is right noetherian then every category $\sigma[M]$ is lfp. If M is such that for every $I \in \mathcal{F}_M$ we have I finitely generated then $\sigma[M]$ is lfp. If M is a coherent module then $\sigma[M]$ is lfp. In particular the category of comodules over a K-coalgebra where K is a field is lfp. More generally [17], the category of C-comodules is locally finitely presented provided C is an R-coalgebra where R is right noetherian and C_R is projective. If $\mathcal{F}_M$ has a minimal element then $\sigma[M]$ is lfp, indeed, it is a module category.

PROPOSITION 1.5. *If $\mathcal{F}_M$ has a minimal element I then $\sigma[M] \simeq$ Mod-R/I.*

Proof. First we show that I is an ideal of R. Let $a \in R$. Then $R/(I : a) \simeq (aR + I)/I \leq R/I \in \sigma[M]$ so $(I : a) \in \mathcal{F}_M$ and hence $I \leq (I : a)$. This is true for every $a \in R$ so I is a two-sided ideal of R.

For any R/I-module N there is a surjection from a direct sum, $(R/I)^{(\kappa)}$, to N and hence $N \in \sigma[M]$ $(= \sigma[R/I])$. Conversely, every member of $\sigma[M]$ is a submodule of a surjective image of some direct sum $(R/I)^{(\kappa)}$ and hence is an R/I-module. So the subcategories $\sigma[M]$ and Mod-R/I of Mod-R are equal. ■

We give a related criterion for $\sigma[M]$ to be locally finitely presented.

PROPOSITION 1.6. *The category $\sigma[M]$ is locally finitely presented iff for every finitely presented module $F \in$ Mod-R and every morphism $f : F \to A \in \sigma[M]$ there is a factorisation of f through a member of $\sigma[M]^{\mathrm{fp}}$.*

Proof. We claim that it is enough to prove the result in the case that F is cyclic. For there is a ring R' and a Morita equivalence α : Mod-$R \to$ Mod-R' such that $\alpha(F)$ is cyclic. All the other terms in the statement are Morita invariant and so if we obtain a factorisation for $\alpha(f)$ then we obtain one for f.

Suppose, then, that $\sigma[M]$ is locally finitely presented. Take $f : R/K \to A \in \sigma[M]$ with K finitely generated and set $I = \mathrm{ann}_R\, f(1) \in \mathcal{F}_M$. Since $\sigma[M]$ is locally finitely presented there is $J \leq I$ in $\mathcal{F}_M$ with J $\mathcal{F}_M$-finitely generated. We claim that $K+J$ is $\mathcal{F}_M$-finitely generated. If $K+J \geq J' \in \mathcal{F}_M$ then we have $(J+J')/J' \simeq J/(J \cap J')$, which is finitely generated since $J \cap J' \in \mathcal{F}_M$ and by choice of J. Also $(K+J)/(J'+J)$, being an epimorphic image of K, is finitely generated. Therefore $(K+J)/J'$ is finitely generated, as claimed. Then, since $I \geq K+J \geq K$, f factors through the natural projection $R/K \to R/(K+J)$ and the latter is, by 1.2, in $\sigma[M]^{\mathrm{fp}}$, as required.

For the converse, suppose that we have the condition and let $A \in \sigma[M]$. Take an epimorphism $p : R^{(\kappa)} \to A$. Each component of p factors through some finitely presented object of $\sigma[M]$ by hypothesis, so p factors through a direct sum of objects of $\sigma[M]^{\mathrm{fp}}$. That is, every object of $\sigma[M]$ is an epimorphic image of a coproduct of objects in $\sigma[M]^{\mathrm{fp}}$ and this is enough for local finite presentation. ■

For contrast, we give an example of a category of the form $\sigma[M]$ where the only finitely presented object is the zero object.

EXAMPLE 1.7. Let $R = K[X_n : n \geq 0]$ be the polynomial ring over a field K in countably many indeterminates. Set $I_n = \langle X_{k2^n} : k \geq 1\rangle$. So $I_0 > I_1 > \cdots$ forms a decreasing sequence of ideals with each factor I_n/I_{n+1} an infinitely generated R-module. Let $\mathcal{F}$ be the filter of ideals generated by the I_n. So, if $M = \bigoplus\{R/I : I \in \mathcal{F}\}$ then $\mathcal{F} = \mathcal{F}_M$ (since R is commutative, $a \in R/I$ implies $\mathrm{ann}_R(a) \geq I$). Then there is no finitely presented object in $\sigma[M]$ other than 0.

Proof. If there is a finitely presented object then there is a cyclic one (see below), so, for a contradiction and using 1.4, suppose that there is $I \in \mathcal{F}_M$ such that I is $\mathcal{F}_M$-finitely generated. Since $I \in \mathcal{F}_M$ we have

$I \geq I_n$ ($> I_{n+1}$) for some n and so, since I is $\mathcal{F}_M$-finitely generated, we have $I = I_{n+1} + \sum_{i=1}^{t} a_i R$ for some $a_i \in R$.

Let m be such that all X_j appearing in $a_1, \ldots, a_t$ have $j < m$ and such that m has the form $m = k2^n$ with k odd. So $X_m \in I_n \backslash I_{n+1}$. Therefore $X_m \in I \setminus I_{n+1}$ and we claim, for a contradiction, that there is no representation $X_m = f + \sum_i a_i g_i$ with $f \in I_{n+1}$ and the $g_i \in R$. In order to prove this claim consider

$$I' = I \cap K[X_0, \ldots, X_{m-1}] = \Big(I_{n+1} + \sum_{i=1}^{t} a_i R \Big) \cap K[X_0, \ldots, X_{m-1}].$$

Since $I \neq R$, I' is a proper ideal in $K[X_0, \ldots, X_{m-1}]$ and hence there is a maximal ideal, J, of $K[X_0, \ldots, X_{m-1}]$ with $I' \subseteq J$. Define $L = K[X_0, \ldots, X_{m-1}]/J$, regarded as an extension field of K. Consider the projection from $R = K[X_0, \ldots, X_{m-1}][X_m, \ldots]$ to $L[X_m, \ldots]$ with kernel $J \cdot R$, followed by the projection to L with kernel $\langle X_m - 1\rangle + \langle X_n : n > m\rangle$. Denote the composite morphism as $\theta : R \to L$.

Since $f \in I_{n+1}$ and $X_m \notin I_{n+1}$ we have $f = f_0 + f_1$ where $f_0 \in (K[X_0, \ldots, X_{m-1}] \cap I_{n+1}) \cdot R$ (that is, every monomial of f_0 is divisible by some $X_j \in I_{n+1}$ with $j < m$) and where every monomial of f_1 is divisible by some X_j with $j > m$. Then $\theta(f_1) = 0$ and $\theta(f_0) = 0$ since $f_0 \in I'$. Moreover each a_i is in I' and hence $\theta(\sum_i a_i g_i) = 0$. But this is a contradiction because $\theta(f + \sum_i a_i g_i = X_m) = 1$.

Hence there is no finitely presented cyclic object. Now suppose that A were a non-zero finitely presented object of $\sigma[M]$. Choose some minimal generating set $a_1, \ldots, a_n$ for A. Then $A/\sum_{i=2}^{n} a_i R$ is a non-zero cyclic object in $\sigma[M]$ and is finitely presented.

We conclude that $\sigma[M]^{\mathrm{fp}}$ has only the zero object. ∎

We have the following characterisation of projective objects in locally finitely presented $\sigma[M]$.

Proposition 1.8. *Let M be an R-module and $I \in \mathcal{F}_M$. Then R/I is a projective object of $\sigma[M]$ if and only if I is complemented in $\mathcal{F}_M$ in the sense that for all $I' \leq I$ with $I' \in \mathcal{F}_M$ there exists a right ideal $J \geq I'$ such that $I + J = R$ and $I \cap J = I'$.*

Proof. "$\Rightarrow$" Let $I' \leq I$ be in $\mathcal{F}_M$. Then the projection $p : R/I' \to R/I$ splits, that is, $\ker(p) = I/I'$ has a complement, isomorphic to R/I, in R/I'—say J is such that $I' \leq J \leq R$ and $I/I' \cap J/I' = 0$ and $I/I' + J/I' = R/I'$. That is, $I \cap J = I'$ and $I + J = R$, as required.

"$\Leftarrow$" Suppose that $A \in \sigma[M]$ and that $p : A \to R/I$ is an epimorphism. Choose an epimorphism $p' : \bigoplus_\lambda R/I_\lambda \to A$ with the $I_\lambda \in \mathcal{F}_M$. If a morphism $g : R/I \to \bigoplus_\lambda R/I_\lambda$ splits pp' then the composite $p'g$ splits p. So without loss of generality $A = \bigoplus_\lambda R/I_\lambda$. Since R/I is cyclic we may choose a preimage of $1+I$ in A and this generates a submodule, R/I', of A such that the restriction of p to R/I' is epi. Therefore it is enough to split this map.

By assumption there is J with $I+J=R$ and $I\cap J=I'$, that is, such that $R/I'=I/I'\oplus J/I'$ and, in particular, with $J/I'\simeq R/I$, yielding a splitting as required. Hence R/I is projective. ∎

It follows that there are enough cyclic projectives in $\sigma[M]$ (enough to generate every module in $\sigma[M]$) if and only if $\mathcal{F}_M$ contains a cofinal set of right ideals as in 1.8. From this it is immediate that $\sigma[\mathbb{Q}/\mathbb{Z}]$, for example, does not have enough cyclic projectives.

2. Purity in $\sigma[M]$ versus Mod-R. Recall that an exact sequence $0\to A\to B\to C\to 0$ in a Grothendieck category $\mathcal{C}$ is *pure* if for every finitely presented object F of $\mathcal{C}$ every morphism from F to C lifts through $B\to C$ (see [15, 33.1]). In this case we also say that the monomorphism $A\to B$ is a *pure embedding*. If $\sigma[M]$ were an elementary localisation of Mod-R (in the sense of [9]) then an exact sequence in $\sigma[M]$ would be pure in $\sigma[M]$ iff it were pure in Mod-R. However, as we have remarked, $\sigma[M]$, even if locally finitely presented, is not in general even a localisation of Mod-R. So now we investigate the relation between purity in $\sigma[M]$ and purity in Mod-R.

PROPOSITION 2.1. *Suppose that $\sigma[M]$ is locally finitely presented. Let $f:A\to B$ be a pure monomorphism in $\sigma[M]$. Then f is a pure monomorphism in* Mod-R.

Proof. Let $C=\mathrm{coker}(f)$ and let $h:F\to C$ with $F\in$ mod-R. By 1.6 there is a factorisation $h=h'p$, with $p:F\to F'$ and $h':F'\to C$, of h through some $F'\in\sigma[M]^{\mathrm{fp}}$. Since the sequence $0\to A\to B\to C\to 0$ is pure in $\sigma[M]$ the map h' lifts to $g:F'\to B$ say and then the composition gp lifts h, as required. ∎

This result also follows from the fact that every pure exact sequence in $\sigma[M]$ is a direct limit of split exact sequences and it also has a short model-theoretic proof (see [13] or apply [1, 2.30]).

The converse to 2.1 is not in general true: an exact sequence in $\sigma[M]$ which is pure in Mod-R need not be pure in $\sigma[M]$ even if $\sigma[M]$ is locally finitely presented.

EXAMPLE 2.2. Let R be a von Neumann regular ring which is not semisimple. Suppose that R has simple modules S,T (possibly isomorphic) such that $\mathrm{Ext}^1(S,T)\neq 0$ (so, because every exact sequence of R-modules is pure, S cannot be finitely presented), say M is a non-split extension of S by T. The category $\sigma[M]$ is locally of finite length and hence is locally finitely presented and both S and T are finitely presented objects of $\sigma[M]$. The non-split exact sequence $0 \to T \to M \to S \to 0$ cannot, therefore, be pure in $\sigma[M]$—otherwise it would split. On the other hand every short exact sequence in Mod-R is pure, because R is von Neumann regular.

For example we may take R to be $k^{\mathbb{N}} \oplus 1.k$ where k is a field, and let $S = R/J$ where $J = k^{\mathbb{N}}$. Since S is not finitely presented, hence does not embed in R, we have $\mathrm{Ext}(S, J) \neq 0$. The ideal J is a direct sum of simple modules T_i, $i \in \mathbb{N}$, so we may take T to be one of these.

For $M, N \in$ Mod-R let $T^M(N) = \sum\{N' \leq N : N' \in \sigma[M]\}$ be the largest submodule of N which is in $\sigma[M]$. This induces a functor (subfunctor of the identity) $T^M : \text{Mod-}R \to \sigma[M]$ which is right adjoint to the inclusion $\sigma[M] \to$ Mod-R (see [15, 45.11]).

Say that $\sigma[M]$ is *closed under inverse images of small epimorphisms* if for any epimorphism $f : P \to N$ in Mod-R with superfluous kernel and $N \in \sigma[M]$, we have $P \in \sigma[M]$.

Proposition 2.3. *Assume the functor $T^M : \text{Mod-}R \to \sigma[M]$ to be exact. Then*:

(1) *$\sigma[M]$ is closed under inverse images of small epimorphisms.*
(2) *If P is finitely presented in $\sigma[M]$, then P is finitely presented in* Mod-R.
(3) *If P is projective in $\sigma[M]$, then P is projective in* Mod-R.

Proof. Notice that exactness of T^M implies that $\sigma[M]$ is closed under extensions in Mod-R, and the class of "torsion free" modules (i.e., modules X with $T^M(X) = 0$) is closed under factor modules. This is because any exact sequence $0 \to K \to L \to N \to 0$ in Mod-R induces an exact sequence

$$0 \to T^M K \to T^M L \to T^M N \to 0.$$

Now $T^M K = K$ and $T^M N = N$ imply $T^M L = L$, showing that $\sigma[M]$ is closed under extensions, and $T^M L = 0$ implies $T^M N = 0$.

(1) Assume K to be superfluous in L and let $N \in \sigma[M]$, and consider the commutative exact diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d] & L \arrow[r] \arrow[d, "="] & N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K + T^M L \arrow[r] & L \arrow[r] & L/(K + T^M L) \arrow[r] & 0
\end{tikzcd}$$

Clearly $L/(K + T^M L) \in \sigma[M]$ and by the above $T^M(L/(K + T^M L)) = 0$. This implies $L = K + T^M L$, hence $L = T^M L$, i.e. $L \in \sigma[M]$.

(2) It is enough to show this for any cyclic module $P \in \sigma[M]$ which is finitely presented in $\sigma[M]$. For this let $R \to P$ be an epimorphism. We can, by the above observation, choose a suitable finitely generated submodule L_1

of $T^M R$ to obtain a commutative exact diagram

$$\begin{tikzcd}
0 \arrow[r] & L_0 \arrow[r] \arrow[d] & L_1 \arrow[r] \arrow[d] & P \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & I \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & P \arrow[r] & 0 \\
 & I/L_0 \arrow[r, "\simeq"] & R/L_1 & &
\end{tikzcd}$$

where L_0 is a finitely generated module (in $\sigma[M]$). Hence I/L_0 is finitely generated and hence so is I. So P is finitely presented in Mod-R.

(3) Let P be projective in $\sigma[M]$ and $f : L \to P$ an epimorphism in Mod-R. Then $f|_{T^M L} : T^M L \to P$ is an epimorphism in $\sigma[M]$ and hence is split by some morphism $g : P \to T^M L$ which obviously also splits f. This shows that P is projective in Mod-R. ∎

Recall that a ring R is *semiperfect* if every finitely generated R-module has a projective cover, and R is *f-semiperfect* if every finitely presented R-module has a projective cover in Mod-R (see, e.g., [15, 42.6, 42.11]).

COROLLARY 2.4. *Let M be an R-module for which T^M is exact. Assume*

(i) *R is a semiperfect ring, or*

(ii) *R is an f-semiperfect ring and $\sigma[M]$ is locally finitely presented.*

Then $\sigma[M]$ has a set of cyclic generators which are projective in Mod-R.

Proof. Assume (i). Every finitely generated module N in $\sigma[M]$ has a projective cover $P \to N$ in Mod-R. By 2.3, P belongs to $\sigma[M]$.

Now assume (ii). Then by 2.3(2), the finitely presented modules in $\sigma[M]$ are finitely presented in Mod-R. Since R is f-semiperfect they have a projective cover in Mod-R which lies in $\sigma[M]$ (by 2.3(1)). ∎

Notice that the above observation has a nice application for the category Comod-C of right comodules over a coalgebra C which is over a quasi-Frobenius ring R, where C is projective as R-module. In this case Comod-C can be identified with $\sigma[_{C^*}C]$, where C^* is the dual algebra, and is locally noetherian (hence locally finitely presented). Moreover C^* is f-semiperfect (being the endomorphism ring of the self-injective module $_{C^*}C$). Then the functor

$$\mathcal{T}^C : C^*\text{-Mod} \to \text{Comod-}C$$

(called the *rational functor*) is exact if and only if there are enough projectives in Comod-C (C is right semiperfect, see [16, 6.3]).

We recall that every locally finitely presented Grothendieck category has pure-injective envelopes, that is, for every object C of the category there is a pure-essential, pure embedding $C \leq N$ where N is pure-injective (see [6], [14], [2]). In particular the category $\sigma[M]$ has pure-injective envelopes.

If A is any module then we use the notation $\overline{A}$ for the pure-injective hull of A: the "smallest" pure-injective module into which A embeds purely. For more detail, see e.g. [5].

Proposition 2.5. *Suppose that $\sigma[M]$ is locally finitely presented and that $\sigma[M]^{\mathrm{fp}} \subseteq \text{mod-}R$. Then an embedding $A \to B$ in $\sigma[M]$ is pure in $\sigma[M]$ if and only if it is pure in* Mod-R. *In particular if $A \in \sigma[M]$ then the canonical embedding $A \to T^M\overline{A}$ is pure in $\sigma[M]$. Indeed, if $f : C \to D$ is a pure embedding in* Mod-R *then $T^M f : T^M C \to T^M D$ is a pure embedding.*

Proof. An embedding $f : A \to B$ in a locally finitely presented category is pure iff given any morphism $g : A' \to B'$ between finitely presented objects and any morphisms $h : A' \to A$ and $h' : B' \to B$ with $fh = h'g$, there is a morphism $k : B' \to A$ such that $kg = h$ (see [1, 2.27]). So, since we already have 2.1, the first statement is immediate. The second statement then follows directly since the canonical embedding $A \to \overline{A}$ is pure in Mod-R and hence so is the embedding $A \to T^M\overline{A}$.

For the third statement, suppose we have a morphism $g : A' \to B'$ between finitely presented objects of $\sigma[M]$ and morphisms $h : A' \to T^M C$ and $h' : B' \to T^M D$ with $T^M f.h = h'g$. Composing with the embeddings $i : T^M C \to C$ and $j : T^M D \to D$ there is, by purity of f and by hypothesis, a morphism $k : B' \to C$ such that $kg = ih$. But the image of k, being in $\sigma[M]$, must be contained in $T^M C$ and hence we can regard k as a morphism from B' to $T^M C$, as required. ∎

The condition that $\sigma[M]^{\mathrm{fp}}$ be contained in mod-R is, by 1.2, equivalent to the condition that every $\mathcal{F}_M$-finitely generated right ideal be finitely generated. For instance we have this if M is coherent in $\sigma[M]$ (since every finitely presented object of $\sigma[M]$ has the form A/B for some finitely generated modules $B \leq A \leq M^n$). Note that the combined conditions that $\sigma[M]$ be locally finitely presented and that $\sigma[M]^{\mathrm{fp}}$ be contained in mod-R are equivalent to there being a cofinal set of finitely generated right ideals in $\mathcal{F}_M$, and so they are satisfied if R is right noetherian.

By Corollary 2.4 the conditions of Proposition 2.5 are also satisfied provided R is semiperfect and T^M is exact.

In order to obtain the first conclusion of 2.5 a weaker assumption will suffice.

Proposition 2.6. *Suppose that $\sigma[M]$ is locally finitely presented. Suppose that for every $I \in \mathcal{F}_M$ there is a finitely generated right ideal $I^0 \leq I$ such that for every $I' \in \mathcal{F}_M$, if $I^0 \leq I'$ then $I \leq I'$. Then for short exact sequences in $\sigma[M]$ purity in $\sigma[M]$ is equivalent to purity in* Mod-R.

Proof. In view of 2.1 we must show that, under the given condition on $\mathcal{F}_M$, purity in Mod-R implies purity in $\sigma[M]$ for short exact sequences in $\sigma[M]$.

So suppose we have the condition on $\mathcal{F}_M$ and let $f : A \to B$ be a monomorphism in $\sigma[M]$ with cokernel $\pi : B \to C$ and suppose that the sequence $0 \to A \to B \to C \to 0$ is pure as a sequence in Mod-R. Let $g : F \to C$ be a morphism with $F \in \sigma[M]^{\mathrm{fp}}$. Choose an epimorphism $p : \bigoplus_{i=1}^n R/J_i \to F$ where each J_i is $\mathcal{F}_M$-finitely generated (by 1.4 this is possible, since F is finitely generated as an object of Mod-R). Also choose a further epimorphism

$$q : \bigoplus_{i=1}^n R/J_i^0 \to \bigoplus_{i=1}^n R/J_i, \qquad 1 + J_i^0 \mapsto 1 + J_i,$$

where J_i^0 is chosen for J_i as in the statement of the result. Because $\bigoplus_{i=1}^n R/J_i^0$ is finitely presented in Mod-R and the sequence is pure in Mod-R there is a lifting $h : \bigoplus_{i=1}^n R/J_i^0 \to B$, with $\pi h = gpq$.

Set $b_i = h(e_i)$ and $I_i = \mathrm{ann}_R\, b_i$. Since $I_i \geq J_i^0$ for each i and $I_i \in \mathcal{F}_M$ we have, by choice of J_i^0, $I_i \geq J_i$ (and hence gp lifts). Also, setting $d_i = p(1+J_i)$, let $\sum_i d_i t_{ij} = 0$, $j = 1, \dots, m$, be a finite presentation of F relative to $\bigoplus_{i=1}^n R/J_i$ (that is, the elements $(t_{1j}, \dots, t_{nj})$, $j = 1, \dots, m$, generate the kernel of p). For each i let r_{il}, $l = 1, \dots, m'$, be a finite generating set for J_i^0. Since $\sum_i g(d_i)t_{ij} = 0$ for each j we have $\sum_i b_i t_{ij} = a'_j \in A$, say, for each j.

Consider the system of linear equations in unknowns $y_1, \dots, y_n$:

$$y_i r_{il} = 0, \quad (i, l), \qquad \sum_i y_i t_{ij} = a'_j, \quad (j).$$

This system has a solution, $b_1, \dots, b_n$, in B so, since A is a pure submodule of B, it has a solution, $a_1, \dots, a_n$ say, in A. Set $b'_i = b_i - a_i$. Then we have:

- $b'_i r_{il} = 0$ for all i, l and hence $\mathrm{ann}_R\, b'_i \geq J_i^0$ and hence, by choice of J_i^0, $\mathrm{ann}_R\, b'_i \geq J_i$;
- moreover $\sum_i b'_i t_{ij} = 0$ for all j and hence sending d_i to b'_i gives a well defined morphism from F to B which lifts g, as required. ∎

We do not know the exact condition on $\mathcal{F}_M$ necessary and sufficient for purity in Mod-R and $\sigma[M]$ to coincide.

3. Pure-injectivity in $\sigma[M]$ versus Mod-R. An object $N \in \sigma[M]$ is injective in $\sigma[M]$ iff $N = T^M E(N)$, where $E(N)$ denotes the injective hull of N in Mod-R. We can obtain similar, though weaker, results for pure-injective objects.

PROPOSITION 3.1. *Suppose that* $\sigma[M]$ *is locally finitely presented and that* $N \in \text{Mod-}R$ *is pure-injective. Then* $T^M N$ *is a pure-injective object of* $\sigma[M]$.

Proof. Let $f : A \to B$ be a pure monomorphism in $\sigma[M]$ and take $g : A \to T^M N$. Compose g with the inclusion i of $T^M N$ in N. By 2.1, f is pure in Mod-R so there is $h : B \to N$ such that $hf = ig$. But the image of h is an object of $\sigma[M]$, hence is contained in $T^M N$ and so g factors through f, as required. ∎

An alternative proof, given in [13], is to use the characterisation of pure-injectivity from [5, 7.1(vi)] together with the fact that T^M commutes with direct sum and the description of direct product in $\sigma[M]$.

EXAMPLE 3.2. Even if we assume $\sigma[M]^{\text{fp}} \subseteq \text{mod-}R$ it does not follow that an object which is pure-injective in $\sigma[M]$ is pure-injective in Mod-R. Take R to be the first Weyl algebra over a field of characteristic 0 and let S be a simple R-module. Since R is (right) noetherian, S is finitely presented. Then the category $\sigma[S]$ is semisimple and S is even an injective object. But, as an R-module, S is not pure-injective [12, 3.2].

COROLLARY 3.3. *Suppose that* $\sigma[M]$ *is locally finitely presented and that we have* $\sigma[M]^{\text{fp}} \subseteq \text{mod-}R$. *Let* $A \in \sigma[M]$. *Then the pure-injective hull of* A *in* $\sigma[M]$ *is a direct summand of* $T^M \overline{A}$. *In particular, if* A *is a pure-injective object of* $\sigma[M]$ *then* A *is a direct summand of* $T^M \overline{A}$.

Proof. By 2.5 the embedding $A \to T^M \overline{A}$ is pure in $\sigma[M]$ and by 3.1 the latter module is pure-injective in $\sigma[M]$, hence has the pure-injective hull of A in $\sigma[M]$ as a direct summand. ∎

COROLLARY 3.4. *Suppose that* $\sigma[M]$ *is locally finitely presented and suppose that purity in* $\sigma[M]$ *coincides with purity in* Mod-R *for short exact sequences in* $\sigma[M]$. *Then the pure-injective objects of* $\sigma[M]$ *are exactly the direct summands of modules of the form* $T^M N$ *where* N *is a pure-injective* R*-module.*

Proof. The proof of 3.3 needs only this weaker assumption. ∎

Can one omit the phrase "direct summand of" in the above description of pure-injective objects, in particular when is the pure-injective hull of A in 3.3 equal to $T^M \overline{A}$? If we assume that $T^M \overline{A}$ is pure in $\overline{A}$ then this follows directly.

LEMMA 3.5. *Suppose that* $\sigma[M]$ *is locally finitely presented and let* $A \in \sigma[M]$. *If the embedding of* $T^M \overline{A}$ *in* $\overline{A}$ *is pure then* $T^M \overline{A}$ *is the pure-injective hull of* A *in* $\sigma[M]$.

Proof. If $T^M\overline{A} = A' \oplus A''$ with $A \subseteq A'$ then, since the composition $A \to (A' \oplus A'')/A'' \to \overline{A}$ is pure and A is pure-essential in $\overline{A}$, we have $A'' = 0$, as required. ∎

The assumption that $T^M N$ is pure in N for every pure-injective R-module N is a very strong one (satisfied for $M = \mathbb{Q}/\mathbb{Z}$ for instance but not for $M = \mathbb{Z}_{p^n}$ as $\mathbb{Z}$-modules) but that assumption is considerably stronger than that used in 3.5.

For the remainder of this section we make the following assumption and investigate the relation between the Ziegler spectrum (see, e.g., [10]) of $\sigma[M]$ and that of Mod-R:

(∗) $\sigma[M]$ is locally finitely presented, $\sigma[M]^{\mathrm{fp}} \subseteq$ mod-R, and $T^M\overline{A}$ is pure in $\overline{A}$ for every $A \in \sigma[M]$.

We do not know a good alternative characterisation of the classes $\sigma[M]$ satisfying the last part of condition (∗) but there are many of them, not least all those $\sigma[M]$ which are closed under pure-injective hulls in Mod-R.

Let $\mathrm{Zg}(\sigma[M])$ denote the Ziegler spectrum of the lfp category $\sigma[M]$. Its points are the (isomorphism classes of) indecomposable pure-injective objects of $\sigma[M]$ and a basis of open sets for the topology is given by the sets

$$(f) = \{N \in \mathrm{Zg}(\sigma[M]) : (f, N) : (A, N) \to (B, N) \text{ is not epi}\},$$

where $f : A \to B$ ranges over morphisms in $\sigma[M]^{\mathrm{fp}}$.

PROPOSITION 3.6. *Assume $\sigma[M]$ satisfies (∗). Then $C \in \mathrm{Zg}(\sigma[M])$ implies $\overline{C} \in \mathrm{Zg}_R$.*

Proof. If $\overline{C}$ decomposes as $\overline{C} = N \oplus N'$ then $C = T^M\overline{C} = T^M N \oplus T^M N'$ (by 3.5) so, since C is indecomposable, we have, say, $T^M N' = 0$ and so $C \leq N$. Therefore, $\overline{C} \leq N$, and hence $N' = 0$, as required. ∎

Therefore we have an embedding $j : \mathrm{Zg}(\sigma[M]) \to \mathrm{Zg}_R$. The image of this embedding consists of those indecomposable pure-injective R-modules, N, such that $T^M N$ is non-zero and is pure in N. We show that j is a homeomorphism of $\mathrm{Zg}(\sigma[M])$ onto its image. In the case that $\sigma[M]$ is closed under products, and hence is a definable subcategory of Mod-R, this is just the embedding of a closed subset of Zg_R, with the relative topology, into Zg_R. In general the image of j might not be closed.

EXAMPLE 3.7. Let R be the first Weyl algebra over a field of characteristic zero and let M be the direct sum of all the simple R-modules, so $\sigma[M]$ consists of all the semisimple R-modules. Then $\mathrm{Zg}(\sigma[M])$ is just the set of all simple R-modules and the image of j is the set of pure-injective hulls of these modules. But the latter set is not closed in Zg_R since $\operatorname{im} j$ carries the discrete topology (see [11, §3]) and so, by compactness of Zg_R, there must be at least one more point in the closure of $\operatorname{im} j$.

Note that $\mathrm{Zg}(\sigma[M])$ also carries the discrete topology: given a simple module S let f be the map $S \to 0$ and observe that $(f) = \{S\}$. So $\mathrm{Zg}(\sigma[M])$ need not be a compact space.

THEOREM 3.8. *Assume $\sigma[M]$ satisfies $(*)$. Then j induces a homeomorphism between $\mathrm{Zg}(\sigma[M])$ and its image in Zg_R.*

Proof. Take a morphism $f : A \to B$ in $\sigma[M]^{\mathrm{fp}}$ and consider the basic open set, let us denote it by $((f)) = \{C \in \mathrm{Zg}(\sigma[M]) : (f, C) \text{ is not epi}\}$, that it defines in $\mathrm{Zg}(\sigma[M])$. If $C \in ((f))$ then, since f is also a morphism in mod-R, we have $\overline{C} \in (f)$ (e.g. by the criterion for purity used in the proof of 2.5). Therefore $j((f)) \subseteq (f) \cap \operatorname{im} j$. If, conversely, we have $\overline{C} \in (f) \cap \operatorname{im} j$, say $g : A \to \overline{C}$ does not factor through f, then $\operatorname{im} g \le C$ $(= T^M\overline{C}$ by 3.5) so clearly $C \in (f)$. Therefore j is an open map.

For the converse, let X be a closed subset of Zg_R and let $\mathcal{D}$ be the corresponding definable subcategory of Mod-R. Recall the bijective correspondence, for any locally finitely presented Grothendieck category, between closed subsets of the Ziegler spectrum and definable subclasses of the category (see [4] or [7]). We show that the intersection $\mathcal{D}' = \mathcal{D} \cap \sigma[M]$ is a definable subcategory of $\sigma[M]$. Certainly $\mathcal{D}'$ is closed under taking pure submodules and it is also closed under directed limits, since both these are computed in $\sigma[M]$ just as in Mod-R. It remains, therefore, to show that $\mathcal{D}'$ is closed under products in $\sigma[M]$. This will be enough because, by $(*)$, if $C \in \mathrm{Zg}(\sigma[M])$ then $C \in \mathcal{D}'$ iff $\overline{C} \in \mathcal{D}$.

Therefore let $\{A_\lambda\}_\lambda$ be modules in $\sigma[M]$ and set $A = \prod_\lambda A_\lambda$ to be their product in Mod-R. Then A is pure in its pure-injective hull $\bar{A}$ and hence, by 2.5, so is the embedding of $T^M A$ into $T^M\bar{A}$. Note that $T^M A$ is the product of the A_λ in $\sigma[M]$. By assumption $(*)$, $T^M\bar{A}$ is pure in $\bar{A}$ and hence $T^M A$ is pure in $\bar{A}$. Since $T^M A \le A \le \bar{A}$ it follows that $T^M A$ is pure in A. Therefore $T^M A \in \mathcal{D}'$ (because $A = \prod A_\lambda \in \mathcal{D}$), as required. ∎

COROLLARY 3.9. *Assume that $\sigma[M]$ satisfies $(*)$. Then the definable subcategories of $\sigma[M]$ are exactly those of the form $\mathcal{D} \cap \sigma[M]$ where $\mathcal{D}$ is a definable subcategory of* Mod-R.

Proof. The statement follows immediately from 3.8. ∎

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Received 28 July 2003;
revised 28 January 2004 (4364)