COMMUNICATIONS IN ALGEBRA

THE LATTICE STRUCTURE OF HEREDITARY PRETORSION CLASSES

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Abstract

In this paper we continue the investigation of the lattice structure of hereditary pretorsion classes ([4],[5]). We show the existence of pseudocomplements and study right supplements for every hereditary pretorsion class. Moreover we investigate relations between these concepts and characterize a class of modules by means of these relations.

1 PRELIMINARIES

Let R be an associative ring with unit and R-Mod the category of unital left R-modules. In this paper we are going to work in the full subcategory $\sigma[M]$ R-modules whose objects consist of the submodules of M-generated modules. Notice that this class is closed under direct sums, submodules and factor modules.

A subclass τ of $\sigma[M]$ is called a *hereditary pretorsion class* if it is closed under direct sums, submodules and factor modules. Such classes are of type $\sigma[U]$, for some R-module U. We will denote by M-ptors the complete lattice of hereditary pretorsion classes on $\sigma[M]$. For any $\tau \in M$-ptors we write τN for the corresponding pretorsion submodule of N, for any R-module N, and we denote by $\mathcal{L}_\tau$ the corresponding linear filter of left ideals in R.

If $\mathcal{C}$ is a subclass of $\sigma[M]$, $\sigma[\mathcal{C}]$ will be the unique minimal element of M-ptors relative to which the elements of $\mathcal{C}$ are pretorsion, and $\mathbb{F}_\tau$ stands for the torsion free class in $\sigma[M]$ that corresponds to any element $\tau \in M$-ptors,

$$\mathbb{F}_\tau = \{N \in \sigma[M] \mid \tau N = 0\}\ .$$

For any couple of elements $\tau_1, \tau_2 \in M$-ptors we will denote by $(\tau_1 : \tau_2) \in M$-ptors the element such that

$$(\tau_1 : \tau_2)N/\tau_1 N = \tau_2(N \mid \tau_1 N) \quad \text{for all } N \in \sigma[M]\,.$$

Notice that the corresponding hereditary pretorsion class is given by all elements $N \in \sigma[M]$ such that there exists an exact sequence

$$0 \to N' \to N \to N'' \to 0\,,$$

where $N' \in \tau_1$ and $N'' \in \tau_2$. This operation is associative and (M-ptors,$(\_:\_)$) is a semigroup.

We shall denote by M-tors the lattice of all hereditary torsion classes defined in $\sigma[M]$. Notice that an element $\tau \in M$-ptors is a hereditary torsion class if, and only if $(\tau : \tau) = \tau$. It is easy to see that M-tors is a frame.

For any subclass $\mathcal{C}$ of $\sigma[M]$, we will denote by $\xi(\mathcal{C})$ the minimal hereditary torsion class in $\sigma[M]$, relative to which every element in $\mathcal{C}$ is a torsion module, and by $\chi(\mathcal{C})$ the maximal hereditary torsion class in $\sigma[M]$ relative to which every element of $\mathcal{C}$ is a torsion free module. For any $N \in \sigma[M]$, the injective hull of N in $\sigma[M]$ will be denoted by $\widehat{N}$.

For all other concepts, notation and terminology concerning hereditary pretorsion classes, hereditary torsion classes and lattice theory, the reader is referred to [1], [2], [3] and [7].

2 PROPERTIES OF M-ptors.

LEMMA 1. *Let $\{\tau_\alpha \,|\, \alpha \in X\}$ be a family of hereditary pretorsion classes, then*

$$\mathcal{L}_{\vee\tau_\alpha} = \{ {}_R I \leq R \,|\, \forall\, \alpha\, \exists\, I_\alpha \in \mathcal{L}_{\tau_\alpha} \textit{ with } I_\alpha = R \textit{ for almost all } \alpha, \cap I_\alpha \subset I \} .$$

Proof: Let $\mathcal{L}=\{ {}_R I \leq R \,|\, \forall\, \alpha\, \exists\, I_\alpha \in \mathcal{L}_{\tau_\alpha}$ with $I_\alpha = R$ for almost all $\alpha, \cap I_\alpha \subset I\}$. It is obvious that $\mathcal{L}$ is a linear filter. On the other hand, since $\mathcal{L}_{\tau_\alpha} \leq \mathcal{L}$ for all $\alpha \in X$, we have $\mathcal{L}_{\vee\tau_\alpha} \leq \mathcal{L}$. The inequality $\mathcal{L} \leq \mathcal{L}_{\vee\tau_\alpha}$ is immediate. $\square$

THEOREM 2. (Modular Law). *Let ρ, τ, η be elements of M-ptors such that $\rho \leq \tau$. Then $\rho \vee (\tau \wedge \eta) = \tau \wedge (\rho \vee \eta)$.*

Proof: The inequality $\rho \vee (\tau \wedge \eta) \leq \tau \wedge (\rho \vee \eta)$ is immediate.

Now let $I \in \mathcal{L}_{\tau\wedge(\rho\vee\eta)}$, by Lemma 1, there exits $J \in \mathcal{L}_\rho$ and $K \in \mathcal{L}_\eta$ with $J \cap K \leq I$. Hence $J \cap I \in \mathcal{L}_\tau$, so $(I \cap J) + K \in \mathcal{L}_{\tau\wedge\eta}$. Finally,

$$J \cap [(I \cap J) + K] = (I \cap J) + (J \cap K) \leq I .$$

Therefore $I \in \mathcal{L}_{\rho\vee(\tau\wedge\eta)}$. $\square$

In [2] an infinite product of hereditary pretorsion classes is defined as follows.

DEFINITION 3. Let $\{\tau_\beta \in M\text{-ptors} \mid \beta \in X\}$ where X is a well ordered set of type ε. Write

(a) $s_1 = \tau_1$

(b) $s_{\alpha+1} = (s_\alpha : \tau_{\alpha+1})$.

(c) $s_\alpha = \vee \{s_\beta \mid \beta < \alpha\}$ if α is a limit ordinal,

and put $:\{\tau_\alpha \mid \alpha \in X\} = s_\varepsilon$.

For our next result we will use the following generalization of Proposition 2.5 of [2].

PROPOSITION 4. *For each $\tau_1, \tau_2 \in M$-ptors*

$$\tau_1 (\tau_2(N)) = (\tau_1 \wedge \tau_2)(N) = \tau_2 (\tau_1(N)) \;, \textit{ for any } N \in \sigma[M] \,.$$

LEMMA 5. *For any τ_1, τ_2 and $\tau_3 \in M$-ptors,*

$$\tau_1 \wedge (\tau_2 : \tau_3) \leq (\tau_1 \wedge \tau_2) : (\tau_1 \wedge \tau_3) \;.$$

Proof: Let $N \in \sigma[M]$. We have

$$\begin{aligned}
[\tau_1 \wedge (\tau_2 : \tau_3)](N) / (\tau_1 \wedge \tau_2)(N) &= (\tau_2 : \tau_3)(\tau_1 N) / (\tau_2 \tau_1)(N) \\
&= \tau_1 [(\tau_2 : \tau_3)(\tau_1 N)/(\tau_2 \tau_1)(N)] \\
&= \tau_1 [\tau_3 (\tau_1 N/(\tau_2 \tau_1)(N))] \\
&= (\tau_1 \tau_3) [\tau_1 N/(\tau_1 \wedge \tau_2)(\tau_1 N)] \\
&= [(\tau_1 \wedge \tau_2) : (\tau_1 \wedge \tau_3)](\tau_1 N)/(\tau_1 \wedge \tau_2)(\tau_1 N) \\
&\leq [(\tau_1 \wedge \tau_2) : (\tau_1 \wedge \tau_3)](N)/(\tau_1 \wedge \tau_2)(N) \,.
\end{aligned}$$

Therefore $[\tau_1 \wedge (\tau_2 : \tau_3)](N) \leq [(\tau_1 \wedge \tau_2) : (\tau_1 \wedge \tau_3)](N)$ for all $N \in \sigma[M]$.

From this the lemma follows. □

THEOREM 6. *Let τ be an element of M-ptors, and $\{\tau_\alpha \mid \alpha \in X\}$ a family of elements of M-ptors, where X is a well ordered set of type ε. Then*

$$\tau \wedge (\,:\{\tau_\alpha \mid \alpha \in X\}) \leq : \{\tau \wedge \tau_\alpha \mid \alpha \in X\} \,.$$

Proof: We proceed by induction over the ordinal ε.

If $\varepsilon = 1$, then the inequality is obvious.

Now, let us assume that $\varepsilon > 1$ and the result is true for any ordinal $\nu < \varepsilon$.

Let us write $\overline{\tau}_\alpha = \tau \wedge \tau_\alpha$ and $\overline{s}_\alpha$ as in Definition 3 corresponding to the family $\{\overline{\tau}_\alpha\}$.

Now $\tau \wedge s_{\alpha+1} = \tau \wedge (s_\alpha : \tau_{\alpha+1})$ and, by Lemma 5,

$$\tau \wedge (s_\alpha : \tau_{\alpha+1}) \leq (\tau \wedge s_\alpha) : (\tau \wedge \tau_{\alpha+1}) \leq (\overline{s}_\alpha : \overline{\tau}_{\alpha+1}) = \overline{s}_{\alpha+1} \, .$$

Now let α be a limit ordinal. Then

$$\tau \wedge s_\alpha = \tau \wedge \left(\bigvee \{s_\beta \,|\, \beta < \alpha\}\right) = \bigvee \{\tau \wedge s_\beta \,|\, \beta < \alpha\} \ ,$$

the last equality holds because M-ptors is an upper continuous lattice (see [5, Proposition 4.7]). To finish the proof notice that

$$\bigvee \{\tau \wedge s_\beta \,|\, \beta < \alpha\} \leq \bigvee \{\overline{s}_\beta \,|\, \beta < \alpha\} = \overline{s}_\alpha \, .$$

□

Observe that if $\tau_\alpha \leq \eta_\alpha$ for all $\alpha \in X$, then:

$$: \{\tau_\alpha \,|\, \alpha \in X\} \leq : \{\eta_\alpha \,|\, \alpha \in X\} \ .$$

NOTATION 7. Let $\{\tau_\alpha \,|\, \alpha \in X\}$ be a family of hereditary pretorsion classes, where X is a well ordered set. We write $\tau_X =: \{\tau_\alpha \,|\, \alpha \in X\}$, provided $\tau_\alpha = \tau$, for all $\alpha \in X$.

The following theorem characterizes those elements of M-ptors which are hereditary torsion classes.

THEOREM 8. *Let τ be a hereditary pretorsion class. Then the following conditions are equivalent:*

(1) $\tau \in M$*-tors.*

(2) $\tau \wedge (\eta : \eta) = (\tau \wedge \eta) : (\tau \wedge \eta)$ *for all* $\eta \in M$*-ptors.*

(3) $\tau \wedge (\eta_1 : \eta_2) = (\tau \wedge \eta_1) : (\tau \wedge \eta_2)$ *for all* $\eta_1, \eta_2 \in M$*-ptors.*

(4) $\tau \wedge \eta_X = (\tau \wedge \eta)_X$ *for all* $\eta \in M$*-ptors and for any well ordered set* X*.*

(5) $\tau \wedge : \{\eta_\alpha \,|\, \alpha \in X\} =: \{\tau \wedge \eta_\alpha \,|\, \alpha \in X\}$ *for any family* $\{\eta_\alpha \,|\, \alpha \in X\}$ *in* M*-ptors, where* X *is a well ordered set.*

Proof: 1)$\Rightarrow$ 5) Since for all $\alpha \in X$ we have the inequalities $\tau \wedge \tau_\alpha \leq \tau_\alpha$ and $\tau \wedge \tau_\alpha \leq \tau$, we get

$$: \{\tau \wedge \tau_\alpha \,|\, \alpha \in X\} \leq : \{\tau_\alpha \,|\, \alpha \in X\}$$

and also

$$: \{\tau \wedge \tau_\alpha \,|\, \alpha \in X\} \leq \tau_X = \tau$$

by hipothesis. So we obtain

$$: \{\tau \wedge \tau_\alpha \,|\, \alpha \in X\} \leq \tau \wedge : \{\tau_\alpha \,|\, \alpha \in X\} \ .$$

By Theorem 6 we have the other inequality.

The implications 5) $\Rightarrow$ 4), 5) $\Rightarrow$ 3), 4) $\Rightarrow$ 2) and 3) $\Rightarrow$ 2) are straightforward.

2) $\Rightarrow$ 1) Take $\eta = \tau$, then $\tau \wedge (\tau : \tau) = (\tau \wedge \tau) : (\tau \wedge \tau)$. Hence $\tau = (\tau : \tau)$ which implies that τ is a hereditary torsion class. □

3 PSEUDOCOMPLEMENTS AND RIGHT SUPPLEMENTS.

Let ξ be the smallest element of M-ptors, and Ω be the largest element of M-ptors. Notice that ξ is the class $\{0\}$, and Ω is the class of all objects in $\sigma[M]$.

DEFINITION 9. Let $\tau \in M$-ptors. An element $\rho \in M$-ptors is called the *right supplement* of τ if $(\tau : \rho) = \Omega$ and ρ is the smallest element of M-ptors with respect to this property. For the existence of such an element, notice that $(\tau : \wedge \eta_\alpha) = \wedge(\tau : \eta_\alpha)$ for all $\tau, \eta_\alpha \in M$-ptors for all α and [2, Proposition 3.13]).

We will denote by $\tau^{(1)}$ the right supplement of τ.

The following theorem gives us several characerizations of $\tau^{(1)}$.

THEOREM 10. *Let $\tau, \rho \in M$-ptors. The following conditions are equivalent:*

(1) $\rho = \tau^{(1)}$.

(2) $\rho = \sigma\left[\{N/\tau N \mid N \in \sigma[M]\}\right]$.

(3) $\rho = \{N/N' | N \in \sigma[M], \tau N \subset N'\}$

(4) $\rho = \sigma[M/\tau M]$.

Proof: 1) $\Rightarrow$ 2) Since $(\tau : \rho) = \Omega$, we have for all $N \in \sigma[M]$, $(\tau : \rho)N = N$, therefore $\rho\,(N/\tau N) = N/\tau N$ for all $N \in \sigma[M]$, hence $\sigma\left[\{N/\tau N \mid N \in \sigma[M]\}\right] \leq \rho$. To prove the other inequality, let us denote by $\rho' = \sigma\left[\{N/\tau N \mid N \in \sigma[M]\}\right]$. Now for all $N \in \sigma[M]$ we have that $N/\tau N \in \rho'$, so $N \in (\tau : \rho')$ which means that $(\tau : \rho') = \Omega$. Thus $\rho \leq \rho'$.

2) $\Rightarrow$ 3) $\rho' = \{N/N' \mid N \in \sigma[M], \tau N \subset N'\}$. For each $N \in \sigma[M]$ we have an an epimorphism $N/\tau N \to N/N'$, implying $\rho' \leq \rho$. The other inequality is immediate from the fact that $N/\tau N \in \rho'$, for each $N \in \sigma[M]$.

3) $\Rightarrow$ 4) Let $\rho' = \sigma\,[M/\tau M]$. From (3) we have that $M/\tau M \in \rho$, so $\rho' \leq \rho$. Now for each $N \in \sigma[M]$ and $N' \subset N$ such that $\tau N \subset N'$ there exists $K \in R$-Mod with a monomorphism $N/N' \to K$ and a epimorphism $(M/\tau M)^{(X)} \to K$. This implies that $\rho \leq \rho'$.

4) $\Rightarrow$ 1) Let $N \in \sigma[M]$. By (4) we have $N/\tau N \in \rho$, so $(\tau : \rho) = \Omega$ which implies $\tau^{(1)} \leq \rho$.

Let $N \in \rho$. Then there is a monomorphism $N \to K$ and an epimorphism $(M/\tau M)^{(X)} \to K$. Now since $M/\tau M \in \tau^{(1)}$ we conclude $N \in \tau^{(1)}$, hence $\rho \leq \tau^{(1)}$. □

COROLLARY 11. *Let $\tau, \rho \in M$-ptors, then $\tau M \subseteq \rho M$ implies $\rho^{(1)} \leq \tau^{(1)}$.*

For the special case $M = R$ we have the following results.

COROLLARY 12. *Let $\tau, \rho \in R$-ptors, then $\tau R \subseteq \rho R$ if, and only if $\rho^{(1)} \leq \tau^{(1)}$.*

COROLLARY 13. *For each $\tau \in R$-ptors, $\tau^{(1)}$ is Jansian. Moreover,*

$$\mathfrak{L}_{\tau^{(1)}} = \{ {}_RJ \leq R \mid \tau R \subseteq J \} .$$

DEFINITION 14. Let $\varphi \colon M$-ptors $\to M$-tors be given by $\varphi(\tau) = \xi(\tau)$.

Notice that $\varphi(\tau)$ may be obtained as the hereditary torsion class corresponding to the hereditary torsion free class $\mathbb{F}_\tau$. It is well known that $\varphi(\tau)$ can also be obtained by means of the Levitzki-Amitsur transfinite process.

REMARK 1. In [3, VI, Propositions 2.5 and 3.3] the hereditary torsion class generated by a class which is closed under quotients and submodules is characterized. It is clear that this characterization is valid in $\sigma[M]$.

COROLLARY 15. *Let $\tau, \rho \in M$-ptors, then $\varphi(\tau \wedge \rho) = \varphi(\tau) \wedge \varphi(\rho)$.*

Proof: Since φ is order preserving we have $\varphi(\tau \wedge \rho) \leq \varphi(\tau) \wedge \varphi(\rho)$.

Now, consider $N \in \varphi(\tau) \wedge \varphi(\rho)$ and let N'' is a nonzero quotient of N. Since $N'' \in \varphi(\tau)$ it contains a nonzero submodule $K \in \tau$. Moreover $N'' \in \varphi(\rho)$ and so it contains a nonzero submodule $K' \in \rho$. So we have $K' \in \tau \wedge \rho$ and hence $N \in \varphi(\tau \wedge \rho)$. □

Recall that a *pseudocomplement* for an element x in any lattice with minimal element 0 is a element y of the lattice, which is maximal with respect to $x \wedge y = 0$.

We shall use the standard notation $\tau^\perp$ to denote the unique pseudocomplement of any $\tau \in M$-tors.

COROLLARY 16. *For any $\tau \in M$-ptors, $\varphi(\tau)^\perp$ is the unique pseudocomplement of τ in M-ptors.*

Proof: Since $\varphi(\tau) \wedge \varphi(\tau)^\perp = \xi$ we have that $\tau \wedge \varphi(\tau)^\perp = \xi$. If $\tau \wedge \rho = \xi$, then $\varphi(\tau) \wedge \varphi(\rho) = \xi$, therefore $\varphi(\rho) \leq \varphi(\tau)^\perp$, which implies that $\rho \leq \varphi(\tau)^\perp$. □

From now on we will denote by $\tau^\perp = \varphi(\tau)^\perp$ for any $\tau \in M$-ptors.

REMARK 2. In many lattices a pseudocomplement does not exist, and when exist it is almost never unique. Since M-ptors is not even distributive, the existence of a unique pseudocomplement for each element is a remarkable fact.

The usual properties of the pseudocomplement in M-tors are also valid in R-ptors, but we want to point out the following one:

COROLLARY 17. *Let $\tau \in M$-ptors, then*

$$\tau^\perp = \chi\{S \in M\text{-}simp \mid S \in \tau\}$$

where M-simp denotes a set of representatives of the simple objects in $\sigma[M]$.

LEMMA 18. *Let $\tau \in M$-ptors. Then $\tau^\perp \leq \tau^{(1)}$.*

Proof: Let $N \in \tau^\perp$, then $\tau N \in \tau^\perp$ and since $\tau N \in \tau$ we have that $\tau N = 0$, so $N \in \tau^{(1)}$. □

LEMMA 19. *For any $\tau \in M$-ptors,*

$$\varphi(\tau)^{(1)} = \{N'' \in \sigma[M] \mid N'' \text{ is an image of some } N \in \mathbb{F}_\tau\} = \sigma[\mathbb{F}_\tau]\,.$$

Proof: First note that $\mathbb{F}_\tau = \mathbb{F}_{\varphi(\tau)}$. Now, let A be the family of homomorphic images of elements of $\mathbb{F}_\tau$. Hence if $N \in A$, by Theorem 10, we have that $N \in \varphi(\tau)^{(1)}$, and so $A \subset \varphi(\tau)^{(1)}$.

Finally take $N \in \varphi(\tau)^{(1)}$, by Theorem 10, there exists an epimorphism $(M/\varphi(\tau)M)^{(X)} \to N$. Since $(M/\varphi(\tau)M) \in \mathbb{F}_\tau$ we get $N \in A$. □

and so $A \subseteq \varphi(\tau)^{(1)}$. Finally take $N \in \varphi(\tau)^{(1)}$, by Theorem 10, there exists an epimorphism $(R/\varphi(\tau)R)^{(X)} \to N$. Since $R/\varphi(\tau)R \in F_\tau$ we get that $N \in A$.

PROPOSITION 20. *Let $\tau \in M$-ptors. Then $\tau^\perp = \chi(\tau)$.*

Proof: By Corollary 17, we know that $\tau^{\perp} = \varphi(\tau)^{\perp} = \chi(\tau \cap M\text{-simp})$, so we have $\tau^{\perp} \geq \chi(\tau)$.

To show the other inequality, take $K \in \tau^{\perp}$ and $0 \neq f\colon K \to \widehat{N}$ a morphism with $N \in \tau$. Let $N' \neq 0$ be a finitely generated submodule of $N \cap \operatorname{im} f$, and take $S \in M$-simp a factor module of N'. Then there exists a nonzero morphism $h\colon K \to \widehat{S}$ which is a contradiction. $\square$

The following theorem gives us information about the "distance" between $\tau^{\perp}$ and $\tau^{(1)}$.

THEOREM 21. *For any $\tau \in M$-ptors*

$$\tau^{\perp} \subseteq \mathbb{F}_{\tau^{\perp\perp}} \subseteq \mathbb{F}_{\tau} \subseteq \varphi(\tau)^{(1)} \subseteq \tau^{(1)} .$$

Moreover, we have the following properties:

(1) $\mathbb{F}_{\chi(\tau^{\perp})} = \mathbb{F}_{\tau^{\perp\perp}}$.

(2) $\sigma\left[\mathbb{F}_{\tau}\right] = \varphi(\tau)^{(1)}$.

(3) *Let $\eta \in M$-ptors be such that $\eta \subseteq \mathbb{F}_{\eta^{\perp\perp}}$. Then $\eta \leq \tau^{\perp}$.*

Proof: The first inequality is obvious, and (1) follows from Proposition 20.

Clearly $\tau \leq \tau^{\perp\perp}$ implies $\mathbb{F}_{\tau^{\perp\perp}} \subseteq \mathbb{F}_{\tau}$. Now, by Lemma 21, $\mathbb{F}_{\tau} \subseteq \varphi(\tau)^{(1)}$, and it also implies (2).

Since $\tau \leq \varphi(\tau)$ we have that $\varphi(\tau)^{(1)} \leq \tau^{(1)}$.

To show (3). Let $\eta \in M$-ptors be such that $\eta \subseteq \mathbb{F}_{\tau^{\perp\perp}}$. Then $\eta \subseteq \mathbb{F}_{\tau}$ and so $\eta \wedge \tau = \{0\}$ which implies $\eta \leq \tau^{\perp}$. $\square$

The following theorem characterizes those hereditary pretorsion classes for which the pseudocomplement and the right supplement coincide.

THEOREM 22. *Let $\tau \in M$-ptors. The following conditions are equivalent:*

(1) $\tau^{\perp} = \tau^{(1)}$.

(2) (i) $\tau^{\perp}$ *is stable and Jansian,*

(ii) $\varphi(\tau) = \tau^{\perp\perp}$,

(iii) *for every* $N \in \sigma[M]$ *we have that* $\varphi(\tau)^{(1)}N = N$ *if and only if* $\mathrm{Hom}_R(K, \widehat{N}) = 0$ *for all* $K \in \tau$.

(iv) $M/\tau M \in \varphi(\tau)^{(1)}$.

Proof: Notice that in (2), condition (i) is equivalent to $\tau^{\perp} = \mathbb{F}_{\tau^{\perp\perp}}$, (ii) is equivalent to $\mathbb{F}_{\tau^{\perp\perp}} = \mathbb{F}_{\tau}$, (iii) is equivalent to $\mathbb{F}_{\tau} = \varphi(\tau)^{(1)}$ and (iv) is equivalent to $\varphi(\tau)^{(1)} = \tau^{(1)}$.

Now the assertions follow from Theorem 21. □

For the special case $M = R$ we obtain another characterization for τ being a cohereditary torsion class in R-Mod (see [8, 4.6], [9, 2.6]):

THEOREM 23. *Let* $\tau \in R$*-ptors. The following conditions are equivalent:*

(1) $\tau^{\perp} = \tau^{(1)}$

(2) $\tau N = (\tau R)N$ *for all* $N \in R$*-Mod.*

Proof: (1) $\Rightarrow$ (2) Let $N \in R$-Mod. By Theorem 10, $\tau N/(\tau R)N \in \tau \wedge \tau^{(1)}$. Now since $\tau^{(1)} = \tau^{\perp}$ we have that $\tau N/(\tau R)N = 0$, so $\tau N = (\tau R)N$.

(2) $\Rightarrow$ (1) By Lemma 20 $\tau^{\perp} \leq \tau^{(1)}$, so it remains to show $\tau^{(1)} \leq \tau^{\perp}$.

Take $N \in \tau^{(1)}$. Since $\tau^{(1)}$ is closed under taking submodules, it is enough to consider a morphism $f\colon N \to S$ with $S \in \tau \cap R$-Simp. Now

$$f(N) = \tau f(N) = (\tau R)f(N) = f((\tau R)N) = f(0) = 0\,,$$

hence $N \in \tau^{\perp}$. □

The following theorem classifies semisimple modules ($\sigma[M]$ is a spectral discrete Grothendieck category) in terms of the pseudocomplement, the right supplement, and lattice structure.

THEOREM 24. *Let* $M \in R$*-Mod. The following conditions are equivalent:*

(1) *M is a semisimple module.*

(2) *$\tau^{\perp} = \tau^{(1)}$, for all $\tau \in M$-ptors.*

(3) *M-ptors is a Boolean lattice.*

Proof: (1) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (2) Let $\tau \in M$-ptors and $\tau^* \in M$-ptors be a complement for τ, then $\tau \wedge \tau^* = \xi$, therefore $\tau^* \leq \tau^{\perp}$. On the other hand, $\Omega = \tau \vee \tau^* \leq (\tau\colon \tau^*)$ which implies that $\tau^{(1)} \leq \tau^*$. Now by Lemma 18 we get $\tau^{\perp} = \tau^{(1)}$.

(2) $\Rightarrow$ (1) Let $\tau \in M$-ptors be the class of semisimple modules in $\sigma[M]$. Then by Corollary 19, $\tau^{\perp} = \xi$. Hence by (2), $\tau^{(1)} = \xi$. So we have that $N \,|\, \tau N = 0$ for all $N \in \sigma[M]$, hence N is semisimple for all $N \in \sigma[M]$, in particular M is a semisimple module. $\square$

NOTATION. We will denote by $\mathbb{Z}_{(p)}$ the localization of $\mathbb{Z}$ at the prime p, and $\mathbb{Z}_{p^\infty}$ the p-primary component of $\mathbb{Q}/\mathbb{Z}$ (Prüfer groups).

The following gives us an example where the inclusion relations in Theorem 21 are all strict:

EXAMPLE 25. Let $R = \mathbb{Z} \times \mathbb{Z} \times (\mathbb{Z}_{(p)} \ltimes \mathbb{Z}_{p\infty})$ where $\mathbb{Z}_{(p)} \ltimes \mathbb{Z}_{p\infty}$ is the trivial extension, with p any prime number.

Take $\tau = \tau_1 \times \tau_2 \times \tau_3 \in R$-ptors, where τ_1 is the class of semisimple p-groups in $\mathbb{Z}$-Mod, τ_2 is the class of singular $\mathbb{Z}$-modules and $\tau_3 = \sigma[(\{0\} \ltimes \mathbb{Z}_{p^\infty})]$ in $\mathbb{Z}_{(p)} \ltimes \mathbb{Z}_{p^\infty}$-Mod.

Notice that the first factor implies $\tau^{\perp} \neq \mathbb{F}_{\tau^{\perp\perp}}$, the second factor implies $\mathbb{F}_{\tau^{\perp\perp}} \neq \mathbb{F}_{\tau}$ and $\mathbb{F}_{\tau} \neq \varphi(\tau)^{(1)}$, and the last factor implies $\varphi(\tau)^{(1)} \neq \tau^{(1)}$.

The following is an example where the converse of Corollary 11 is not valid.

EXAMPLE 26. Let $M = \mathbb{Z}_{p^\infty}$ where p is any prime number and let τ be the class of semisimple objects in $\sigma[M] \subseteq \mathbb{Z}$-Mod. Then $\tau^{(1)} = (\tau : \tau)^{(1)}$ but $(\tau : \tau)M \not\subseteq \tau M$.

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