# HOMOLOGICAL PROPERTIES OF QUANTUM POLYNOMIALS

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Abstract. In the paper we study the endomorphim semigroup of a general quantum polynomial ring, its finite groups of automorphisms and homological properties of this ring as a module over the skew group ring of a finite group of automorphisms. Moreover properties of the division ring of fractions are considered.

## Introduction

The study of quantum polynomial rings was initiated by J. C. McConnell and J. J. Pettit [MP] as multiplicative analog of the Weyl algebra. They are of considerable interest in non-commutative algebraic geometry.

The action of automorphism groups was studied by J. Alev and M. Chamarie [AC]. The extended action of finite automorphism groups on division rings of fractions (for two indeterminates) and its subrings of invariants were studied by J. Alev and F. Dumas [AD]. Similar topics are considered in [M1], [Kh], [KPS], [OP]. An extensive investigation of various properties of general quantum polynomials was performed by the first author in [A1], [A2], [A3], [A4].

The purpose of the present paper is to study actions of finite automorphism groups on such rings under the assumption that the number of indeterminates is at least 3. In Section 1 basic properties of general quantum rings are collected. Then the form of ring endomorphisms of such rings is determined in Section 2. The subsequent section is devoted to the description of invariants under finite automorphism groups. In this context the trace map is an important tool and related results are provided is Section 4. More properties of the quantum polynomial rings as modules over skew group rings are given in the final section.

## 1. General quantum polynomials

Let $D$ be a division ring with a fixed set $\alpha_1, \dots, \alpha_n, \quad n \geq 2$, of its automorphisms. We shall also fix elements $q_{ij} \in D^*, \quad i, j = 1, \dots, n$, satisfying the equalities

$$q_{ii} = q_{ij}q_{ji} = Q_{ijr}Q_{jri}Q_{rij} = 1, \quad \alpha_i(\alpha_j(d)) = q_{ij}\alpha_j(\alpha_i(d))q_{ji}, \tag{1}$$

where

$$Q_{ijr} = q_{ij}\alpha_j(q_{ir}), \text{ and } d \in D.$$

Put

$$Q = (q_{ij}) \in \mathrm{Mat}(n, D) \text{ and } \alpha = (\alpha_1, \dots, \alpha_n).$$

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**Definition 1.1.** The entries $q_{ij}$ of the matrix $Q$ form a *system of multiparameters.*

**Definition 1.2.** Denote by

$$\Lambda = D_{Q,\alpha}[X_1^{\pm 1}, \dots, X_r^{\pm 1}, X_{r+1}, \dots, X_n] \tag{2}$$

the associative ring generated by elements of $D$ and by the elements

$$X_1, \dots, X_n, X_1^{-1}, \dots, X_r^{-1}, \tag{3}$$

subject to the defining relations

$$\begin{aligned} X_i X_i^{-1} &= X_i^{-1} X_i = 1, \quad 1 \le i \le r; \\ X_i d &= \alpha_i(d) X_i, \qquad d \in D, \quad i = 1, \dots, n; \\ X_i X_j &= q_{ij} X_j X_i, \qquad i, j = 1, \dots, n. \end{aligned} \tag{4}$$

The ring (2) is called a *quantum polynomial ring.* If $r = n$ the ring (2) is said to be a *quantum Laurent polynomial ring.*

Such rings first appeared in [MP] as a mulitplicative analog of the Weyl algebra. It is assumed in [MP] that $D$ is a field, $\alpha_1 = \cdots = \alpha_n$ are identical automorphisms of $D$ and $r = n$. In this particular case (1) is equivalent to the equalities $q_{ii} = q_{ij} q_{ji} = 1$ for all $i, j = 1, \dots, n$. The importance of quantum polynomials in noncommutative geometry is explained in [D]. It can be viewed as the coordinate ring $\mathcal{O}_Q(\mathbb{A}^n)$ of the quantum affine plane $\mathbb{A}^n$ of dimension $n$ [BG], [GL1],[GL2]. A survey of some results on a structure of projective modules is exposed in [A3]. As it follows from [A1] the ring (2) is a crossed product $\Lambda = D\sharp_t H$, where the bialgebra $H$ is a tensor product of an integral group ring of a free abelian group with the basis $\{X_i | 1 \le i \le r\}$ and an integral semigroup ring of a free abelian semigroup with the basis $\{X_i | i \le n\}$.

**Proposition 1.3** ([A3], §2)**.** *The ring $\Lambda$ from (2) is a left and a right vector space over $D$ whose basis consists of monomials*

$$u = X_1^{m_1} \cdots X_n^{m_n},$$

*where $m_i \in \mathbb{Z}$ if $1 \le i \le r$ and $m_i \in \mathbb{N} \cup 0$, if $i \le n$. In particular the ring $\Lambda$ from (2) is a left and right Noetherian domain with the division ring of fractions*

$$F = D_{Q,\alpha}(X_1, \dots, X_n).$$

*Each automorphism $\alpha_i$ of $D$ can be extended to $F$ in such a way that*

$$\alpha_i(f) = X_i f X_i^{-1}$$

*for all $f \in F$.*

It is shown in [MP] that if $D$ is a field and $\alpha$ is a set of identical automorphisms of $D$, then

$$D_{Q,\alpha}[X_1^{\pm 1}, \dots, X_n^{\pm 1}] \simeq D_{Q',\alpha'}[Y_1^{\pm 1}, \dots, Y_n^{\pm 1}],$$

if and only if there exists a matrix $M = (m_{ij}) \in GL(n, \mathbb{Z})$, such that

$$q'_{ij} = \prod_{r,s} q_{rs}^{m_{ri} m_{sj}}.$$

**Definition 1.4.** Let $N$ be the subgroup in the multiplicative group $D^*$ of the division ring $D$ generated by the derived subgroup $[D^*, D^*]$ and the set of all elements of the form $z^{-1}\alpha_i(z)$, where $z \in D^*$ and $i = 1, \dots, n$.

It is fairly obvious that $N$ is a normal subgroup in the multiplicative group $D^*$ and $D^*/N$ is a multiplicative abelian group. The normal subgroup $N$ always appears when we multiply monomials in the ring $\Lambda$. The following formulae will be used throughout the paper. For any two monomials in $\Lambda$ we have

$$(X_1^{m_1} \cdots X_n^{m_n})(X_1^{s_1} \cdots X_n^{s_n}) = \left[\prod_{i \leq j} q_{ji}^{m_j s_i}\right] u \cdot X_1^{m_1+s_1} \cdots X_n^{m_n+s_n}. \tag{5}$$

where $u \in N \triangleleft D^*$. The proof follows immedialety from (4).

**Theorem 1.5.** *Let $F$ be the division ring of fractions of $\Lambda$. Then $N = [F^*, D^*] \cap D^*$. The subgroup of $D^*/N$ generated by the images of $q_{ij}, 1 \leq i, j \leq n$, is equal to $[F^*, F^*] \cap D^*$.*

Recall that elements $a_1, \ldots, a_m$ of a multiplicative abelian group are *independent* if, for any integers $s_1, \ldots, s_m$, we have

$$a_1^{s_1} \cdots a_m^{s_m} = 1 \iff s_1 = \cdots = s_m = 0.$$

In the paper [A1] the following restriction on the multiparameters is assumed.

(♠) *the images of all multiparameters $q_{ij}, 1 \leq i \leq j \leq n$, are independent in the multiplicative abelian group $D^*/N$.*

The restriction of this form first appeared in the paper [MP]. If the restriction (♠) is satisfied we call $\Lambda$ the *ring of general quantum polynomials.*

The following example shows that rings of general quantum Laurent polynomials are naturally related to group rings of some soluble groups.

**Example 1.6.** Let a soluble group $W$ be generated by elements

$$Y = \{Y_i | 1 \leq i \leq n\}.$$

Suppose that $W$ has a normal series

$$W = W_0 > W_1 > W_2 > \ldots > W_{t+1} = 1,$$

with finitely generated free abelian factors $W_i/W_{i+1}$, such that the elements

$$\tilde{Y} = \{Y_i W_1 | \quad 1 \leq i \leq n, \quad Y_i \in W\} \subset W/W_1$$

form a basis of the free abelian group $W/W_1$. Suppose also that the elements

$$\{Y_{ij} W_2 | 1 \leq i \leq j \leq n, Y_{ij} = [Y_i, Y_j]\} \subset W_1/W_2.$$

form a basis of the free abelian group $W_1/W_2$. Let $\alpha_1, \ldots, \alpha_n$ be the inner automorphism of the group $W$ of conjugation by $Y_1, \ldots, Y_n$. Note in particular that

$$\alpha_i(Y_{pq}) \equiv Y_{pq} \pmod{W_1}.$$

Let $k$ be a field and $kW, kW_1$ group algebras of the groups $W, W_1$, respectively. According to [B, §11], the multiplicative semigroup $S = kW_1 \setminus 0$ is an Ore set in the group algebra $kW$. Hence there exists the division ring of fractions $D$ of the group ring $kW_1$, and therefore we can consider the ring $S^{-1}kW$ as a general quantum Laurent polynomial ring $A$ from (2) with $r = n$.

Some other examples of general quantum polynomials can be found in example 3.20 and in example 5.4.

In what follows we are going to study the endomorphism semigroup of $\Lambda$.

**Notation 1.7.** Denote by $\operatorname{End}\Lambda$ the semigroup of all ring endomorphisms of $\Lambda$ acting identically on $D$. Denote by $\operatorname{Aut}\Lambda$ the group of all ring automorphisms of $\Lambda$, identical on $D$, i.e., the invertible elements of the semigroup $\operatorname{End}\Lambda$.

We recall some related results on this subject.

**Proposition 1.8** ([AC]). *Let $\Lambda$ be a quantum polynomial ring in which $r = 0, n = 2$, and $\alpha_1 = \alpha_2$ are identical automorphisms of $D$. Denote in this case the corresponding ring $\Lambda$ by $A_q$. If $q \neq \pm 1$, then $\operatorname{Aut}(A_q)$ consists of the torus $(D^*)^2$ with its natural action on $DX_1 \oplus DX_2$. If $q = -1$, then $\operatorname{Aut}(A_q)$ is the semi-direct product of $(D^*)^2$ and $\langle\tau\rangle$ where*

$$\tau(X_1) = X_2, \quad \tau(X_2) = X_1.$$

**Theorem 1.9** ([AC]). *Let*

$$\Lambda = A_{q_1} \otimes \cdots \otimes A_{q_m},$$

*where $A_{q_i}$ are from Proposition 1.8. Then the automorphism group $\operatorname{Aut}\Lambda$ is the semi-direct product of the torus $(D^*)^{2m}$ and the group of permutations of the variables $X_1, \dots, X_{2m}$.*

**Theorem 1.10** ([AC]). *Let $\Lambda$ be a quantum polynomial ring over a field $D$ with identical automorphisms $\alpha_1, \dots, \alpha_n$. Suppose that $r = 0$ and $q_{ij} = q$ for all $1 \leq i \leq j \leq n$, where $q$ is not a root of unit.*

*If $n \neq 3$, then $\operatorname{Aut}(\Lambda) \simeq (D^*)^n$.*

*If $n = 3$, then $\operatorname{Aut}(\Lambda)$ is isomorphic to the semi-direct product of the additive group of the field $D$ and the multiplicative group $(D^*)^3$. The additive group of $D$ has the following action on $\Lambda$: an element $\beta \in D$ induces the automorphism*

$$X_1 \mapsto X_1, \quad X_2 \mapsto X_2 + \beta X_1 X_3, \quad X_3 \mapsto X_3,$$

*of the ring $\Lambda$.*

**Theorem 1.11** ([OP], Proposition 3.2). *Let $D$ be a field in which the automorphisms $\alpha_1, \dots, \alpha_n$ are identical. Suppose that $r = 0$ and the localized ring*

$$D_Q[X_1^{\pm 1}, \dots, X_n^{\pm 1}] = \Lambda_{X_1 \cdots X_n} = D_Q[X_1, \dots, X_n]_{X_1 \cdots X_n}$$

*is simple. If $\gamma \in \operatorname{Aut}\Lambda$, then there exist elements $\gamma_1 \dots, \gamma_n \in D^*$ and a permutation $\sigma \in S_n$ such that*

$$\gamma(X_i) = \gamma_i X_{\sigma(i)}.$$

Similar problems were considered in [KPS]. The following result is related to the previous ones. We quote it in a slighty modified way.

**Theorem 1.12** ([A2], Theorem 3.7). *Let $D$ be a field with a set of identical automorphisms $\alpha_1, \dots, \alpha_n$. Suppose that $r = 0$ and the mutiparameters*

$$q_{ij}, \quad 1 \leq i \leq j \leq n, \quad n \geq 3,$$

*are independent in the multiplicative group $D^*$. If*

$$\gamma \in \operatorname{End}\Lambda \text{ and all } \gamma(X_1), \dots, \gamma(X_n) \neq 0$$

*then*

$$\gamma \in \operatorname{Aut}\Lambda \text{ and } \operatorname{Aut}(\Lambda) = (D^*)^n.$$

**Theorem 1.13** ([A3], Ch 3.). *Let $\Lambda$ be a general quantum polynomial ring with $r = n \geq 2$. Then $\Lambda$ is a simple ring.*

Although the proof is exposed in [A3] under a slightly weaker setting we present the proof in the special case of a general quantum polynomial ring.

*Proof.* Let $I$ be a nonzero two-sided ideal in $\Lambda$. Choose in $I$ a nonzero element

$$f = \sum a_{l_1,\dots,l_n} X_1^{l_1} \cdots X_n^{l_n}, \quad l_1, \dots, l_n \geq 0, \tag{6}$$

whose leading term

$$a_{s_1,\dots,s_n} X_1^{s_1} \cdots X_n^{s_n}, \quad a_{s_1,\dots,s_n} \in D^*,$$

is minimal with respect to lexicographic order of multi-indices.

Let, say $s_1 > 0$. Then

$$\begin{aligned} X_2 f X_2^{-1} &= \sum \alpha_2(a_{l_1,\dots,l_n})(q_{21}X_1)^{l_1} \cdots (q_{2n}X_n)^{l_n} \\ &= \sum \alpha_2(a_{l_1,\dots,l_n}) q_{21}^{l_1} \cdots q_{2n}^{l_n} d_{l_1,\dots,l_n} X_1^{l_1} \cdots X_n^{l_n}, \end{aligned}$$

where $d_{l_1\dots,l_n} \in N$ (the normal subgroup from Definition 1.4). Put

$$z = \alpha_2(a_{s_1,\dots,s_n}) q_{21}^{s_1} \cdots q_{2n}^{s_n} d_{s_1,\dots,s_n} a_{s_1,\dots,s_n}^{-1} \in D^*.$$

Then $g = zf - X_2 f X_2^{-1} \in I$ and if $g \neq 0$, the leading term of $g$ is less than the leading term of $f$, which is impossible. Hence we have for each multi-index in (6),

$$z a_{l_1,\dots,l_n} = \alpha_2(a_{l_1,\dots,l_n}) q_{21}^{l_1} \cdots q_{2n}^{l_n} d_{l_1,\dots,l_n}. \tag{7}$$

Suppose that $(l_1, \dots, l_n) \leq (s_1, \dots, s_n)$ with respect to lexicographic order and $a_{l_1,\dots,l_n} \neq 0$. We obtain from (6), (7)

$$\alpha_2(a_{s_1,\dots,s_n}) q_{21}^{s_1} \cdots q_{2n}^{s_n} d_{s_1,\dots,s_n} a_{s_1,\dots,s_n}^{-1} a_{l_1,\dots,l_n} = \alpha_2(a_{l_1,\dots,l_n}) q_{21}^{l_1} \cdots q_{2n}^{l_n} d_{l_1,\dots,l_n},$$

and therefore in $D^*/N$,

$$q_{21}^{s_1} \cdots q_{2n}^{s_n} N = q_{21}^{l_1} \cdots q_{2n}^{l_n} N.$$

Since $q_{21} = q_{12}^{-1}, \quad q_{22} = 1$ and $q_{12}, q_{23}, \dots, q_{2n}$ are independent in $D^*/N$, we obtain $s_1 = l_1, s_3 = l_3, \dots, s_n = l_n$. Similarly considering the conjugation by $X_1$ we can also obtain $s_2 = l_2$. Thus $(l_1, \dots, l_n) = (s_1, \dots, s_n)$, a contradiction.

We have proved that $f$ is a monomial. However, $r = n$ and each variable $X_i$ is invertible in $\Lambda$. Then any monomial is invertible in $\Lambda$ and $I = \Lambda$. $\square$

In the next section we shall generalize Theorem 1.12 to arbitrary general quantum polynomial rings $\Lambda$, and study finite groups $G$ of automorphisms of $\Lambda$.

## 2. Endomorphisms of general quantum polynomial rings

In this section we shall assume that $\Lambda$ is a general quantum polynomial ring from (2), i.e., the images of $q_{ij}, \quad 1 \leq i \leq j \leq n$, are independent in $D^*/N$.

**Theorem 2.1.** *Suppose that $\gamma \in \operatorname{End} \Lambda$ and there exist at least three distinct indices $1 \leq i, j, t \leq n$ such that $\gamma(X_i), \gamma(X_j), \gamma(X_t) \neq 0$. Then there exist elements $\gamma_1, \dots, \gamma_n \in D$ and an integer $\epsilon = \pm 1$ such that $\gamma_1, \dots, \gamma_r \neq 0$, and*

$$\gamma(X_w) = \gamma_w X_w^{\epsilon}, \quad w = 1, \dots, n. \tag{8}$$

*If $r < n$, then $\epsilon = 1$.*

*Proof.* We shall modify the proof of Theorem 3.7 from [A2]. Consider the natural lexicographic order on the set of multi-indices $\mathbb{Z}^n$ and on the set of monomials in $X_1, \dots, X_n$. Let $\gamma \in \operatorname{End}\Lambda$ and denote by $a_i, i = 1, \dots, n$, the smallest (the leading) term of $\gamma(X_i)$ provided $\gamma(X_i) \neq 0$.

Suppose that $\gamma(X_i), \gamma(X_j) \neq 0$. Observe that the smallest (the leading) term of a product of non zero polynomials in $\Lambda$ is equal to the product of the smallest (the leading) terms of factors. Thus (4) implies

$$a_i a_j = q_{ij} a_j a_i. \tag{9}$$

Suppose that

$$a_i = \beta X_1^{l_1} \cdots X_n^{l_n}, \quad a_j = \xi X_1^{t_1} \cdots X_n^{t_n}, \text{ where } \beta, \gamma \in k^*.$$

Comparing the coefficients and using (5) we obtain

$$\beta\xi \left( \prod_{r>s} q_{rs}^{l_r t_s} \right) = \beta\xi q_{ij} \left( \prod_{r>s} q_{rs}^{t_r l_s} \right) d,$$

where $d \in N$, $N$ is from Definition 1.4. Hence in $D^*/N$ we have

$$\left( \prod_{r>s} q_{rs}^{l_r t_s} \right) \equiv q_{ij} \left( \prod_{r>s} q_{rs}^{t_r l_s} \right) \mod N. \tag{10}$$

Suppose that $i > j$. Since the images of $q_{rs}, n \geq r > s \geq 1$ in $D^*/N$ are independent (10) we have for $r \neq s$

$$l_r t_s = \delta_{ri}\delta_{sj} + t_r l_s. \tag{11}$$

Consider the matrix

$$\begin{pmatrix} l_1 & \cdots & l_j & \cdots & l_i & \cdots & l_n \\ t_1 & \cdots & t_j & \cdots & t_i & \cdots & t_n \end{pmatrix}, \quad n \geq 3.$$

Let for example $l_p \neq 0$ and $p \neq j, i$. For each index $q \neq p$ by (11) we have

$$\begin{vmatrix} l_p & l_q \\ t_p & t_q \end{vmatrix} = 0$$

and therefore $t_q = t_p l_q l_p^{-1}$. In particular

$$t_i = t_p l_i l_p^{-1}, \quad t_j = t_p l_j l_p^{-1},$$

that is

$$l_i t_j - l_j t_i = l_i t_p l_j l_p^{-1} - l_j t_p l_i l_p^{-1} = 0,$$

which contradicts (11). Thus $l_p = 0$ for all $p \neq i, j$. Similarly one can prove that $t_p = 0$ if $p \neq i, j$.

Hence

$$a_i = \beta X_i^{l_i} X_j^{l_j}, \quad a_j = \xi X_i^{t_i} X_j^{t_j}, \qquad \text{where } l_i t_j - l_j t_i = 1.$$

By the assumption there exists a third variable $X_u$ such that $\gamma(X_u) \neq 0$. The preceding argument applied to the pairs of indices $(i, u), (j, u)$ shows that

$$a_u = \delta X_u^{r_u} X_i^{r_i} = \lambda X_u^{d_u} X_j^{d_j}, \text{ where } \delta, \lambda \in k^*.$$

Finally $r_i = d_j = 0$, that is $a_u = \delta X_u^{r_u}$. Similarly

$$a_i = \beta X_i^{l_i}, \quad a_j = \xi X_j^{t_j}, \quad l_i t_j = 1,$$

and therefore $l_i = t_j = \epsilon = \pm 1$.

If we apply the corresponding argument for the leading term of $\gamma(X_i)$ we obtain a similar result with some $\epsilon' = \pm 1$ for the leading terms. Thus if either $r < n$ or $r = n$ and $\epsilon = \epsilon'$ the theorem is proved.

Let now $r = n$, $\epsilon = -1, \epsilon' = 1$. We have now to show that the least and the leading terms of $\gamma(X_i)$ coincide and therefore they are equal to $a_i$.

From above we know that if $i = 1, \dots, n$, then

$$\gamma(X_i) = \gamma_i' X_i^{-1} + \sum_s \gamma_i''(s) X_1^{m_{i1}(s)} \cdots X_n^{m_{in}(s)} \tag{12}$$

where $\gamma_i', \gamma_i''(s) \in D^*$ and the sum is taken over some multi-indices

$$(m_{i1}(s), \dots, m_{in}(s)) \in \mathbb{Z}^n$$

such that

$$(0, \dots, 0, \overset{i}{-1}, 0, \dots, 0) < (m_{i1}(s), \dots, m_{in}(s)) \leq (0, \dots, 0, \overset{i}{1}, 0, \dots, 0).$$

Thus

$$\begin{aligned} m_{i1}(s) = \cdots = m_{i,i-1}(s) = 0, \quad m_{ii}(s) = -1, 0, 1; \\ m_{i,i+1}(s) \begin{cases} > 0, \quad i < n, & \text{if } m_{ii}(s) = -1, \\ \leq 0, \quad i < n, & \text{if } m_{ii}(s) = 1. \end{cases} \end{aligned} \tag{13}$$

Pick for each index $i = 1, \dots, n$ the least monomial $\gamma_i'' X_i^{m_{ii}} \cdots X_n^{m_{in}}$ in (12). Then $\gamma(X_i)\gamma(X_j) = q_{ij}\gamma(X_j)\gamma(X_i), i < j$, implies

$$\gamma_i' X_i^{-1} \gamma_j' X_j^{-1} + \gamma_i' X_i^{-1} \gamma_j'' X_j^{m_{jj}} \cdots X_n^{m_{jn}} + \tag{14}$$
$$\gamma_i'' X_i^{m_{ii}} \cdots X_n^{m_{in}} \gamma_j'' \gamma_j' X_j^{-1} + \cdots =$$
$$q_{ij} \gamma_j' X_j^{-1} \gamma_i' X_i^{-1} + q_{ij} \gamma_j' X_j^{-1} \gamma_i'' X_i^{m_{ii}} \cdots X_n^{m_{in}} + \tag{15}$$
$$q_{ij} \gamma_j'' X_j^{m_{jj}} \cdots X_n^{m_{jn}} \gamma_i'' \gamma_i' X_i^{-1} + \cdots, \tag{16}$$

where $+\cdots$ is a sum of monomials $\delta X_1^{l_1} \cdots X_n^{l_n}, \quad \delta \in D^*$ such that

$$(l_1, \dots, l_n) > \min((0, \dots, 0, m_{ii}, \dots, m_{i,j-1}, m_{ij} - 1, m_{i,j+1}, \dots, m_{in}),$$
$$(0, \dots, 0, \overset{i}{-1}, 0, \dots, 0, m_{jj}, \dots, m_{jn})).$$

Since $\gamma_i' X_i^{-1} \gamma_j' X_j^{-1} = q_{ij} \gamma_j' X_j^{-1} \gamma_i' X_i^{-1}$ we deduce from (14) that

$$\gamma_i' X_i^{-1} \gamma_j'' X_j^{m_{jj}} \cdots X_n^{m_{jn}} + \gamma_i'' X_i^{m_{ii}} \cdots X_n^{m_{in}} \gamma_j'' \gamma_j' X_j^{-1} + \cdots$$
$$q_{ij} \gamma_j' X_j^{-1} \gamma_i'' X_i^{m_{ii}} \cdots X_n^{m_{in}} + q_{ij} \gamma_j'' X_j^{m_{jj}} \cdots X_n^{m_{jn}} \gamma_i'' \gamma_i' X_i^{-1} + \cdots. \tag{17}$$

Suppose first that

$$(0, \dots, 0, m_{ii}, \dots, m_{i,j-1}, m_{ij} - 1, m_{i,j+1}, \dots, m_{in}) \leq$$
$$(0, \dots, 0, \overset{i}{-1}, 0, \dots, 0, m_{jj}, \dots, m_{jn}).$$

Then $m_{ii} = -1$. Moreover if $i + 1 \leq j - 1$ then $m_{i,i+1} = 0$, a contradiction with (13). Thus if $i \leq j - 2$, then

$$(0, \dots, 0, m_{ii}, \dots, m_{i,j-1}, m_{ij} - 1, m_{i,j+1}, \dots, m_{in}) >$$
$$(0, \dots, 0, \overset{i}{-1}, 0, \dots, 0, m_{jj}, \dots, m_{jn}),$$

and therefore in (17)

$$\gamma_i' X_i^{-1} \gamma_j'' X_j^{m_{jj}} \cdots X_n^{m_{jn}} = q_{ij} \gamma_j'' X_j^{m_{jj}} \cdots X_n^{m_{jn}} \gamma_i'' \gamma_i' X_i^{-1}.$$

Applying (5) we obtain

$$1 = q_{ji}^{-1} \prod_{r \geq j > i} q_{ri}^{-m_{jr}}.$$

Thus

$$m_{jr} = \begin{cases} -1, & r = j, \\ 0, & r > j, \end{cases}$$

a contradiction with (13).

Thus we have proved that if $1 \leq i \leq j-2 < j \leq n$ then either $\gamma(X_i)$ or $\gamma(X_j)$ has the form (8) for $\epsilon = -1$ and for $w = i, j$. Suppose that (8) holds for some $w = 1, \ldots, n$. Then by the previous considerations the leading term of any $\gamma(X_r)$ has the form $\gamma_r X_r^{-1}$, that is (8) holds for any variable. □

**Corollary 2.2.** *If $\gamma(X_{r+1}), \ldots, \gamma(X_n) \neq 0$, in the situation of Theorem 2.1, then $\gamma$ is an automorphism of $\Lambda$. In particular any injective endomorphism of $\Lambda$ is an automorphism.*

**Remark 2.3.** The assumption in Theorem 2.1 of the existence of three variables with non-zero images is essential. For example, let $D$ be a field, $r = 0, n = 2$ and $\alpha_1, \alpha_2$ identical on $D$. Then there exists a nontrivial endomorphism of the coordinate algebra $\Lambda$ of the quantum plane, for example,

$$X_1 \mapsto X_1^2 X_2, \quad X_2 \mapsto X_1^3 X_2^2.$$

Observe also that there exist automorphisms of the ring $\Lambda$ which are not identical on $D$. In fact if $d \in D$ is a noncentral element then the automorphism of conjugation by $d$ is an automorphism of the ring $\Lambda$ which is not identical on $D$.

**Remark 2.4.** Let $S$ be the set of all $\gamma \in \operatorname{End} \Lambda$ such that $\gamma(X_i) \neq 0$ for at most two indices $i \in \{1, \ldots, n\}$. If $S$ is nonempty, then the ring $\Lambda$ is not a simple one and therefore we have, by Theorem 1.13, $n$.

We claim that $S$ is an ideal in the semigroup $\operatorname{End} \Lambda$. In fact let $\delta \in \operatorname{End} \Lambda$. Then $\delta\gamma(X_i) \neq 0$ implies $\gamma(X_i) \neq 0$. Thus $\delta\gamma \in S$. If $\delta \notin S$, then by Theorem 2.1, since $n$,

$$\delta(X_i) = \delta_i X_i, \quad \delta_i \in D \text{ for all } i.$$

Thus

$$\gamma\delta(X_i) = \gamma(\delta_i X_i) = \delta_i \gamma(X_i),$$

and therefore $\gamma\delta \in S$.

Starting from now we shall always assume the number of variables $n \geq 3$, although some of results are valid for $n = 2$.

**Theorem 2.5.** *Let $\gamma \in \operatorname{Aut} \Lambda$ be of the form* (8).
*If $\epsilon = 1$, then the elements $\gamma_1, \ldots, \gamma_n$ are central in $D$ and*

$$\gamma_i \alpha_i(\gamma_j) = \gamma_j \alpha_j(\gamma_i), \quad i, j = 1, \ldots, n. \tag{18}$$

*If $\epsilon = -1$, then*

$$\alpha_i \alpha_j(q_{ji}) \alpha_i(\gamma_j) \gamma_i = q_{ij} \alpha_j(\gamma_i) \gamma_j \tag{19}$$

$$\alpha_i^2(d) = \gamma_i d \gamma_i^{-1}. \tag{20}$$

*In particular, the elements* $\alpha_i(\gamma_i)\gamma_i^{-1}$ *are central in* $D$.

*Proof.* By definition $\gamma$ respects the defining relations (4), namely,

$$\begin{aligned}(\gamma_i X_i^\epsilon)(\gamma_j X_j^\epsilon) &= q_{ij}(\gamma_j X_j^\epsilon)(\gamma_i X_i^\epsilon);\\ (\gamma_i X_i^\epsilon)d &= \alpha_i(d)(\gamma_i X_i^\epsilon), \quad d \in D.\end{aligned}$$

This means that

$$\begin{aligned}\gamma_i\alpha_i^\epsilon(\gamma_j)X_i^\epsilon X_j^\epsilon &= q_{ij}\gamma_j\alpha_j^\epsilon(\gamma_i)X_j^\epsilon X_i^\epsilon;\\ \gamma_i\alpha_i^\epsilon(d) &= \alpha_i(d)\gamma_i, \quad d \in D.\end{aligned}$$

If $\epsilon = 1$, then

$$\gamma_i\alpha_i(\gamma_j) = q_{ij}\gamma_j\alpha_j(\gamma_i)q_{ji}; \tag{21}$$
$$\gamma_i\alpha_i(d) = \alpha_i(d)\gamma_i, \quad d \in D. \tag{22}$$

If $\epsilon = -1$, then

$$\gamma_i\alpha_i^{-1}(\gamma_j) = q_{ij}\gamma_j\alpha_j^{-1}(\gamma_i)\alpha_i^{-1}\alpha_j^{-1}(q_{ji}); \tag{23}$$
$$\gamma_i\alpha_i^{-1}(d) = \alpha_i(d)\gamma_i, \quad d \in D. \tag{24}$$

Suppose that $\epsilon = 1$. Then (22) means that each coefficient $\gamma_i$ is central, since $\alpha_i$ is an automorphisms of $D$. Moreover, from (21) one can easily deduce (18), since $q_{ij}q_{ij} = 1$ and all $\gamma_t$ are central.

Consider now the case $\epsilon = -1$. Then $r = n \geq 3$ and the ring $\Lambda$ is simple by Theorem 1.13. Hence each endomorphism has a trivial kernel. This means that each coefficient $\gamma_i \neq 0$. Applying (24) we obtain in (23)

$$\alpha_i(\gamma_j)\gamma_i = q_{ij}\alpha_j(\gamma_i)\gamma_j\gamma_i^{-1}\alpha_i(\gamma_j)^{-1}\alpha_i\alpha_j(q_{ji})\alpha_i(\gamma_j)\gamma_i,$$

or

$$1 = q_{ij}\alpha_j(\gamma_i)\gamma_j\gamma_i^{-1}\alpha_i(\gamma_j)^{-1}\alpha_i\alpha_j(q_{ji}),$$

and (19) holds. Moreover from (24) one can easily deduce (20).

Note that

$$\begin{aligned}\gamma^2(X_i) &= \gamma(\gamma_i X_i^{-1}) = \gamma_i\gamma(X_i)^{-1}\\ &= \gamma_i(\gamma_i X_i^{-1})^{-1} = \gamma_i X_i\gamma_i^{-1} = \gamma_i\alpha_i(\gamma_i)^{-1}X_i.\end{aligned}$$

Hence

$$\alpha_i(\gamma_i)\gamma_i^{-1} = [\gamma_i\alpha_i(\gamma_i)^{-1}]^{-1}$$

is central in $D$. □

**Notation 2.6.** Denote by $\operatorname{End}^+\Lambda$ the subsemigroup of all $\gamma \in \operatorname{End}\Lambda$ of the form (8) with $\epsilon = 1$. Put $\operatorname{Aut}^+\Lambda = \operatorname{End}^+\Lambda \cap \operatorname{Aut}\Lambda$.

**Corollary 2.7.** *The group* $\operatorname{Aut}^+\Lambda$ *is commutative.*

*Proof.* Let $\gamma, \delta$ be from $\operatorname{Aut}^+\Lambda$ and

$$\gamma(X_i) = \gamma_i X_i, \quad \delta(X_i) = \delta_i X_i, \quad \gamma_i, \delta_i \in D^*.$$

Then

$$(\gamma\delta)(X_i) = \gamma(\delta_i X_i) = (\delta_i\gamma_i)X_i = (\gamma_i\delta_i)X_i = (\delta\gamma)(X_i),$$

since $\gamma_i, \delta_i$ are central. □

**Corollary 2.8.** *If* $r < n$ *, then* $\operatorname{Aut}\Lambda$ *is commutative.*

*Proof.* If $n$, then $\operatorname{Aut}\Lambda = \operatorname{Aut}^+\Lambda$ by Theorem 2.1 □

**Corollary 2.9.** *Let $G$ be a subgroup in* $\operatorname{Aut}^+\Lambda$ *of order $d$ and $\gamma \in G$ of the form* (8). *Then $\gamma_i^d = 1$ for any $i$.*

**Proposition 2.10.** *Let $p = \operatorname{char} D > 0$. Then* $\operatorname{Aut}^+\Lambda$ *has no elements of order $p$.*

*Proof.* Let $\gamma \in \operatorname{Aut}^+\Lambda$ have order $p$ and $\gamma$ has the representation (8). Then $\gamma_i^p = 1$ by Corollary 2.9, for any $i$. Therefore $\gamma_i = 1$ and $\gamma$ is the identical automorphism. □

**Proposition 2.11.** *Let $r = n$. Then* $\operatorname{End}\Lambda = \operatorname{Aut}\Lambda \cup 0$. *If $\zeta \in \operatorname{Aut}\Lambda \setminus \operatorname{Aut}^+\Lambda$, then $\zeta$ has the form* (8) *with $\epsilon = -1$ and*

$$\zeta^2 \in \operatorname{Aut}^+\Lambda, \quad \zeta(\operatorname{Aut}^+\Lambda)\zeta^{-1} = \operatorname{Aut}^+\Lambda, \quad \operatorname{Aut}\Lambda = \operatorname{Aut}^+\Lambda \cup \zeta \operatorname{Aut}^+\Lambda.$$

*Proof.* Since the ring $\Lambda$ is simple by Theorem 1.13, any nonzero endomorphism has zero kernel. Thus if $\gamma \neq 0$, then each coefficient $\gamma_i \neq 0$, for any $i$. In this case $\gamma$ is an automorphism. □

**Remark 2.12.** It follows from Corollary 2.8 and Proposition 2.11 that the group $\operatorname{Aut}\Lambda$ is metabelian, i.e., it is a soluble group of a class at most 2.

**Proposition 2.13.** *Let $r = n$ and $G$ a finite subgroup in* $\operatorname{Aut}\Lambda$. *If*

$$\zeta \in G \setminus \operatorname{Aut}^+\Lambda,$$

*then* $G = (G \cap \operatorname{Aut}^+\Lambda) \cup \zeta(G \cap \operatorname{Aut}^+\Lambda)$.

**Proposition 2.14.** *Let $p = \operatorname{char} D > 2$ and $r = n$. Then* $\operatorname{Aut}\Lambda$ *has no non-identical elements of order $p$.*

*Proof.* If $\zeta \in \operatorname{Aut}\Lambda$ has order $p > 2$, then $\zeta^2 \in \operatorname{Aut}^+\Lambda$ has also order $p$, a contradiction to Proposition 2.10. □

**Corollary 2.15.** *Let $p = \operatorname{char} D > 0$ and $G$ a finite subgroup in* $\operatorname{Aut}\Lambda$ *such that $|G|$ is divisible by $p$. Then $p = 2, r = n$, and* $G \not\subseteq \operatorname{Aut}^+\Lambda$.

**Definition 2.16.** Let $\mathbb{Z}^n$ be the free addivite abelian group of a rank $n$, whose elements are identified with vectors $(l_1, \dots, l_n)$, $l_j \in \mathbb{Z}$. We consider the ring $\Lambda$ with the natural $\mathbb{Z}^n$-grading

$$\Lambda = \oplus_{(l_1,\dots,l_n)} \Lambda_{l_1,\dots,l_n},$$

where

$$\Lambda_{l_1,\dots,l_n} = \begin{cases} DX_1^{l_1} \cdots X_n^{l_n}, & \text{if } l_{r+1}, \dots, l_n \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

**Remark 2.17.** Let $p = \operatorname{char} D$ and

$$k = \begin{cases} \mathbb{Q}, & \text{the field of rationals, if } p = 0; \\ \mathbb{F}_p, & \text{the residue field of } \mathbb{Z} \text{ modulo } p, \text{ if } p > 0. \end{cases}$$

Let $H$ be a group $k$-algebra of a free abelian group with the basis $X_1, \dots, X_n$ with comultiplication $\Delta(X_i) = X_i \otimes X_i$ for any $i$. Thus $\Lambda$ is $kH$-comodule by

$$\rho : \Lambda \to \Lambda \otimes_k kH, \quad \rho(dX_1^{l_1} \cdots X_n^{l_n}) = dX_1^{l_1} \cdots X_n^{l_n} \otimes X_1^{l_1} \cdots X_n^{l_n}, \quad d \in D. \quad (25)$$

Then $\operatorname{Aut}^+\Lambda$ coincides with the automorphism group of $\Lambda$ as a right $kH$-comodule algebra, since these automorphisms - and only these - preserve the grading from Definition 2.16 (see [M2, example 4.1.7]).

**Remark 2.18.** For a left ideal (a subring) $I$ in $\Lambda$ the following are equivalent (see [M2]):

1. $I$ is homogeneous with respect to the grading from Definition 2.16;
2. if $f \in I$, then all monomials occuring in $f$ belong to $I$;
3. if $\rho$ is the structure map from (25) for $\Lambda$ as a right the $kH$-comodule $\Lambda$, then $I$ is a subcomodule, that is, $\rho(I) \subseteq I \otimes kH$.
4. $I$ is generated as a left ideal (as a ring) by some monomials.

**Corollary 2.19.** *Let $G$ be a subgroup of* $\operatorname{Aut}^+ \Lambda$. *If $I$ is a homogeneous left ideal with respect to the grading from Definition 2.16, then $I$ is $G$-invariant.*

**Definition 2.20.** An automorphism $\gamma \in \operatorname{Aut} \Lambda$ is *inner* if there exists an invertible element $u \in \Lambda$ such that $\gamma(x) = uxu^{-1}$ for all $x \in \Lambda$.

**Remark 2.21.** Because of the grading from Definition 2.16 an element $u \in \Lambda$ is invertible if and only if

$$u = gX_1^{l_1} \cdots X_r^{l_r}, \quad g \in D^*, \quad l_1, \dots, l_r \in \mathbb{Z}. \tag{26}$$

Notice that an automorphism $\gamma \in \operatorname{Aut} \Lambda$ is inner if and only if, for some $u \in \Lambda^*$,

$$\gamma(z) = uzu^{-1}, \quad \gamma(X_i) = uX_iu^{-1}, \tag{27}$$

for every $z \in D$ and each index $i = 1, \dots, n$. In fact elements of $D$ and the variables from (3) generate the ring $\Lambda$.

**Theorem 2.22.** *Let $\gamma \in \operatorname{Aut} \Lambda$ be of the form* (8). *If $\gamma$ is inner, then for any index $i = 1, \dots, n$ and every $z \in D$ we have*

$$\gamma_i \in q_{1i}^{l_1} \cdots q_{ri}^{l_r} N, \quad \alpha_1^{l_1} \cdots \alpha_r^{l_r}(z) = g^{-1}zg, \tag{28}$$

*where $N$ is the normal subgroup of $D^*$ from Definition 1.4.*

*Proof.* Let $\gamma$ be of the form (27), where $u$ from (26). We have for each index $i = 1, \dots, n$,

$$\begin{aligned} \gamma_i X_i^{\epsilon} = uX_iu^{-1} &= gX_1^{l_1} \cdots X_r^{l_r} X_i X_r^{-l_r} \cdots X_1^{-l_1} g^{-1} \\ &= q_{1i}^{l_1} \cdots q_{ri}^{l_r} g' X_i, \quad g' \in N. \end{aligned}$$

Hence $\epsilon = 1$ and for each index $i = 1, \dots, n$ we have

$$\gamma_i \in q_{1i}^{l_1} \cdots q_{ri}^{l_r} N.$$

Moreover, if $z \in D$, then

$$\gamma(z) = gX_1^{l_1} \cdots X_r^{l_r} z X_r^{-l_r} \cdots X_1^{-l_1} g^{-1} = g\alpha_1^{l_1} \cdots \alpha_r^{l_r}(z) g^{-1} = z,$$

because $\gamma$ acts identically on $D$. ∎

**Corollary 2.23.** *Let $\gamma \in \operatorname{Aut}^+ \Lambda$ be inner of the form* (27)*, where $u$ from* (26). *Suppose that $\gamma$ has finite order. Then $u = g$ is a central element of $D$ and $\gamma_i = g\alpha_i(g)^{-1}$ is a root of 1, for every $i = 1, \dots, n$.*

*Proof.* Let $\gamma$ have order $d \geq 1$. By Corollary 2.9, $\gamma_i^d = 1$ for every $i = 1, \dots, n$. Thus by (28), $q_{1i}^{dl_1} \cdots q_{ri}^{dl_r} \in N$. From Definition 1.4 we know that in this case $l_1 = \ldots = l_r = 0$ and therefore $u = g \in D^*$.

Also from (28) we have $z = g^{-1}zg$, for any $z \in D$, that is the element $g$ belongs to the center of $D$. Now

$$\gamma(X_i) = gX_ig^{-1} = g\alpha_i(g)^{-1}X_i,$$

and so $\gamma_i = g\alpha_i(g)^{-1}$. □

**Corollary 2.24.** *Suppose that the automorphisms $\alpha_1, \dots, \alpha_n$ act identically on the center of $D$, and let $G$ be a finite subgroup in $\operatorname{Aut}\Lambda$. Then any inner automorphism in $G$ is identical.*

*Proof.* Let $\gamma \in G$ be inner. By Theorem 2.22, $\gamma$ is of the form (8) with $\epsilon = 1$ and by Corollary 2.23,

$$\gamma_i = g\alpha_i(g)^{-1}, \quad i = 1, \dots, n,$$

where $g$ is a central element of $D$. By the assumption $\alpha_i(g) = g$ and therefore $\gamma_i = 1$, for any $i = 1, \dots, n$. □

## 3. Invariants of groups of automorphisms

In this section we study invariants of various subgroups $G$ of $\operatorname{Aut}\Lambda$, where $\Lambda$ is a general quantum polynomial ring as in the previous section.

**Notation 3.1.** If $G$ is a subgroup of $\operatorname{Aut}\Lambda$ then by $\Lambda^G$ we denote the subring of all elements $a \in \Lambda$ which are stable under the action of any element $g \in G$, that is $g(a) = a$ for each $g \in G$.

**Proposition 3.2.** *Let $f \in \Lambda \setminus 0$ and $\gamma \in \operatorname{Aut}\Lambda$ have the form (8) with $\epsilon = 1$. If $\gamma(f) = f$, then $\gamma(g) = g$ for every monomial $g$ occuring in $f$.*

*Proof.* Let $\gamma$ be of the form (8) with $\epsilon = 1$ and

$$f = \sum \beta_{l_1,\dots,l_n} X_1^{l_1} \cdots X_n^{l_n}, \quad \beta_{l_1,\dots,l_n} \in D. \tag{29}$$

Then

$$\begin{aligned} \gamma(f) &= \sum \beta_{l_1,\dots,l_n} (\gamma_1 X_1)^{l_1} \cdots (\gamma_n X_n)^{l_n} \\ &= \sum \beta_{l_1,\dots,l_n} \gamma_{l_1,\dots,l_n} X_1^{l_1} \cdots X_n^{l_n}, \quad \gamma_{l_1,\dots,l_n} \in D^*. \end{aligned}$$

Since $\gamma(f) = f$, we have

$$\begin{aligned} \beta_{l_1,\dots,l_n} X_1^{l_1} \cdots X_n^{l_n} &= \beta_{l_1,\dots,l_n} \gamma_{l_1,\dots,l_n} X_1^{l_1} \cdots X_n^{l_n} \\ &= \gamma(\beta_{l_1,\dots,l_n} X_1^{l_1} \cdots X_n^{l_n}). \end{aligned}$$

□

**Proposition 3.3.** *Let $\gamma \in \operatorname{Aut}\Lambda \setminus \operatorname{Aut}^+\Lambda$, $\quad r = n$, and $f \in \Lambda \setminus 0$. Then the following are equivalent:*

1. $\gamma(f) = f$;
2. $$f = \sum_{l_1,\dots,l_n \in \mathbb{Z}} (\beta'_{l_1,\dots,l_n} X_1^{l_1} \cdots X_n^{l_n} + \beta''_{l_1,\dots,l_n} X_1^{-l_1} \cdots X_n^{-l_n}),$$

   *where*

$$\begin{aligned} \gamma(\beta'_{l_1,\dots,l_n} X_1^{l_1} \cdots X_n^{l_n}) &= \beta''_{l_1,\dots,l_n} X_1^{-l_1} \cdots X_n^{-l_n}, \\ \gamma(\beta''_{l_1,\dots,l_n} X_1^{-l_1} \cdots X_n^{-l_n}) &= \beta'_{l_1,\dots,l_n} X_1^{l_1} \cdots X_n^{l_n} \end{aligned}$$

*Proof.* We need only to prove that 1) implies 2). Let $f$ be from 1) and suppose that $f$ has representation (29). Then

$$\begin{aligned} f \;=\; \gamma(f) &= \sum \beta_{l_1,\dots,l_n}(\gamma_1 X_1^{-1})^{l_1}\cdots(\gamma_n X_n^{-1})^{l_n} \\ &= \sum \hat{\beta}_{l_1,\dots,l_n} X_1^{-l_1}\cdots X_n^{-l_n}, \quad \hat{\beta}_{l_1,\dots,l_n}\in D. \end{aligned}$$

Now the proof follows. □

**Proposition 3.4.** *Let $\gamma \in \mathrm{Aut}^+\Lambda$ have finite order $d$. Then*

$$\gamma(X_i^{\phi(d)d}) = X_i^{\phi(d)d},$$

*where $\phi$ is the Euler function.*

*Proof.* Let

$$\gamma(X_i) = \gamma_i X_i, \quad \gamma_i \in D.$$

From Theorem 2.5 and Corollary 2.9 we know that $\gamma_i$ is a central element of $D$ and $\gamma_i^d = 1$.

Let $\gamma_i$ be a primitive root of one with degree $m$ dividing $d$. Then for any $j \in \mathbb{Z}$ the element $\alpha_i^j(\gamma_i)$ is again a primitive root of one with degree $m$. The set $T$ of these roots consists of all elements $\{\gamma_i^t | (t,m) = 1\}$ and $|T| = \phi(m)$. Put

$$\alpha_i(\gamma_i) = \gamma_i^r, \quad (r,m) = 1, \quad 1 \le m.$$

The cyclic group generated by the automorphism $\alpha_i$ acts on $T$ and the orbits of this action have length $t$, where $t$ is the smallest positive integer such that $\gamma_i = \alpha_i^t(\gamma_i)$. Thus $\phi(m) = tw, w \in \mathbb{N}$. The number $t$ is equal to the minimal positive integer such that $m|(r^t - 1)$.

**Claim.** $\phi(m)|\phi(d)$.

*Proof.* Consider the prime decompositions of $m$ and $d$,

$$m = p_1^{l_1}\cdots p_k^{l_k}, \quad d = p_1^{v_1}\cdots p_k^{v_k}, \quad 0 \le l_i \le v_i.$$

Suppose that $l_1, \dots, l_j > 0$ and $l_{j+1} = \cdots = l_k = 0$. Then

$$\begin{aligned} \phi(m) &= p_1^{l_1-1}\cdots p_j^{l_j-1}(p_1-1)\cdots(p_j-1), \\ \phi(d) &= p_1^{v_1-1}\cdots p_k^{v_k-1}(p_1-1)\cdots(p_k-1), \end{aligned}$$

and the proof follows. □

Now we are able to complete the proof of the Proposition. We have

$$\gamma(X_i^{\phi(d)d}) = (\gamma_i X_i)^{\phi(d)d} = \gamma_i\alpha_i(\gamma_i)\cdots\alpha_i^{\phi(d)d-1}(\gamma_i)X_i^{\phi(d)d}$$

and

$$\gamma_i\alpha_i(\gamma_i)\cdots\alpha_i^{\phi(d)d-1}(\gamma_i) = \gamma_i^M,$$

where

$$M = 1 + r + \cdots + r^{\phi(d)d-1}. \tag{30}$$

By the Claim we have $\phi(d) = \phi(m)h$, $h \in \mathbb{N}$, and $\phi(m) = tw$, $m|(r^t - 1)$. Thus $\phi(d)d = twhd$, and in (30) we can write

$$M = (1 + r + \cdots + r^{t-1})(1 + r^t + \cdots + r^{t(whd-1)}).$$

But $\gamma_i^{r^t} = \gamma_i$ and $\gamma_i^d = 1$. Thus

$$\gamma_i^M = \gamma_i^{(1+r^t+\cdots+r^{t(whd-1)})(1+r+\cdots+r^{t-1})} = \gamma_i^{whd(1+r+\cdots+r^{t-1})} = 1.$$

□

**Corollary 3.5.** *Let $G$ be a finite subgroup of* $\operatorname{Aut}^+ \Lambda$ *of order $d$. Then $\Lambda^G$ contains the subring $\Phi$ generated by $D$ and by all elements $X_i^{\phi(d)d}$, $i = 1, \dots, n$. In particular, $\Lambda$ is a finitely generated left and right $\Phi$-module.*

**Corollary 3.6.** *Let $G$ be a finite subgroup of* $\operatorname{Aut}^+ \Lambda$. *Then $\Lambda^G$ is left and right Noetherian.*

*Proof.* By Proposition 3.4 we know that $\Lambda^G \supseteq \Phi$, where $\Phi$ is from Corollary 3.5. The ring $\Phi$ is again a quantum polynomial ring with variables $X_i^{\phi(d)d}$, $i = 1, \dots, n$, and their inverses if $1 \le i \le r$. Hence $\Phi$ is left and right Noetherian by Proposition 1.3.

The ring $\Lambda$ is a finitely generated free $\Phi$-module. In fact the monomials

$$X_1^{l_1} \cdots X_n^{l_n}, \quad 0 \le l_i \le \phi(d)d \text{ for every } i,$$

form a basis of $\Lambda$ as a free finitely generated $\Phi$-module. Thus $\Lambda^G$ is a finitely generated left and right $\Phi$-module too, and therefore it is left and right Noetherian. □

The following Corollary is related to [P], Proposition 5, p. 60 and Proposition 5, p. 68.

**Corollary 3.7.** *Let $I$ be a nonzero left ideal in $\Lambda$ and $G, \Phi$ from Corollary 3.5. Then $I \cap \Phi \neq 0$.*

*Proof.* As already mentioned above the ring $\Phi$ is a quantum polynomial ring over $D$ with the variables $X_i^{\phi(d)d}$ and their inverses if $1 \le i \le r$. Thus the ring $\Phi$ is left and right Noetherian. The ring $\Lambda$ is a finitely generated $\Phi$-module and therefore it is a Noetherian $\Phi$-module.

Let $f \in I \setminus 0$. Put

$$M_j = \sum_{s=0}^{j} \Phi f^s.$$

Thus we obtain in $\Lambda$ an ascending chain of $\Phi$-submodules

$$M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots .$$

Then there exists an integer $k > 1$ such that $M_{k-1} = M_k$. It means that $f^k \in M_{k-1}$ and therefore there exist elements $a_0, \dots, a_{k-1} \in \Phi$ such that

$$f^k = \sum_{j=0}^{k-1} a_j f^j.$$

Since $\Lambda$ is a domain we can always assume that $a_0 \neq 0$. Then

$$a_0 = f^k - \sum_{j=1}^{k-1} a_j f^j \in \Phi \cap \Lambda f \subseteq \Phi \cap I.$$

□

**Remark 3.8.** Up to now we considered finite subgroups of ring automorphisms $G$ which are contained in $\operatorname{Aut}^+ \Lambda$. The next step is to study subgroups $G \subseteq \operatorname{Aut} \Lambda$ which are not contained in $\operatorname{Aut}^+ \Lambda$. Observe that if $\operatorname{Aut} \Lambda \neq \operatorname{Aut}^+ \Lambda$, then by Theorem 2.1, $r = n$. The subgroup $\operatorname{Aut}^+ \Lambda$ has index 2 in $\operatorname{Aut} \Lambda$ and any element of the coset $\operatorname{Aut}\Lambda \setminus \operatorname{Aut}^+\Lambda$ has the form (8) with $\epsilon = -1$, see Proposition 2.11. Thus if $G$ is a subgroup of $\operatorname{Aut}\Lambda$ which is not in $\operatorname{Aut}^+\Lambda$, then $G$ contains an element $\zeta$ of the form (8) with $\epsilon = -1$. Moreover $G \cap \operatorname{Aut}^+\Lambda$ is a subgroup of index 2 in $G$. There are two cosets of $G$ with respect to the subgroup $G \cap \operatorname{Aut}^+\Lambda$, namely, the subgroup itself and $\zeta(G \cap \operatorname{Aut}^+\Lambda)$. In particular the order of $G$ is always even.

**Proposition 3.9.** *Let $\gamma \in \operatorname{Aut}\Lambda \setminus \operatorname{Aut}^+\Lambda$ and $r = n$. Suppose that $\gamma$ has the form (8) with $\epsilon = -1$ and*

$$f = \beta X_i^l + \delta X_i^{-l}, \quad \beta, \delta \in D^*, \quad l > 0.$$

*If $l = 2s$, then $\gamma(f) = f$ if and only if*

$$\delta = \beta\gamma_i^s\alpha_i(\gamma_i)^s.$$

*If $l = 2s+1$, then $\gamma(f) = f$ if and only if*

$$\gamma_i = \alpha_i(\gamma_i), \quad \delta = \beta\gamma_i^l.$$

*Proof.* Let $\gamma(f) = f$. By Proposition 3.3 we have

$$\beta(\gamma(X_i))^l = \delta X_i^{-l}, \quad \delta(\gamma(X_i))^{-l} = \beta X_i^l.$$

Then

$$\begin{aligned}
\beta(\gamma(X_i))^l &= \beta(\gamma_i X_i^{-1})^l = \beta\gamma_i\alpha_i^{-1}(\gamma_i)\cdots\alpha_i^{-(l-1)}(\gamma_i)X_i^{-l};\\
\delta(\gamma(X_i))^{-l} &= \delta(\gamma_i X_i^{-1})^{-l} = \delta(X_i\gamma_i^{-1})^l = \delta\alpha_i(\gamma_i)^{-1}\cdots\alpha_i^l(\gamma_i)^{-1}X_i^l.
\end{aligned}$$

Thus

$$\beta = \delta\alpha_i(\gamma_i)^{-1}\cdots\alpha_i^l(\gamma_i)^{-1}, \quad \delta = \beta\gamma_i\alpha_i^{-1}(\gamma_i)\cdots\alpha_i^{-(l-1)}(\gamma_i),$$

and therefore

$$\alpha_i^l(\gamma_i)\cdots\alpha_i(\gamma_i) = \gamma_i\alpha_i^{-1}(\gamma_i)\cdots\alpha_i^{-(l-1)}(\gamma_i). \tag{31}$$

From (20) we have $\alpha_i^{2k}(\gamma_i) = \gamma_i$ for any integers $k$. Hence $\alpha_i^{-k}(\gamma_i) = \alpha_i^k(\gamma_i)$, for any $k$. Moreover since $\alpha_i(\gamma_i)\gamma_i^{-1}$ is central in $D$, the elements $\alpha_i(\gamma_i)$ and $\gamma_i$ commute. Thus in (31) we have

$$\alpha_i^l(\gamma_i)\cdots\alpha_i(\gamma_i) = \gamma_i\alpha_i(\gamma_i)\cdots\alpha_i^{(l-1)}(\gamma_i),$$

or $\gamma_i = \alpha_i^l(\gamma_i)$. This means that if $l = 2s+1$, then $\gamma_i = \alpha_i(\gamma_i)$, and the assertion follows.

If $l = 2s$, then $\delta = \beta(\gamma_i\alpha_i(\gamma_i))^s$. □

**Theorem 3.10.** *Let $G$ be a finite subgroup of order $2d$ in $\operatorname{Aut}\Lambda$ and $G \not\subseteq \operatorname{Aut}^+\Lambda$. As noticed in Remark 3.8, there exists an element*

$$\zeta \in G \setminus \operatorname{Aut}^+\Lambda, \text{ such that } \zeta(X_i) = \zeta_i X_i^{-1}, \quad \zeta_i \in D^* \text{ for all } i = 1, \dots, n. \tag{32}$$

*Then*

$$X_i^{2\phi(d)d} + [\zeta_i\alpha_i(\zeta_i)]^{2\phi(d)d}X_i^{2\phi(d)d} \in \Lambda^G. \tag{33}$$

*Proof.* The group $H = G \cap \operatorname{Aut}^+\Lambda$ has order $d$. Hence

$$X_i^{2\phi(d)d}, X_i^{-2\phi(d)d} \in \Lambda^H$$

by Proposition 3.4. Apply Proposition 3.9. □

**Corollary 3.11.** *Let $G$ be a finite subgroup of order $d$ in $\operatorname{Aut}\Lambda$, $G \not\subseteq \operatorname{Aut}^+\Lambda$, and $r = n$. Denote by $\Psi$ the subring in $\Lambda$, generated by $D$ and by all elements of the form (33), $i = 1, \dots, n$, where $\zeta$ is from (32). Then $\Psi \subseteq \Lambda^G$ and $\Lambda$ is a finitely generated left and right $\Psi$-module.*

To prove that the ring $\Psi$ is Noetherian we need two technical observations.

**Lemma 3.12.** *Let $\zeta_1, \dots, \zeta_n$ be from (32), and $\xi_i = \zeta_i \alpha_i(\zeta_i)$. Then $\alpha_i(\xi_i) = \xi_i$.*

*Proof.* By (20) we have

$$\alpha_i(\xi_i) = \alpha_i[\zeta_i \alpha_i(\zeta_i)] = \alpha(\zeta_i)\alpha_i^2(\zeta_i) = \alpha_i(\zeta_i)\zeta_i.$$

But by Theorem 2.5 the elements $\alpha(\zeta_i), \zeta_i$ commute. □

**Lemma 3.13.** *Let $R$ be a left Noetherian ring with an automorphism $\beta$. Consider in a skew Laurent polynomial extension $R[Y^{\pm 1}; \beta]$ a subring $\Gamma$ generated by $R$ and by an element $Z = Y + \nu Y^{-1}$ such that $\nu = \beta(\nu)$ is an invertible element of $R$. Then $\Gamma$ is left Noetherian.*

*Proof.* Observe first that

$$Z^m = Y^m + \sum_{j=2`1}^{m-1} \binom{m}{j} \nu^j Y^{m-j} Y^{-j} + \nu^m Y^{-m}.$$

This means that if $aY^m$ is the leading term of some $f \in \Gamma$, then $a\nu^m Y^{-m}$ is the smallest term of $f$.

Let $I$ be a left ideal in $\Gamma$ and let $I_m$ be the set consisting of zero and all leading coefficients of polynomials of degree $m$ in $I$. It is clear that $I_m \subseteq \beta^{-1}(I_{m+1})$ for any $m$, and each $I_m$ is a left ideal in $R$. Since $R$ is left Noetherian, the set of left ideals $\beta^i(I_j)$, $i \geq 0, j > 0$, has a maximal element, say $\beta^s(I_M)$. Then $\beta^{k+s}(I_M) = I_{M+k+s}$ for all $k \geq 0$.

Since $R$ is left Noetherian, the ideal $\beta^s(I_M)$ is finitely generated. Let $a_1, \dots, a_t$ be a set of generators of the left $R$-module $I_{M+s}$, and let $f_1, \dots, f_t \in I$ have leading coefficients $a_1, \dots, a_t$ , respectively.

If $g \in I$ has degree $M + s + k, k \geq 0$, and $b$ is the leading coefficient of $g$, then $b = \beta^k(c)$ for some $c \in I_{M+s}$. So $c = c_1 a_1 + \cdots + c_t a_t$ for some $a_i \in R$, and

$$g - Y^k(c_1 f_1 + \cdots + c_t f_t) \in I$$

has degree less then $g$. Thus finally,

$$I = \sum_{i=1}^{t} \Gamma f_i + \left[ I \cap \left( \sum_{j=-m+1}^{m} RY^j \right) \right].$$

Since $R$ is left Noetherian, the ideal $I$ is finitely generated. □

**Theorem 3.14.** *The ring $\Psi$ from Corollary 3.11 is left and right Noetherian.*

*Proof.* The ring $\Psi$ is contained in the subring $K$ of $\Lambda$ generated by $D$ and by the elements

$$Y_i = X_i^{2\phi(d)d}, \quad i = 1, \dots, n.$$

Then $K$ itself is a quantum polynomial ring with variables $Y_1^{\pm 1}, \dots, Y_n^{\pm 1}$. Thus we have to prove that in

$$K = D_{Q', \alpha'}[Y_1^{\pm 1}, \dots, Y_n^{\pm 1}]$$

for the corresponding $Q', \alpha'$, the subring $\Psi$ generated by $D$ and by

$$Z_i = Y_i + \nu_i Y_i^{-1}, \quad i = 1, \dots, n, \quad \nu_i = \xi_i^{2\phi(d)d},$$

is left and right Noetherian. Observe that by Lemma 3.12, $Y_i \nu_i = \nu_i Y_i$.

We shall proceed by induction on the number of variables $n$. The case $n = 0$ is trivial, since in this case $\Psi = D$.

In order to apply induction we refer to Lemma 3.13. Now the proof of Theorem follows. ☐

With the same proof as for Corollary 3.6, Corollary 3.7 we obtain

**Corollary 3.15.** *Let $G$ be a finite subgroup of $\operatorname{Aut} \Lambda$. Then $\Lambda^G$ is left and right Noetherian.*

**Corollary 3.16.** *Let $I$ be a nonzero left ideal in $\Lambda$ and $G, \Psi$ from Corollary 3.11. Then $I \cap \Psi \neq 0$.*

The next theorem and its corollaries contain in some sense a converse statement to Corollary 3.7, Theorem 3.14, Corollary 3.15.

**Theorem 3.17.** *Let $\Lambda$ be a quantum polynomial ring, not necessarily a general one, with the grading from Definition 2.16. Let $R$ be a homogeneous subring in $\Lambda$, containing $D$. If $\Lambda$ is a finitely generated left $R$-module, then there exists a positive integer $m$ such that*

$$X_1^{\pm m}, \dots, X_r^{\pm m}, X_{r+1}^m, \dots, X_n^m \in R.$$

*Proof.* Let $U$ be the set of all $n$-tuples

$$(l_1, \dots, l_n) \in \mathbb{Z}^r \times (\mathbb{N} \cup 0)^{n-r} \text{ such that } X_1^{l_1} \cdots X_n^{l_n} \in R.$$

Since $R$ is a ring, the set $U$ is a subsemigroup in the additive semigroup

$$\mathbb{Z}^r \times (\mathbb{N} \cup 0)^{n-r}.$$

By assumption there exist elements $f_1, \dots, f_s \in \Lambda$ such that

$$\Lambda = Rf_1 + \cdots + Rf_s. \tag{34}$$

Without loss of generality we can assume that $f_1, \dots, f_s$ are monomials with coefficient 1. If $g \in \Lambda$ is a monomial, then by (34) it can be represented in the form

$$g = \sum_{i=1}^{s} \sum_j u_{ji} f_i$$

for some monomials $u_{ji}$. But $g$ is itself a monomial. Hence $g = uf_i$ for some $f_i$ and for some monomial $u \in R$. It we apply (5) and look at the set of corresponding multi-indices, we can observe that if $G$ is the multi-index of $g$ then $G \in F_i + U$, where $F_i$ is the multi-index of $f_i$. Thus

$$\mathbb{Z}^r \times (\mathbb{N} \cup 0)^{n-r} = (U + F_1) \cup \cdots \cup (U + F_s). \tag{35}$$

In particular for any index $i = 1, \dots, s$ there exists an index $i' = 1, \dots, s$ such that

$$2F_i = F_{i'} + u(i), \quad u(i) \in U. \tag{36}$$

By induction we set $i(0) = i, i(k+1) = i(k)'$.

**Claim.** *For any $k \geq 1$ we have*

$$2^k F_{i(m)} = F_{i(k+m)} + w, \quad w = w(i, m, k) \in U.$$

*Proof.* By (36) the affirmation is valid for $k = 1$. If it holds for some $k$ then by (36) we have

$$2F_{i(k+m)} = F_{i(k+m+1)} + u, \quad u \in U.$$

Then by induction

$$2^{k+1} F_{i(k+m)} = 2F_{i(k+m)} + 2w = F_{i(k+m+1)} + u + 2w, \quad u + 2w \in U.$$

□

We now start with an arbitrary index $i = 1, \dots, s$, and consider the sequence $i = i(0), i(1), \dots$. Then $i(t) = i(r)$ for some $t \leq r$. By the Claim we have

$$2^{r-t} F_{i(t)} = F_{i(r)} + w, \quad w \in U.$$

It follows that $(2^{r-t} - 1)F_{i(t)} \in U$. Then by the Claim,

$$(2^{r-t} - 1)2^t F_i = (2^{r-t} - 1)(F_{i(t)} + u) = (2^{r-t} - 1)F_{i(t)} + (2^{r-t} - 1)u \in U.$$

If we take now

$$e_j = (0, \dots, 0, \overset{j}{1}, 0, \dots, 0) \in \mathbb{Z}^r \times (\mathbb{N} \cup 0)^{n-r},$$

then $e_j = F_i + v_{ij}$, for some $v_{ij} \in U$. By the preceding considerations $me_j \in U$ for some $m > 0$. Similarly, if $1 \leq i \leq r$, then $-me_i \in U$ for some $m > 0$. □

Starting from now we shall again assume that $\Lambda$ is a general quantum polynomial ring.

**Corollary 3.18.** *Let $G$ be a subgroup in $\operatorname{Aut}^+ \Lambda$ such that $\Lambda$ is a finitely generated $\Lambda^G$-module. Then there exists an integer $m > 0$ such that*

$$X_1^{\pm m}, \dots, X_r^{\pm m}, X_{r+1}^m, \dots, X_n^m \in \Lambda^G.$$

*Proof.* By Proposition 3.2, $\Lambda^G$ is a homogeneous subring in $\Lambda$ contaning $D$. Apply Theorem 3.17. □

**Theorem 3.19.** *Let $G$ be a subgroup in $\operatorname{Aut} \Lambda$, $G \nsubseteq \operatorname{Aut}^+ \Lambda$ and $r = n$. Suppose that $\Lambda$ is a finitely generated left $\Lambda^G$-module. Then there exist an integer $m > 0$ and elements $\tau_1, \dots, \tau_n \in D^*$ such that*

$$X_1^m + \tau_1 X_1^{-m}, \dots, X_n^m + \tau_n X_n^{-m} \in \Lambda^G.$$

*Proof.* Let $H = G \cap \operatorname{Aut}^+ \Lambda$. Then $\Lambda^H \supseteq \Lambda^G$ and therefore $\Lambda$ is a finitely generated left $\Lambda^H$-module. By Corollary 3.18 there exists a positive integer $m$ such that

$$X_1^{\pm m}, \dots, X_r^{\pm m}, X_{r+1}^m, \dots, X_n^m \in \Lambda^H.$$

Let $\zeta \in G$ be of the form (32). Then

$$\zeta(X_i^m) = \tau_i X_i^{-m}, \quad \tau_i \in D^*.$$

But $\zeta^2 \in H$, so $X_i^m = \zeta(\tau_i X_i^{-m})$ and therefore

$$X_i^m + \tau_i X_i^{-m} \in \Lambda^G.$$

□

**Example 3.20.** Let $D = \mathbb{C}$ be the field of complex numbers with automorphisms $\alpha_1(z) = \cdots = \alpha_n(z) = \overline{z}$, the complex conjugation. Suppose that $q_{ij} \in \mathbb{N}, \quad 1 \le i \le j \le n$, are different primes and $\Lambda$ is the corresponding quantum polynomial ring from (2). Observe that the equations (1) are satisfied and the subgroup $N$ in $\mathbb{C}^*$ from Definition 1.4 is generated by all complex numbers of the form $z^{-1}\overline{z}$, i.e. it consists of all complex numbers with absolute value 1. It means that $q_{ij}, \quad 1 \le i \le j \le n$, are indepedent in $\mathbb{C}^*/N$ and $\Lambda$ is a general quantum polynomial ring.

Let $\xi \in \mathbb{C}^*, \quad |\xi| = 1$ and $\xi \neq 1$. Denote by $\zeta$ the automorphims of $\Lambda$ such that $\zeta(X_i) = \xi X_i \quad i = 1, \ldots, n$, and $G = \langle \zeta \rangle$. Then we have, for any $i, j = 1, \ldots, n$ and $\epsilon_i, \epsilon_j = \pm 1$.

$$\zeta(X_i^{\epsilon_i} X_j^{\epsilon_j}) = \xi\overline{\xi} X_i^{\epsilon_i} X_j^{\epsilon_j} = X_i^{\epsilon_i} X_j^{\epsilon_j}.$$

Hence $\Lambda^G$ contains the subring $\Gamma$, generated by

$$X_i^{\epsilon_i} X_j^{\epsilon_j}, \quad 1 \le i, j \le n, \quad \epsilon_i, \epsilon_j = \pm 1.$$

**Claim.** $\Gamma = \Lambda^G$.

*Proof.* If $f \in \Lambda^G$ then $f$ has a unique representation of the form

$$f = a_0 + \sum_{i=1}^{n} a_i X_i, \quad a_i \in \Gamma.$$

Then

$$f = \zeta(f) = a_0 + \sum_{i=1}^{n} a_i \xi X_i.$$

Hence $a_i \xi = a_i$ and $a_i = 0$ for each $i = 1, \ldots, n$, which means that $f \in \Gamma$. □

This example shows that taking different $\xi \in \mathbb{C}^* \setminus 1$ with absolute value 1 we obtain the same $G$-invariant subring $\Lambda^G$. If $\xi$ is not a root of 1, then $G$ is infinite. However $\Lambda$ is obviously a finitely generated $\Lambda^G$-module and $\Lambda^G$ is left and right Noetherian.

**Example 3.21.** Let $D, \alpha_1, \ldots, \alpha_n, q_{ij}, \Lambda$ be from example 3.20 and $\lambda \in \mathbb{C}^*$, with $|\lambda| > 1$. Suppose that $r = 0$ and $\gamma$ is automorphism of $\Lambda$ such that $\gamma(X_i) = \lambda X_i, \quad i = 1, \ldots, n$. Then for any monomial

$$X_1^{l_1} \cdots X_n^{l_n}, \quad l_1 + \cdots + l_n > 0,$$

we have

$$\gamma(X_1^{l_1} \cdots X_n^{l_n}) = \lambda^s \overline{\lambda}^t X_1^{l_1} \cdots X_n^{l_n}, \quad s + t = l_1 + \cdots + l_n > 0.$$

This means by Proposition 3.2 that $\Lambda^G = D$ and $\Lambda$ is not a finitely generated left $\Lambda^G$-module.

## 4. The trace map

As before we assume in this section that $\Lambda$ is a general quantum polynomial ring. Let $G$ be a finite subgroup of $\operatorname{Aut}\Lambda$. Consider the trace map (e. g. [W2], p. 309)

$$\operatorname{tr}_G : \Lambda \to \Lambda^G, \quad a \mapsto \sum_{\gamma \in G} \gamma(a).$$

**Theorem 4.1.** *Let $a$ be a monomial in $\Lambda$ and $G$ a finite subgroup in $\operatorname{Aut}^+ \Lambda$. If $a \notin \Lambda^G$, then $\operatorname{tr}_G a = 0$. If $a \in \Lambda^G$, then $\operatorname{tr}_G a = |G| a$.*

*Proof.* Let
$$a = X_1^{l_1} \cdots X_n^{l_n} \notin \Lambda^G.$$
Then there exists $\gamma \in G$ such that $\gamma(a) \neq a$. It suffices to prove the affirmation in the case when $G = \langle \gamma \rangle$ is a cyclic group of order $d > 1$. Let $\gamma$ be of the form (8) with $\epsilon = 1$. Then
$$\gamma(a) = (\gamma_1 X_1)^{l_1} \cdots (\gamma_n X_n)^{l_n} = \tau X_1^{l_1} \cdots X_n^{l_n},$$
where
$$\tau = \prod_{i=1}^{n} \left( \alpha_1^{l_1} \cdots \alpha_{i-1}^{l_{i-1}} [\gamma_n \alpha_n(\gamma_n) \cdots \alpha_n^{l_n - 1}(\gamma_n)] \right)$$
is a central element of $D$. Observe that $\gamma^s(a) = \tau^s a$ for any integer $s$. Thus $\tau^d = 1$ since $\gamma^d = 1$. If $a \notin \Lambda^G$, then $\tau \neq 1$ and therefore
$$1 + \tau + \cdots + \tau^{d-1} = 0.$$
It follows that
$$\operatorname{tr}_G a = (1 + \tau + \cdots + \tau^{d-1}) a = 0.$$
□

**Corollary 4.2.** *If $G$ is a finite subgroup in $\operatorname{Aut}^+ \Lambda$, then $\operatorname{tr}_G \Lambda = \Lambda^G$.*

*Proof.* Apply Theorem 4.1 and Proposition 2.10. □

**Definition 4.3.** Let $G$ be a subgroup in $\operatorname{Aut} \Lambda$. Then the group $G$ acts on $\Lambda$. Denote by $\Lambda' G$ the corresponding *skew group ring*. This is a free left $\Lambda$-module with the basis $\{g | g \in G\}$ and multiplication (e. g. [W2, §37]).
$$(ag)(bh) = ag(b)gh, \quad a, b \in \Lambda, \quad g, h \in G.$$

**Corollary 4.4.** *Let $G$ be a finite subgroup in $\operatorname{Aut}^+ \Lambda$. Then the ring $\Lambda$ is a projective left $\Lambda' G$-module.*

*Proof.* By Proposition 2.10 the order of $G$ is invertible in $D$, and by [W2], 39.17, $\Lambda$ is a projective left $\Lambda' G$-module. □

**Theorem 4.5.** *Let $G$ be a finite subgroup in $\operatorname{Aut} \Lambda$ not contained in $\operatorname{Aut}^+ \Lambda$ and a monomial*
$$a = X_1^{l_1} \cdots X_n^{l_n}, \quad l_j \in \mathbb{Z}. \tag{37}$$
*Put $H = G \cap \operatorname{Aut}^+ \Lambda$. If $a \notin \Lambda \setminus \Lambda^H$, then $\operatorname{tr}_G a = 0$.*
*If $a \in \Lambda^H$ and $\zeta \in G \setminus \operatorname{Aut}^+ \Lambda$ from Remark 3.8, then*
$$\begin{aligned} \operatorname{tr}_G a &= |H|(a + \zeta(a)), \\ \zeta(a) &= \tau X_1^{-l_1} \cdots X_n^{-l_n}, \quad \tau \in D^*. \end{aligned}$$

*Proof.* If $a \notin \Lambda^H$, then by Theorem 4.1 we have $\operatorname{tr}_H a = \operatorname{tr}_G a = 0$.
Let $a \in \Lambda^H$. Then $G = H \cup \zeta H$. Thus $\operatorname{tr}_G a = |H|(a + \zeta a)$ and
$$\zeta(a) = (\zeta_1 X_1^{-1})^{l_1} \cdots (\zeta_n X_n^{-1})^{l_n} = \tau X_1^{-l_1} \cdots X_n^{-l_n}, \quad \tau \in D^*.$$
□

**Corollary 4.6.** *Let $G$ be a finite subgroup in $\operatorname{Aut} \Lambda$, not contained in $\operatorname{Aut}^+ \Lambda$. Then the following are equivalent:*

1. $\operatorname{tr}_G \Lambda = \Lambda^G$;

2. $1 \in \operatorname{tr}_G \Lambda$;
3. $\operatorname{char} D \neq 2$.

*Proof.* If $a$ is a monomial from (37) and $(l_1, \ldots, l_n) \neq (0, \ldots, 0)$, then $\operatorname{tr}_G a = |H|(a + \zeta(a))$. By Proposition 2.10, we have $(|H|, \operatorname{char} D) = 1$ and therefore

$$a + \zeta(a) = \frac{\operatorname{tr}_G a}{|H|} \in \operatorname{tr}_G \Lambda.$$

But if $a = 1$ then $\operatorname{tr}_G a = |G|a$. Thus by Corollary 2.15

$$1 \in \operatorname{tr}_G \Lambda \iff (|G|, \operatorname{char} D) = 1 \iff \operatorname{char} D \neq 2.$$

□

**Corollary 4.7.** *Let $G$ be a finite subgroup in* $\operatorname{Aut} \Lambda$, *not contained in* $\operatorname{Aut}^+ \Lambda$. *Then the ring $\Lambda$ is a projective left $\Lambda' G$-module if and only if* $\operatorname{char} D \neq 2$.

*Proof.* By Corollary 4.6, $\operatorname{tr}_G \Lambda = \Lambda^G$ if and only if $\operatorname{char} D \neq 2$. By [W2], 39.17, $\Lambda$ is a projective left $\Lambda' G$-module if and only if $\operatorname{tr}_G \Lambda = \Lambda^G$. □

## 5. Homological properties

In this section, as before, we shall assume that $\Lambda$ is a general quantum polynomial ring. If $r = n$, $\Lambda$ is a simple ring and hence by [M1, Theorem 2.4], for any outer automorphism group $G$, $\Lambda$ is a generator for the $\Lambda' G$-modules (see also [W2, 40.7]). The next theorem characterizes the projectivity of $\Lambda$ in this case.

**Theorem 5.1.** *Let $r = n$ and $G$ be a finite subgroup in* $\operatorname{Aut} \Lambda$ *of outer automorphisms, i.e., any inner automorphism in $G$ is identical. Then the following are equivalent:*

1. *$\Lambda$ projective (and a generator) in the category of left $\Lambda' G$-modules;*
2. *$\Lambda$ is a generator in the category of right $\Lambda^G$-modules;*
3. *$\Lambda^G$ is a simple ring;*
4. *$\Lambda' G$ and $\Lambda^G$ are Morita-equivalent (by* $\operatorname{Hom}_{\Lambda' G}(\Lambda, -)$*);*
5. *$G \subseteq \operatorname{Aut}^+ \Lambda$ or* $\operatorname{char} D \neq 2$.

*Proof.* By Theorem 1.13, the ring $\Lambda$ is simple in the case $r = n$. Apply [W2], 40.8, and Corollary 4.4, Corollary 4.7. See also [M1, Theorem 2.5]. □

**Corollary 5.2.** *Let $r = n$ and $\alpha_1, \ldots, \alpha_n$ act identically on center of $D$. Suppose that $G$ is a finite subgroup of* $\operatorname{Aut} \Lambda$. *The the following are equivalent:*

1. *$\Lambda$ is projective (and a generator) in the category of left $\Lambda' G$-modules;*
2. *$\Lambda$ is a generator in the category of right $\Lambda^G$-modules;*
3. *$\Lambda^G$ is a simple ring;*
4. *$\Lambda' G$ and $\Lambda^G$ are Morita-equivalent (by* $\operatorname{Hom}_{\Lambda' G}(\Lambda, -)$*);*
5. *$G \subseteq \operatorname{Aut}^+ \Lambda$ or* $\operatorname{char} D \neq 2$.

*Proof.* By Corollary 2.24 any inner automorphism in $G$ is identical. So we have only to apply Theorem 5.1. □

**Remark 5.3.** It follows from condition 1 in Theorem 5.1 that $\Lambda' G \simeq \operatorname{End}_{\Lambda^G} \Lambda$ (see [M1, Theorem 2.4]).

**Example 5.4.** If $n$, $\Lambda$ may not be a (self-) generator as a left $\Lambda'G$-module.

In fact, let $0 \leq n$ and $D = \mathbb{C}$ be the field of complex numbers. Suppose that $q_{ij}$, $1 \leq i \leq j \leq n$, are different primes and $\alpha_1 = \cdots = \alpha_n$ are identical automorphisms of $\mathbb{C}$. It is routine to check that the equations (1) are satisfied.

Denote by $\Lambda$ the corresponding quantum polynomial ring (2). Put $\xi = \exp(\dfrac{2\pi}{10}i)$ and denote by $\gamma$ the automorphism of $\Lambda$ such that

$$\gamma(X_i) = \begin{cases} X_i, & \text{if } i \leq n; \\ \xi Y_n, & \text{if } i = n. \end{cases}$$

Then the homogeneous ideal $I = \Lambda X_n$ is invariant under the action of the cyclic group $G = \langle\gamma\rangle$. By Proposition 3.2, $I^G$ is spanned by all monomials

$$X_1^{l_1} \cdots X_n^{l_n}, \quad l_n > 0, \quad 10|l_n.$$

However, $X_n \notin I^G$ and $I \neq \Lambda I^G$.

From [W2, 39.5], it follows that $\Lambda$ is not a self-generator as a left $\Lambda'G$-module.

Now we shall turn our attention to a division ring $\Delta$ of fractions of an Ore domain $R$ and an action of ring automorphism groups $G \subseteq \operatorname{Aut} R$ on $\Delta$. These results can be applied to the case $D = \Lambda$, the quantum polynomial ring and $\Delta = F$, the division ring of fractions of $\Lambda$.

The action of group $G$ of group automorphisms of $R$ can be extended to an action on $\Delta$. It means that we can look at $\Delta$ as a left $R'G$-module, where $R'G$ is the skew group ring from Definition 4.3.

**Proposition 5.5.** *Let $I$ be a $G$-invariant left ideal of $R$ and $\phi : I \to \Delta$ an $R'G$-module homomorphism. Then $\phi$ can be extended to $R'G$-module homomorphim $\psi : R \to \Delta$.*

*Proof.* Let $a, b \in I \setminus 0$. By the Ore condition there exist elements $u, v \in R \setminus 0$ such that $ua = vb$. Then

$$u\phi(a) = v\phi(b); \tag{38}$$

$$v^{-1}u = ba^{-1} \in \Delta. \tag{39}$$

If $\phi(a) = 0$, then $v\phi(b) = 0$ by (38) and therefore $\phi(b) = 0$ since $\Delta$ is a division ring. Thus if $\phi(a) = 0$, then $\phi = 0$ and in this case we can put $\psi = 0$.

Suppose that $\phi(a) \neq 0$. Then we have by (38), (39)

$$ba^{-1} = v^{-1}u = \phi(b)\phi(a)^{-1}.$$

This means that

$$a^{-1}\phi(a) = b^{-1}\phi(b). \tag{40}$$

Define $\psi : R \to \Delta$ by setting

$$\psi(x) = xa^{-1}\phi(a). \tag{41}$$

Clearly $\psi$ is an $R$-module homomorphism. If $b \in I \setminus 0$, then by (40) and (41),

$$\psi(b) = ba^{-1}\phi(a) = bb^{-1}\phi(b) = \phi(b).$$

We need to show that $\psi$ is a $R'G$-module homomorphism. If $\sigma \in G$, then we obtain, taking $b = \sigma(a)$ in (40),

$$a^{-1}\phi(a) = (\sigma(a))^{-1}\phi(\sigma(a)) = \sigma(a^{-1}\phi(a)).$$

Thus we have for any $x \in R$
$$\psi(\sigma(x)) = \sigma(x)a^{-1}\phi(a) = \sigma(x)\sigma(a^{-1}\phi(a)) = \sigma(\psi(x)).$$
$\square$

We shall now apply these results to the case $R = \Lambda$ - a general quantum polynomial ring with $n \geq 3$ variables, and $\Delta = F$ its division ring of fractions. The action of $G$ in $\Lambda$ can be extended to the action on $F$.

**Corollary 5.6.** *Let $\Lambda$ be a general quantum polynomial ring with $n \geq 3$ variables, $G$ a finite group of automorphisms such that $\Lambda$ is a faithful left $\Lambda' G$-module. Then $\Lambda$ is injective as left $\Lambda' G$-module.*

*Proof.* We know from Corollary 3.5 and Corollary 3.11 that $\Lambda$ is a finitely generated left and right $\Lambda^G$-module. Therefore $\Lambda$ is a subgenerator in the category of $\Lambda' G$-modules. By Proposition 5.5 the division ring of fractions $F$ is $\Lambda$-injective as $\Lambda' G$-module. Apply [W1, 16.3]. $\square$

The next theorem is well know and its proof follows from Corollary 3.7 and Corollary 3.16,

**Theorem 5.7.** *Let $G$ be a finite subgroup in $\operatorname{Aut}\Lambda$. Then $F^G$ is the division ring of fractions of $\Lambda^G$.*

Notice that Theorem 5.7 can be considered as a special case of [W2, 11.6]. The forthcoming theorem is inspired by the main results of [AD].

**Theorem 5.8.** *Let $G$ be a finite subgroup in $\operatorname{Aut}^+\Lambda$. Put*
$$\Gamma = \Lambda_{X_{r+1}\cdots X_n} = D_{Q,\alpha}[X_1^{\pm 1}, \dots, X_n^{\pm 1}].$$
*There exist monomials $Y_1, \dots, Y_n \in \Gamma^G$ such that*

1. *the subalgebra $\Omega$ in $\Gamma^G$ generated by $D, Y_1^{\pm 1}, \dots, Y_n^{\pm 1}$ is a general quantum polynomial algebra $D_{Q',\alpha'}[Y_1^{\pm 1}, \dots, Y_n^{\pm 1}]$ which coincides with $\Gamma^G$;*
2. *if $F$ is the division ring of fraction of $\Lambda$ then $F^G$ is the division ring of fractions of $\Omega$.*

*Proof.* Without loss of generality we can assume that $r = n$ and $\Lambda = \Gamma$. By Corollary 3.6, the ring $\Lambda$ is a finitely generated left (right) $\Lambda^G$-module, and by Proposition 3.2, the subring $\Lambda^G$ is homogeneous with respect to $\mathbb{Z}^n$-grading from Definition 2.16. Denote by $U$ the set of all multi-indicies $(m_1, \dots, m_n) \in \mathbb{Z}^n$ such that $X_1^{m_1} \cdots X_n^{m_n} \in \Lambda^G$. It is not difficult to observe that $U$ is a subgroup in $\mathbb{Z}^n$. By Corollary 3.18, there exists an integer $m > 0$ such that $X_1^m, \dots, X_n^m \in \Lambda^G$. This means that $U$ is a free Abelian subgroup in $\mathbb{Z}^n$, of rank $n$. Let monomials $Y_1, \dots, Y_n$ in $X_1, \dots, X_n$ have multi-indices in $\mathbb{Z}^n$ which form a base of $U$. Then $\Lambda^G$ is generated by $D, Y_1^{\pm 1}, \dots, Y_n^{\pm 1}$. Moreover, by (5) we have
$$Y_i Y_j = q'_{ij} Y_j Y_i,$$
where the image of $q'_{ij}$ in $D^*/N$ belongs to the subgroup $L$ generated by images of all $q_{st}, 1 \leq s < t \leq n$. On the other hand each $X_i^m, i = 1, \dots, n$, is a product of $Y_1^{\pm 1}, \dots, Y_n^{\pm 1}$, and
$$X_i^m X_j^m = q_{ij}^{m^2} n' X_j^m X_i^m, \quad n' \in N.$$
By (5) the image of $q_{ij}^{m^2}$ in $D^*/N$ belongs to the subgroup $L'$ generated by images of all $q'_{ij}, 1 \leq i < j \leq n$. According to the assumption the group $L$ is a free Abelian

group of rank $\frac{n(n-1)}{2}$ and the subgroup $L'$ in $L$ has the same rank. This means that the multiparameters $q'_{ij}, 1 \leq i < j \leq n$, form a set of free generators of $L'$. Taking $\alpha' = (\alpha'_1, \dots, \alpha'_n)$, where each $\alpha'_i$ is the automorphism of $D$ induced by conjugation by $Y_i$ $(i = 1, \dots, n)$, we see that $\Omega$ is a general quantum polynomial algebra. □

**Corollary 5.9.** *Let $r = n$ and $\alpha_1, \dots, \alpha_n$ be identical on the center of $D$. Suppose that $G$ is a finite subgroup in* $\operatorname{Aut}\Lambda$. *Then the left and right global dimensions of $\Lambda' G$ are equal to 1.*

*Proof.* By Theorem 5.8 the ring $\Omega = \Lambda^G$ is a general quantum polynomial ring. So the global dimension of $\Omega$ is equal to 1 by [MP, Corollary in 3.10] (and [A1]). Apply Morita-equivalence of the rings $\Lambda' G$ and $\Omega$ (see Corollary 5.2). □

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## References

[A1] Artamonov V.A., Projective modules over quantum polynomial algebras, Mat. Sb. 185(7) (1994), 3-12.

[A2] Artamonov V.A., Quantum polynomial algebras, J. Math. Sci. (New York), Algebra 4, 87(3) (1997), 3441-3462.

[A3] Artamonov V.A., Quantum Serre's conjecture, Uspekhi Mat. Nauk 53(4) (1998), 3-78.

[A4] Artamonov V.A., General quantum polynomials: irreducible modules and Morita-equivalence. Izv. RAN, Ser. Math. 63 (1999), N 5, 3-36.

[AC] Alev J., Chamarie M., Dérivations et automorphismes de quelques algèbres quantiques, Comm. Algebra 20(6) (1992), 1787-1802.

[AD] Alev J., Dumas F., Invariants du corps de Weyl sous l'action de groupes finis, Comm. Algebra 25(5) (1997), 1655-16723.

[B] Bovdi A.A., Group rings, Kiev UMK. 1988, 156p.

[BG] Brown K.A., Goodearl K.R., Prime spectra of quantum semisimple groups, Trans. Amer. Math. Soc. 348(6) (1996), 2465-2501

[D] Demidov E.E., Quantum groups, Moscow, Factorial, 1998, 127p.

[GL1] Goodearl K.R., Letzter E.S., Prime factor of the coordinate ring of quantum matrices, Proc. Amer. Math. Soc. 121(4) (1994), 1017- 1025.

[GL2] Goodearl K.R., Letzter E.S., Prime and primitive spectra of multiparameter quantum affine spaces, "Trends in ring theory" (Miskolc), 1996, 39-58. CMS Conf. Proc. 22, AMS Providence RI, 1998.

[Kh] Kharchenko V.K., Automorphisms and derivations of associative rings, Kluwer Academic Publs, Dordrecht, 1991.

[KPS] Kirkman E., Procesi C., Small L., A $q$-analog for the Virasoro algebra, Comm. Algebra 20(10) (1994), 3755-3774.

[M1] Montgomery S., Fixed rings of finite automorphism groups of associative rings, Springer Lecture Notes Math. 818, 1980.

[M2] Montgomery S., Hopf algebras and their actions on rings, Regional Conf. Ser. Math. Amer. Math. Soc., Providence RI, 1993.

[MP] McConnell J.C., Pettit J.J., Crossed products and multiplicative analogues of Weyl algebras, J. London Math. Soc. 38(1) (1988), 47-55.

[OP] Osborn J.M., Passman D.S., Derivations of skew polynomial rings, J. Algebra 176(2) (1995), 417-448.

[P] Passman D. S., Group rings, crossed products and Galois theory, Regional Conf. Ser. Math., Amer. Math. Soc, Providence RI, 1986.

[W1] Wisbauer R., Foundations of module and ring theory, Gordon and Breach, Reading, Paris, 1991.

[W2] Wisbauer R., Modules and algebras: bimodule structure and group actions on algebras, Pitman Monographs and Surveys PAM 81, Addison Wesley Longman, Essex, 1996.

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