Strongly Rational Comodules and Semiperfect Hopf Algebras over QF Rings

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Abstract

Let C be a coalgebra over a QF ring R. A left C-comodule is called *strongly rational* if its injective hull embeds in the dual of a right C-comodule. Using this notion a number of characterizations of right semiperfect coalgebras over QF rings are given, e.g., C is right semiperfect if and only if C is strongly rational as left C-comodule. Applying these results we show that a Hopf algebra H over a QF ring R is right semiperfect if and only if it is left semiperfect or - equivalently - the (left) integrals form a free R-module of rank 1.

Introduction

One of the striking differences between categories of modules over algebras and categories of comodules over a coalgebra is the possible lack of projectives in the latter categories. A coalgebra is called *right (left) semiperfect* if the category of right (left) comodules has a projective generator. Right semiperfect coalgebras and Hopf algebras over fields have been investigated thoroughly. While in general right semiperfect coalgebras need not be left semiperfect it is well known that for Hopf algebras over fields semiperfectness is a left right symmetric property.

The purpose of the paper is to study properties and characterizations of right semiperfect coalgebras and Hopf algebras over rings. While the basic

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definitions and elementary properties hold for coalgebras over any ring R, deeper results quite often depend on the special properties of the ring R. To make full use of the Finiteness Theorem for comodules we need R to be noetherian, and to take advantage of left right symmetry we need the functor $\mathrm{Hom}_R,(-,R)$ to be exact, i.e., R should be injective. Rings with these two properties are just QF rings and indeed, over such rings we obtain essentially all the characterizations of semiperfectness known for base fields.

In the first section we recall basic techniques for the study of coalgebras C over QF rings R by considering C as a module over the dual algebra C^*. Assuming C to be projective as an R-module we can identify the category of right C-comodules with the full subcategory $\sigma[_{C^*}C] \subset C^*$-Mod, whose objects are subgenerated by C (see [17]).

In section 2 we concentrate on right semiperfect coalgebras C over QF rings. *Strongly rational* left comodules L are introduced by the property that the injective hull $E(L)$ of L in $\sigma[_{C^*}C]$ can be embedded into the dual M^* of some right C-comodule M. This notion turns out to be helpful for our investigations. For example, C is right semiperfect if and only if every simple left C-comodule is strongly rational.

The last section is devoted to semiperfect Hopf algebras. Applying the previous results we give complete proofs for the characterization of these Hopf algebras over QF rings R, including the Uniqueness Theorem for integrals, which here says that the integrals form a free R-module of rank 1 (see 3.9). Specializing to base fields we obtain results of Beattie-Dăscălescu-Grünenfelder-Năstăsescu [4], Donkin [6], Lin [7], D.E. Radford [9], Sullivan [12], and others as Corollary 3.10.

1 Coalgebras and comodules

In this section we recall some basic definitions for coalgebras and comodules.

By C we always denote a coassociative coalgebra over a commutative ring R defined by the R-linear map (comultiplication) $\Delta : C \to C \otimes_R C$ with counit $\varepsilon : C \to R$. The dual module $C^* = \mathrm{Hom}_R(C,R)$ is an R-algebra with the convolution product and has ε as identity element.

A right comodule over C is defined by an R-linear map $\varrho : M \to M \otimes_R C$ satisfying the coassociativity and the counital condition. Morphisms between comodules M, N are R-linear maps which respect the comodule structure (notation $\mathrm{Com}_C(M,N)$). We denote the category of right C-comodules with these morphisms by $\mathcal{M}^C$. Symetrically the category of left C-comodules is defined and denoted by ${}^C\mathcal{M}$. In particular C itself is a right (left) C-comodule and is a subgenerator in $\mathcal{M}^C$ (and ${}^C\mathcal{M}$). The category $\mathcal{M}^C$ (and ${}^C\mathcal{M}$) is closed under direct sums and factor objects. Moreover it is closed under subobjects (and hence a Grothendieck category) if and only if C is flat as an R-module (see [17, 3.15]).

The following relationship (see [17, 3.12]) indicates that there is a strong interplay between properties of the ring R and the comodule C.

1.1 Hom-Com **relations.** *For $M \in \mathcal{M}^C$ and any R-module X, the map*

$$\mathrm{Com}_C(M, X \otimes_R C) \to \mathrm{Hom}_R(M, X), \quad f \mapsto (id \otimes \varepsilon) \circ f,$$

is an isomorphism with inverse map $h \mapsto (h \otimes id) \circ \varrho$.

In particular, for $X = R$ we have an isomorphism $\mathrm{Com}_C(M, C) \simeq \mathrm{Hom}_R(M, R)$.

Putting $M = C$ we have $\mathrm{Com}_C(C, C) \simeq C^*$ which is in fact an algebra anti-isomorphism.

Any right comodule M over C is a left module over the dual algebra C^* by

$$\rightharpoonup : C^* \otimes_R M \to M, \quad f \otimes m \mapsto (id \otimes f) \circ \rho(m),$$

and by this $\mathcal{M}^C$ is a subcategory of C^*-Mod. In particular C is a (C^*, C^*)-bimodule,

$$\begin{aligned} \rightharpoonup &: C^* \otimes_R C \to C, \quad f \otimes c \mapsto f \rightharpoonup c := (id \otimes f) \circ \Delta(c), \\ \leftharpoonup &: C \otimes_R C^* \to C, \quad c \otimes g \mapsto c \leftharpoonup g := (g \otimes id) \circ \Delta(c)\,. \end{aligned}$$

Moreover M is a subcomodule of $M \otimes_R C$ which is a factor comodule of some $C^{(\Lambda)}$, i.e., C is a *subgenerator* for all right (left) comodules. In case C is projective as an R-module, $\mathcal{M}^C$ is a full subcategory and we have $\mathcal{M}^C = \sigma[{}_{C^*}C]$, the full subcategory of C^*-Mod whose objects are submodules of C-generated modules (and ${}^C\mathcal{M} = \sigma[C_{C^*}]$, see [17, 4.1, 4.3]). To exploit this identification we henceforth will assume that C is projective as an R-module. Although this condition is not always necessary it is indispensable when (right) C-comodule properties are derived from (left) C^*-module properties. It also implies that sub-coalgebras of C are essentially the (C^*, C^*)-submodules and we have the most helpful

1.2 Finiteness Theorem. *Every finite subset of any right C-comodule M is contained in a sub-comodule of M which is finitely generated as an R-module.*

$\sigma[_{C^*}C]$ is a fully reflective subcategory of C^*-Mod and the trace functor is right adjoint to this inclusion. In comodule theory this is usually called the

1.3 Rational functor. By the *rational functor* we mean the left exact functor

$$\mathrm{Rat} : C^*\text{-Mod} \to \sigma[_{C^*}C],$$

assigning to any left C^*-module M the *rational submodule*

$$\mathrm{Rat}(M) = \mathrm{Tr}_{C^*}(\sigma[_{C^*}C], M) = \sum \{\mathrm{Im}\, f \mid f \in \mathrm{Hom}_{C^*}(U, M),\, U \in \sigma[_{C^*}C]\}.$$

Clearly $\mathrm{Rat}(M)$ is the largest submodule belonging to $\sigma[_{C^*}C]$ and $M = \mathrm{Rat}(M)$ if and only if $M \in \sigma[_{C^*}C]$ (see [17, 5.2]).

The notation Rat is also used for the corresponding functor Mod-$C^* \to \sigma[C_{C^*}]$.

As shown in [17, 5.3, 5.4], over noetherian rings the trace ideal can be characterized by finiteness conditions and the duals of finitely R-generated C-comodules are again C-comodules:

1.4 Lemma. *Let R be noetherian. Then:*

(1) The left rational ideal $\mathrm{Rat}(_{C^*}C^*)$ *is described by*

$$\begin{aligned} T_1 &= \{f \in C^* \mid C^* * f \ \ \textit{is a finitely generated } R\textit{-module}\}, \\ T_2 &= \{f \in C^* \mid \ \mathrm{Ke}\ f \textit{ contains a left coideal } K, \textit{ such that } C/K \textit{ is} \\ &\quad\ \ \textit{a finitely generated } R\textit{-module} \ \}, \\ T_3 &= \{f \in C^* \mid (id \otimes f) \circ \Delta(C) \ \ \textit{is a finitely generated } R\textit{-module}\}. \end{aligned}$$

(2) For every finitely R-generated right (left) C-comodule M, M^* is a left (right) C-comodule.

Recall that R is a QF ring if it is artinian, injective and a cogenerator in R-Mod. Over such rings, the functor $(-)^* = \mathrm{Hom}_R(-, R)$ is faithful and exact. Moreover every faithful R-module is a generator and cogenerator in R-Mod, and hence every faithful flat R-module is faithfully flat. It follows essentially from 1.1 and 1.2 that coalgebras over such rings have particularly nice properties:

1.5 Coalgebras over QF rings. *Let R be a QF ring. Then:*

(1) C is an injective cogenerator in $\sigma[_{C^*}C]$.

(2) $Q := \mathrm{Soc}_{C^*}C \trianglelefteq C$ *(essential submodule) and*

$$\mathrm{Jac}(C^*) = \mathrm{Hom}_{C^*}(C/Q, C) \simeq \mathrm{Hom}_R(C/Q, R).$$

(3) If M is a projective object in $\mathcal{M}^C$ then M^* is C-injective as right C^*-module and $\mathrm{Rat}(M^*)$ *is injective in ${}^C\mathcal{M}$.*

(4) If $M \in \mathcal{M}^C$ is finitely generated as an R-module then:

- *(i) M is injective in $\mathcal{M}^C$ if and only if M is injective in* C^*-Mod.
- *(ii) M is projective in $\mathcal{M}^C$ if and only if M is projective in* C^*-Mod.

(5) For any simple submodule $S \subset {}_{C^*}C$ and $0 \neq a \in S$, we have that $S^* \simeq a \leftharpoonup C^*$ is a simple right C^*-module and

$$\mathrm{Tr}(S, {}_{C^*}C) = C^* \rightharpoonup a \leftharpoonup C^* = \mathrm{Tr}(S^*, C_{C^*})$$

is a minimal subcoalgebra of C (and hence is finitely generated as R-module).

(6) Let $\{S_\lambda\}_\Lambda$ be a representing set of the simple modules in $\sigma[{}_{C^*}C]$. Then the coradical of C is

$$\sum_\Lambda \mathrm{Tr}(S_\lambda, {}_{C^*}C) = \sum_\Lambda \mathrm{Tr}(S_\lambda^*, C_{C^*}).$$

Proof. (1) Since R is injective in R-Mod we conclude from the Hom-Com relations 1.1 that C is injective in $\sigma[{}_{C^*}C]$.

For any right C-comodule M, there is an R-module monomorphism $M \to R^{(\Lambda)}$ (for some Λ) which yields a comodule monomorphism $M \to R^{(\Lambda)} \otimes_R C \simeq C^{(\Lambda)}$. So C is a cogenerator in $\sigma[{}_{C^*}C]$.

For (2)-(4) we refer to [17, 6.1, 6.2].

(5) Since C is self-injective it is clear that $S^* \simeq \mathrm{Hom}_{C^*}(S, C)$ is a simple right C^*-module, and also that $\mathrm{Tr}(S, {}_{C^*}C) = C^* \rightharpoonup a \leftharpoonup C^*$.

With the inclusion $i : S \to C$, we have $S^* = \{f \circ i \mid f \in C^*\}$, and it is easy to verify that

$$S^* \;\to\; a \leftharpoonup C^*, \;\; f \circ i \mapsto a \leftharpoonup f,$$

is an isomorphism of right C^*-modules. By symmetry we conclude $C^* \rightharpoonup a \leftharpoonup C^* = \mathrm{Tr}(S^*, C_{C^*})$, and by 1.2, $C^* \rightharpoonup a \leftharpoonup C^*$ is finitely generated as an R-module.

(6) The coradical of C is defined to be the sum of minimal sub-coalgebras (=sub-bimodules) and hence the assertion follows from (5). $\square$

1.6 Decomposition of coalgebras over QF rings. *Assume R to be a QF ring, and let $\{S_\lambda\}_\Lambda$ be a representing set of the simple modules in $\sigma[{}_{C^*}C]$. Then:*

(1) *$C = \sum_\Lambda E(S_\lambda)$ in $\sigma[{}_{C^*}C]$, where $E(S_\lambda)$ denotes the injective hull of S_λ in $\sigma[{}_{C^*}C]$.*

(2) *If C is co-commutative then $C = \sum_\Lambda E(S_\lambda)$ is a decomposition of C into irreducible subcoalgebras $E(S_\lambda)$.*

Proof. (1) By the Finiteness Theorem, C is locally of finite length as a C^*-module. So we obtain the decomposition from [16, 32.5].

(2) Since C is co-commutative ${}_{C^*}C$ has a commutative endomorphism ring C^*. Now it follows from [16, 48.16] that all the $E(S_\lambda)$ are fully invariant submodules of C and hence they are (irreducible) sub-coalgebras. $\square$

Recall that for any R-module M, we have the evaluation map

$$\Phi_M : M \to M^{**},\ m \mapsto [\beta \mapsto \beta(m)].$$

If R is a cogenerator in R-Mod then $\Phi_M(M)$ is dense in M^{**} in the finite topology (e.g., [16, 47.6]). Notice that Φ_M is a C^*-module morphism provided M is a C^*-module.

For every subset X of an R-module M and Y of M^*, we denote

$$\begin{aligned} X^\perp &= \{f \in M^* \mid f(X) = 0\}, \text{ and} \\ Y^\perp &= \{m \in M \mid f(m) = 0, \text{ for every } f \in Y\}. \end{aligned}$$

1.7 Proposition. *Let R be QF, $M \in C^*$-Mod and let $K \subset L \in$ Mod-C^*. Assume there exists a monomorphism $i : L \hookrightarrow M^*$ in Mod-C^* and K is finitely generated as R-module. Then:*

(1) *$L^* = K^\perp + i^*(\Phi_M(M))$.*

(2) *If $M \in \mathcal{M}^C$, $L^* = K^\perp + \mathrm{Rat}({}_{C^*}L^*)$.*

Proof. (1) Let $\alpha \in L^*$. Since $i^* : M^{**} \to L^*$ is surjective, there exists $\widehat{\alpha} \in M^{**}$ such that $\alpha = i^*(\widehat{\alpha}) = \widehat{\alpha} \circ i$. Let $B = \{\xi_1, \ldots, \xi_n\}$ be a generating set of K as an R-module. As $\Phi_M(M)$ is dense in M^{**} (with the finite topology), there exists an $x \in M$ such that $\Phi_M(x) - \widehat{\alpha} \in B^\perp$, i.e. $(\Phi_M(x))(\xi_i) = \widehat{\alpha}(\xi_i)$. It follows that

$$\xi_i(x) = (\Phi_M(x))(\xi_i) = \widehat{\alpha}(\xi_i) = \alpha(\xi_i),$$

for every $i \leq n$, as $\xi_i \in K \subset L$. Hence $\alpha - i^*(\Phi_M(x)) = \alpha - (\Phi_M(x) \circ i) \in K^{\perp}$.

(2) If $M \in \mathcal{M}^C$, $i^*(\Phi_M(M)) = (i^* \circ \Phi_M)(M) \in \mathcal{M}^C$, and therefore we have $i^*(\Phi_M(M)) \subset \mathrm{Rat}({}_{C^*}L^*)$. $\square$

1.8 Lemma. *Let R be a QF ring, S a simple object with injective envelope $E(S)$ in ${}^C\mathcal{M}$, and $S^{\perp} = \mathrm{Hom}_R(E(S)/S, R)$. Then $E(S)^*$ is a cyclic projective left C^*-module and $S^{\perp} = \mathrm{Jac}(C^*)E(S)^*$. In particular $S^{\perp}$ is superfluous in $E(S)^*$.*

Proof. Since $E(S)$ is a direct summand of C as a right C^*-module, $E(S)^*$ is a direct summand of ${}_{C^*}C^*$. Moreover we have $S \subset Q \subset C$, where Q is the socle of C_{C^*}. From this we obtain the commutative exact diagram

$$\begin{array}{ccccccccc}
0 & \to & (C/Q)^* & \to & C^* & \to & Q^* & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (E(S)/S)^* & \to & E(S)^* & \to & S^* & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & .
\end{array}$$

Since $(C/Q)^* = \mathrm{Jac}(C^*)$ (by 1.5) is superfluous in C^*, we have that $(E(S)/S)^*$ is superfluous in $E(S)^*$. $\square$

1.9 Lemma. *Let R be a QF ring, S a simple object with injective envelope $E(S)$ in ${}^C\mathcal{M}$, and assume there exists a monomorphism $i : E(S) \hookrightarrow M^*$ in* Mod-C^*, *for some $M \in \mathcal{M}^C$. Then $E(S)^* = i^*(\Phi_M(M))$ and $E(S)$ is finitely generated as an R-module.*

Proof. By Proposition 1.7, $E(S)^* = S^{\perp} + i^*(\Phi_M(M))$, and hence it follows by Lemma 1.8 that $E(S)^* = i^*(\Phi_M(M))$ is a cyclic rational left C^*-module. In particular $E(S)^*$ is finitely R-generated and so is $E(S)$. $\square$

2 Strongly rational comodules

As before C will denote a coassociative R-coalgebra with C_R projective. We introduce the notion of a *strongly rational* comodule and use it to prove old and new characterizations of semiperfect coalgebras.

2.1 Definition. A comodule $L \in {}^C\mathcal{M}$ is called *strongly rational*, or *s-rational* for short, if the injective envelope $E(L)$ of L in ${}^C\mathcal{M}$ embeds in M^* as a right C^*-module, for some $M \in \mathcal{M}^C$.

Of particular interest are (simple) s-rational modules when R is a QF ring as will be seen by the following observations.

2.2 Proposition. *Let R be a QF ring. Then a simple object $S \in {}^C\mathcal{M}$ is s-rational if and only if the injective envelope $E(S)$ of S in ${}^C\mathcal{M}$ is finitely generated as an R-module.*

In this case $E(S)$ is injective in Mod-C^* *and $E(S)^*$ is a cyclic projective and rational left C^*-module.*

Proof. By Lemma 1.9, if $S \in {}^C\mathcal{M}$ is a simple s-rational object, then $E(S)$ is finitely generated as an R-module.

Conversely, if $E(S)$ is a finitely generated R-module, then $E(S) \cong (E(S)^*)^*$ is s-rational (by 1.4(2)). Moreover, by Lemma 1.8, $E(S)^*$ is a cyclic projective left C^*-module, and so $E(S)$ is injective in Mod-C^*. $\square$

2.3 Proposition. *Let R be a QF ring and assume that P is a projective object in $\mathcal{M}^C$. If S is a simple quotient of P in $\mathcal{M}^C$, then S^* is a simple s-rational object in ${}^C\mathcal{M}$ and $E(S^*)^*$ is a cyclic projective left C^*-module and it is a direct summand of P.*

Proof. By Proposition 1.5, we know that $S^* \in {}^C\mathcal{M}$ is simple, and also that $\mathrm{Rat}(P^*)$ is an injective object in ${}^C\mathcal{M}$. Therefore $\mathrm{Rat}(P^*)$ contains a copy of the injective envelope of S^* in ${}^C\mathcal{M}$. In particular, S^* is s-rational so that by Lemma 1.9, the comodule $E(S^*)^* = i^* \circ \Phi_P(P)$ is a quotient of P and by Lemma 1.8 it is a cyclic projective left C^*-module. $\square$

2.4 Proposition. *Let R be a QF ring. For a simple object $S \in \mathcal{M}^C$, the following are equivalent:*

(a) S is a quotient of a projective object of $\mathcal{M}^C$;

(b) S is a quotient of an object of $\mathcal{M}^C$ which is a cyclic projective left C^*-module (and is finitely R-generated);

(c) S^* is s-rational;

(d) the injective envelope of S^* in ${}^C\mathcal{M}$ is finitely generated as R-module.

Proof. Notice that by Proposition 1.5, S^* is a simple object in ${}^C\mathcal{M}$.

$(a) \Rightarrow (c) \Leftrightarrow (d)$ By Propositions 2.3 and 2.2.

$(d) \Rightarrow (b)$ As $E(S^*)$ is a finitely generated R-module we know from Proposition 1.4(2) that $E(S^*)^* \in \mathcal{M}^C$. By Lemma 1.8, $E(S^*)^*$ is a cyclic projective left C^*-module. Moreover $S \cong S^{**}$ is a quotient of $E(S^*)^*$.

$(b) \Rightarrow (a)$ is trivial. $\square$

2.5 Corollary. *If R is a QF ring the following are equivalent.*

(a) $\mathcal{M}^C$ *contains a non zero projective object;*

(b) $\mathcal{M}^C$ *contains a non zero object which is a projective cyclic left C^*-module (of finite length);*

(c) ${}^C\mathcal{M}$ *contains a simple s-rational object;*

(d) ${}^C\mathcal{M}$ *contains a simple object such that its injective envelope in* ${}^C\mathcal{M}$ *is finitely R-generated.*

Proof. Since, by Proposition 1.5, the dual of a simple object in $\mathcal{M}^C$ is a simple object in ${}^C\mathcal{M}$, the conclusion follows by Proposition 2.4 and the fact that, for any projective object $P \in \mathcal{M}^C$, $\mathrm{Rad}(P) \neq P$ (see [16, 22.3]). $\square$

2.6 Right semiperfect coalgebras. *Let R be a QF ring and put* $T := \mathrm{Rat}({}_{C^*}C^*)$. *The following assertions are equivalent and characterize C as a right semiperfect coalgebra (in the sense of [7]):*

(a) Every simple object of $\mathcal{M}^C$ *is a quotient of a projective object of* $\mathcal{M}^C$*;*

(b) every simple object of $\mathcal{M}^C$ *is a quotient of a projective object of* $\mathcal{M}^C$ *that is a cyclic projective left C^*-module (finitely R-generated);*

(c) every simple object of ${}^C\mathcal{M}$ *is s-rational;*

(d) the injective hull of any simple object in ${}^C\mathcal{M}$ *is finitely R-generated;*

(e) C is an s-rational object of ${}^C\mathcal{M}$*;*

(f) every simple module in $\mathcal{M}^C$ *has a projective cover;*

(g) the functor $\mathrm{Rat} : C^*\text{-Mod} \to \mathcal{M}^C$ *is exact;*

(h) for every $N \in \sigma[{}_{C^*}C]$, $TN = N$*;*

(i) for every $N \in \sigma[{}_{C^*}C]$, *the canonical map* $T \otimes_{C^*} N \to N$ *is an isomorphism;*

(j) $TC = C$ *and* C^*/T *is flat as a right C^*-module;*

(k) $T^2 = T$ *and T is a generator in* $\mathcal{M}^C$*;*

(l) $\mathcal{M}^C$ *has a generating set of finitely generated modules which are projective in* C^*-Mod.

In this case the injective envelope in ${}^C\mathcal{M}$ *of every simple object of* ${}^C\mathcal{M}$ *is an injective right C^*-module and the right trace* $\mathrm{Rat}(C^*{}_{C^*}) \subset T$.

Proof. Since by Proposition 1.5 the dual of a simple object in $\mathcal{M}^C$ is a simple object in ${}^C\mathcal{M}$ and conversely, the equivalences $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d)$ follow by Proposition 2.4.

$(c) \Rightarrow (e)$ Let $Soc(C_{C^*}) = \bigoplus_\Lambda S_\lambda$, for suitable simple objects S_λ in ${}^C\mathcal{M}$. For every $\lambda \in \Lambda$, let $i_\lambda : E(S_\lambda) \hookrightarrow (M_\lambda)^*$ be an embedding, where $M_\lambda \in \mathcal{M}^C$. Then

$$C = \bigoplus_\Lambda E(S_\lambda) \hookrightarrow \bigoplus_\Lambda (M_\lambda)^* \hookrightarrow \prod_\Lambda (M_\lambda)^* \simeq \left(\bigoplus_\Lambda M_\lambda\right)^*,$$

where $\bigoplus_\Lambda M_\lambda \in \mathcal{M}^{\mathcal{C}}$. This shows that C is s-rational in ${}^C\mathcal{M}$.

$(e) \Rightarrow (c)$ is trivial as C is an injective cogenerator of ${}^C\mathcal{M}$.

$(f) \Rightarrow (l)$ Clearly the projective covers of simple modules in $\mathcal{M}^C$ are finitely generated as R-modules. By 1.5(4) they are in fact projective in C^*-Mod.

$(l) \Rightarrow (a)$ is trivial and for the remaining equivalences we refer to [17, 4.11, 5.3 and 6.3].

The final assertions follow by Proposition 1.5 and [17, 5.3]. □

For coalgebras over fields the equivalence $(a) \Leftrightarrow (d)$ appears in [7, Theorem 10].

From [17, 6.4] we have the following characterization of coalgebras which are semiperfect on both sides.

2.7 Left and right semiperfect coalgebras. *Let R be a QF ring and put $T := \mathrm{Rat}({}_{C^*}C^*)$ and $T' := \mathrm{Rat}(C^*{}_{C^*})$. The following are equivalent.*

(a) C is a left and right semiperfect coalgebra;

(b) all left C-comodules and all right C-comodules have projective covers;

(c) the injective hulls of simple objects in ${}^C\mathcal{M}$ and $\mathcal{M}^C$ are finitely R-generated;

(d) $T = T'$ and is dense in C^*;

(e) ${}_{C^*}C$ and C_{C^*} are direct sums of finitely generated C^*-modules.

Under these conditions T is a ring with enough idempotents.

The next result extends [4, Lemma 3.2] from base fields to QF rings.

2.8 Lemma. *Let C be a left and right semiperfect coalgebra over a QF ring R. Put $T := \mathrm{Rat}({}_{C^*}C^*)$ and consider the inclusion $i : T \to C^*$ and, for $M \in \mathcal{M}^C$, the map*

$$\mathrm{Hom}(i, M) : \mathrm{Hom}_{C^*}(C^*, M) \to \mathrm{Hom}_{C^*}(T, M).$$

(1) For any $M \in \mathcal{M}^C$, $\mathrm{Hom}(i, M)$ is injective.

(2) If M is finitely generated as R-module, then $\mathrm{Hom}(i, M)$ is bijective.

Proof. (1) Let $f \in \mathrm{Hom}_{C^*}(C^*, M)$ such that $f \circ i = 0$. Then by 2.6(h),

$$0 = f(T) = Tf(\varepsilon) = C^* f(\varepsilon),$$

implying $f(\varepsilon) = 0$ and $f = 0$.

(2) Let $M \in \mathcal{M}^C$ be finitely R-generated with injective hull $\widehat{M}$ in $\mathcal{M}^C$. By [17, 6.2], $\widehat{M}$ is in fact C^*-injective so that we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & T & \xrightarrow{i} & C^* & \xrightarrow{\pi} & C^*/T & \to & 0 \\ & & {\scriptstyle f}\downarrow & & {\scriptstyle \tilde{f}}\downarrow & & {\scriptstyle \bar{f}}\downarrow & & \\ 0 & \to & M & \to & \widehat{M} & \xrightarrow{p} & \widehat{M}/M & \to & 0. \end{array}$$

Since $\widehat{M}/M \in \mathcal{M}^C$ and $\bar{f} \circ \pi \circ i = 0$, we conclude from (1) that $\bar{f} \circ \pi = 0$. This implies $p \circ \tilde{f} = 0$, i.e., $\mathrm{Im}\,\tilde{f} \subset M$. Therefore $\tilde{f} \in \mathrm{Hom}_{C^*}(C^*, M)$ and $\tilde{f} \circ i = f$. □

Recall that a coalgebra C is said to be *left (right) co-Frobenius* if there exists some left (right) C^*-monomorphism $C \hookrightarrow C^*$. More generally C is said to be *left (right) QcF* (Quasi-co-Frobenius) if C is cogenerated by C^* as a left (right) C^*-module (i.e., C is a torsionless C^*-module, see [15]). Over QF rings this class of coalgebras can be characterized in the following way (see [17, 6.5]).

2.9 Left QcF coalgebras. *If R is a QF ring the following are equivalent:*

(a) C is left QcF;

(b) C is a submodule of a free left C^*-module;

(c) in $\mathcal{M}^C$ every (indecomposable) injective object is projective;

(d) C is a projective object in $\mathcal{M}^C$;

(e) C is projective in C^*-Mod.

If these conditions are satisfied, then C is a left semiperfect coalgebra and C is a generator in $\sigma[C_{C^*}]$.

In view of the decomposition of co-commutative coalgebras (see 1.6) we obtain:

2.10 Corollary. *Let R be a QF ring and assume C to be a co-commutative coalgebra. Then C is left QcF if and only if it is left co-Frobenius.*

We finally recall the case when C is a projective generator in $\mathcal{M}^C$ (see [17, 6.6]).

2.11 C **as a projective generator in** $\sigma[{}_{C^*}C]$**.** *Let R be a QF ring and put $T := \mathrm{Rat}({}_{C^*}C^*)$. The following are equivalent:*

(a) C is a left and right QcF coalgebra;

(b) C is a projective generator in $\mathcal{M}^C$;

(c) C is a projective generator in ${}^C\mathcal{M}$;

(d) $C = TC$, T is a ring with enough idempotents and an injective cogenerator in $\mathcal{M}^C$.

3 Semiperfect Hopf algebras

Let H be a Hopf algebra over a ring R with comultiplication Δ and antipode S. We will always assume that H *is projective and faithful* as an R-module.

An R-module M is called a *right H-Hopf module* if it is a

(i) right H-module by $\psi:\ M \otimes_R H \to M$,

(ii) right H-comodule by $\varrho:\ M \to M \otimes_R H$, satisfying $\varrho(mb) = \varrho(m)\Delta b$.

Morphisms between Hopf modules M and N are maps which are both H-module and H-comodule morphisms and we denote these by $\mathrm{Bim}_H(M,N)$. The category of right H-Hopf modules is denoted by $\mathcal{M}_H^H$.

Let M be a right H-Hopf module. The *coinvariants* of H in M are defined as

$$M^{coH} := \{m \in M \mid \ \varrho(m) = m \otimes 1_H\}\ .$$

The importance of this notion follows from the R-module isomorphism

$$\nu_M:\ \mathrm{Bim}_H(H,M) \to M^{coH},\ f \mapsto f(1_H),$$

with inverse map $\omega_M : m \mapsto [b \mapsto (b \otimes \varepsilon) \circ \varrho(m)]$. In particular from this we have $\mathrm{Bim}_H(H,H) \simeq R1_H$ which means that we can identify R with the ring of right Hopf module endomorphisms of H.

3.1 Fundamental Theorem. *Let H be a Hopf algebra over R.*

(1) H is a generator in $\mathcal{M}_H^H$, in particular for any right H-Hopf module M,

$$M^{coH} \otimes_R H \to M,\ m \otimes h \mapsto mh\,,$$

is a Hopf module isomorphism.

(2) If H_R is faithfully flat then H is a projective generator in $\mathcal{M}_H^H$, and hence

$$\mathrm{Bim}_H(H,-) : \mathcal{M}_H^H \to R\text{-Mod}$$

is an equivalence of categories.

Proof. (1) The corresponding proof in Sweedler [14, Theorem 4.1.1] does not depend on the base field (see also Schneider [10, Theorem 2.1]).

(2) Since $\mathcal{M}_H^H$ is a Grothendieck category the proof of [16, 18.5] applies and shows that H is projective in $\mathcal{M}_H^H$. □

3.2 Corollary. *Let H be a Hopf algebra over R.*

(i) Assume R to be a QF ring. Then H is a projective generator and an injective cogenerator in $\mathcal{M}_H^H$ and

$$H = H_1 \oplus \ldots \oplus H_n,$$

where the H_i are non-isomorphic H-Hopf modules which are injective hulls of simple Hopf modules in $\mathcal{M}_H^H$.

(ii) If R is semisimple then H is a direct sum of simple Hopf modules.

(iii) If R is a field then H is a simple Hopf module.

Proof. (i) Since R is a QF ring, H_R is faithfully flat. By 3.1(2), H is a projective generator in the category $\mathcal{M}_H^H$ which is equivalent to R-Mod. Since R is an injective cogenerator in R-Mod, H has the corresponding property in $\mathcal{M}_H^H$.

Since $\mathrm{Bim}_H(H,H) \simeq R1_H$, H is a cogenerator in $\mathcal{M}_H^H$ with commutative endomorphism and hence the decomposition follows from [16, 48.16] (compare 1.6).

(ii) and (iii) follow as in the proof of (i). □

By the ring structure of H a left module structure on H^* is defined by

$$\rightharpoondown : H \otimes_R H^* \to H^*, \qquad b \otimes f \mapsto [c \mapsto f(cb)],$$

and for $a \in H$, $f, g \in H^*$, and $\Delta(a) = \sum_i a_i \otimes \tilde{a}_i$,

$$a \rightharpoondown (f{*}g) \;=\; \textstyle\sum_i (a_i \rightharpoondown f){*}(\tilde{a}_i \rightharpoondown g)\,.$$

Applying the antipode S, a right H-module structure on H^* is defined by

$$\leftharpoondown_s \,: H^* \otimes_R H \to H^*, \quad f \otimes a \mapsto S(a) \rightharpoondown f$$

i.e., for each $c \in H$,

$$[f \leftharpoondown_s a](c) = [(S(a) \rightharpoondown f](c) = f(cS(a)).$$

For $a \in H$ and $g, f \in H^*$, we have the identity

$$(*) \quad g{*}(f \leftharpoondown_s a) = \sum_i [(\tilde{a}_i \rightharpoondown g){*}f] \leftharpoondown_s a_i$$

By the right comodule structure of H we have the left trace ideal T in H^* which plays a central part for our further investigations. It is a Hopf module with respect to the right H-module structure defined by $\leftharpoondown_s$.

3.3 Lemma. *Let H be a Hopf algebra with R noetherian. Then the left trace ideal $T := \mathrm{Rat}({}_{H^*}H^*)$ is a right Hopf module and hence $T^{coH} \otimes_R H \simeq T$.*

In particular T is generated by H as a right H-comodule.

Proof. By definition T is a right H-comodule and we denote by $\varrho : T \to T \otimes_R H$ the structure map. It remains to show that T has a right H-module structure which makes it a Hopf module.

Let $f \in T$ and $\varrho(f) = \sum_j f_j \otimes \tilde{f}_j$. For any $g \in H^*$, $g{*}f = \sum_j f_j g(\tilde{f}_j)$, and for $a \in H$ we obtain by $(*)$,

$$\begin{array}{rcl} g{*}(f \leftharpoondown_s a) & = & \sum_{i,j} (\tilde{a}_i \rightharpoondown g)(\tilde{f}_j)(f_j \leftharpoondown_s a_i) \\ & = & \sum_{i,j} g(\tilde{f}_j \tilde{a}_i)(f_j \leftharpoondown_s a_i)\,. \end{array}$$

This shows that the left ideal in H^* generated by $f \leftharpoondown_s a$ is finitely generated as an R-module by the $f_j \leftharpoondown_s a_i$, and hence $f \leftharpoondown_s a \in T$ by 1.4, proving that T is a right H-submodule in H^*.

Moreover since the identity holds for all $g \in H^*$ it implies

$$\varrho(f \leftharpoondown_s a) = \sum_{i,j} (f_j \leftharpoondown_s a_i) \otimes \tilde{f}_j \tilde{a}_i = \varrho(f)\Delta a\,,$$

which is the condition for T to be a Hopf module. $\square$

Let H be a Hopf algebra. An element $t \in H^*$ is called a *left integral* if it is a left H-comodule morphism.

3.4 Left integrals. *Let R be a noetherian ring, H a Hopf algebra, and $T := \mathrm{Rat}({}_{H^*}H^*)$. For $t \in H^*$ the following are equivalent:*

(a) t *is a left integral;*

(b) $(id \otimes t) \circ \Delta = \iota \circ t$*;*

(c) *for every* $f \in H^*$, $f{*}t = f(1_H)t$*;*

(d) $t \in T$ *and* $\varrho(t) = t \otimes 1_H$*;*

(e) $\alpha : H \to H^*$, $b \mapsto t \leftharpoondown_s b$, *is a left* H^**-morphism.*

Proof. $(a) \Leftrightarrow (b)$ The map t is a left comodule morphism if and only if the following diagram is commutative:

$$\begin{array}{ccccccc} H & \stackrel{t}{\to} & R & & b & \mapsto & t(b) \\ \downarrow_\Delta & & \downarrow_\iota & & \downarrow & & \downarrow \\ H \otimes_R H & \stackrel{id \otimes t}{\longrightarrow} & H & & \Delta(b) & \mapsto & (id \otimes t) \circ \Delta(b) = t(b)1_H. \end{array}$$

The commutativity of this diagram is just expressed by condition (b).

$(b) \Leftrightarrow (c)$ For any $f \in H^*$ and $b \in H$,

$$\begin{array}{rcccl} f{*}t(b) & = & (f \otimes t) \circ \Delta(b) & = & f((id \otimes t) \circ \Delta(b)), \\ f(1_H)t(b) & = & f(t(b)1_H) & = & f(\iota \circ t(b)). \end{array}$$

Now the assertion follows by a dual basis argument.

$(c) \Rightarrow (d)$ Clearly $H^*{*}t = Rt$ and hence $t \in T$ by 1.4. Denote by $\varrho : T \to T \otimes_R H$ the structure map of T and put $\varrho(t) = \sum_i t_i \otimes \tilde{t}_i$. Then for every $f \in H^*$,

$$f{*}t = (id \otimes f) \circ \varrho(t) = \sum\nolimits_i t_i\, f(\tilde{t}_i).$$

Now $f{*}t = f(1_H)t$ implies

$$(id \otimes f)(\sum_i t_i \otimes \tilde{t}_i) = (id \otimes f)(t \otimes 1_H),$$

and hence $\varrho(t) = t \otimes 1_H$.

$(d) \Rightarrow (c)$ Under the given conditions we have for every $f \in H^*$,

$$f{*}t = (id \otimes f) \circ \varrho(t) = (id \otimes f)(t \otimes 1_H) = f(1_H)\, t.$$

$(d) \Rightarrow (e)$ We have $\alpha(H) \subset T$ and the commutative diagram

$$\begin{array}{ccccccc} H & \stackrel{\alpha}{\to} & T & & b & \mapsto & t \leftharpoondown_s b \\ \downarrow_\Delta & & \downarrow_\varrho & & \downarrow & & \downarrow \\ H \otimes_R H & \to & T \otimes_R H & & \Delta(b) & \mapsto & (\alpha \otimes 1_H)\Delta(b) = \varrho(t \leftharpoondown_s b)\,. \end{array}$$

This shows that α is a right H-comodule - and hence a left H^*-module morphism.

$(e) \Rightarrow (d)$ Since α is a left H^*-module morphism its image lies in T and α can be regarded as a right comodule morphism. So we have again the above diagram. $\square$

3.5 Proposition. *Let H be a Hopf algebra over R and $T := \mathrm{Rat}({}_{H^*}H^*)$. Assume there exists a generator P in $\sigma[{}_{H^*}H]$ which is projective in H^*-*Mod. *Then:*

(1) H is a generator in $\mathcal{M}^H$.

(2) If R is artinian then T is projective as an R-module.

(3) If R is QF then T and T^{coH} are faithfully flat R-modules.

Proof. (1) Clearly $T = \mathrm{Tr}(P, H^*)$ and $T = T^2$ and $TP = P$ (by [16, 18.7]). By [17, 2.6 and Corollary 2.7], T is a generator in $\mathcal{M}^H$ (as right H-comodule) and by the Fundamental Theorem, H generates T (as right Hopf module). So H is a generator in $\mathcal{M}^H$.

(2) Since R is artinian and H is projective as an R-module, H^* is also a projective R-module. By [17, 5.3], H^*/T is flat as right H^*-module, and hence is a direct limit of projective H^*-modules which are also projective R-modules. Therefore H^*/T is projective as an R-module and so is T.

(3) Now assume R to be QF. As a faithful R-module H is a generator in R-Mod. Since T generates H it also generates R and both H and T are faithfully flat. From $T^{coH} \otimes_R H \simeq T$ we conclude that T^{coH} is also a faithfully flat R-module. $\square$

Although in general left semiperfect coalgebras need not be right semiperfect, the above proposition implies that for Hopf algebras over QF rings these two notions are equivalent.

3.6 Corollary. *Let H be a right semiperfect Hopf algebra over a QF ring R. Then:*

(1) H is cogenerated by H^* as left H^*-module.

(2) H is left semiperfect as coalgebra and $\mathrm{Rat}({}_{H^*}H^*) = \mathrm{Rat}(H^*_{H^*})$.

Proof. (1) Let $T := \mathrm{Rat}({}_{H^*}H^*)$. For any $t \in T^{coH}$, the map $H \mapsto H^*,\ b \mapsto t \leftharpoondown_s b$, is a left H^*-morphism. Since $T^{coH} \otimes b \neq 0$, we know by the isomorphism

$$T^{coH} \otimes_R H \simeq T,\ t \otimes b \mapsto t \leftharpoondown_s b,$$

that for any $b \in H$ there exists some $t \in T^{coH}$ such that $t \leftharpoondown_s b \neq 0$. Hence H is cogenerated by H^*.

(2) As shown in 2.9, (1) implies that H is left semiperfect. Now it follows by 2.7 that the left and right trace ideals coincide. □

Before listing characterizations of semiperfect Hopf algebras we prove a technical lemma which generalizes the uniqueness theorem of Sullivan [12] to Hopf algebras over QF rings. For this we adapt the proof given in [4, Theorem 3.3] (see also [11]).

3.7 Lemma. *Let H be a right semiperfect Hopf algebra over a QF ring R and $T := \mathrm{Rat}({}_{H^*}H^*)$.*

(1) For every $M \in \mathcal{M}^H$ which is finitely generated as R-module,

$$\mathrm{length}_R(\mathrm{Hom}_{H^*}(H, M)) \leq \mathrm{length}_R(M).$$

(2) In particular, $T^{coH} = R\chi \simeq R \simeq R$, for some $\chi \in T^{coH}$.

(3) There exists $t \in T$ for which $t(1_H) = 1_R$.

(4) For every $\chi \in T^{coH}$ such that $T^{coH} = R\chi$, there exists some $\bar{h} \in H$ such that $\chi \leftharpoondown_s \bar{h}(1_H) = 1_R$.

Proof. (1) By 3.5(3), R is a direct summand of T^{coH}. This implies by the Fundamental Theorem 3.1 that $H \simeq R \otimes_R H$ is a direct summand of $T^{coH} \otimes_R H \simeq T$ in $\mathcal{M}_H^H$ and hence also in $\mathcal{M}^H$. So we have an epimorphism $\mathrm{Hom}_{H^*}(T, M) \to \mathrm{Hom}_{H^*}(H, M)$.

Under the given conditions we know from 2.8 that

$$M \simeq \mathrm{Hom}_{H^*}(H^*, M) \simeq \mathrm{Hom}_{H^*}(T, M).$$

From this the assertion follows.

(2) Considering R as a right H-comodule by $R \to R \otimes_R H, r \mapsto r \otimes 1$, we conclude from (1) that $\mathrm{length}_R(\mathrm{Hom}_{H^*}(H, R)) \leq \mathrm{length}_R(R)$. Since by 3.6, $T = \mathrm{Rat}(H^*_{H^*})$, we know from 3.4 that $\mathrm{Hom}_{H^*}(H, R) = T^{coH}$ and hence we have $\mathrm{length}_R(T^{coH}) \leq \mathrm{length}_R(R)$.

Since R is a direct summand of T^{coH} this implies $R \simeq T^{coH}$.

(3) By 2.6, for any $N \in \mathcal{M}^H$, the canonical map $T \otimes_{C^*} N \to N$ is an isomorphism. In particular

$$T \otimes_R R \to R, \quad t \otimes r \mapsto t \rightharpoonup r = rt(1),$$

is an isomorphism. Therefore there exists $t_1, \dots, t_n \in T$ and $r_1, \dots, r_n \in R$, such that

$$1_R = \sum_{i=1}^{n} t_i \rightharpoonup r_i = \sum_{i=1}^{n} r_i t_i(1_H) = [\sum_{i=1}^{n} r_i t_i](1_H).$$

Hence $t := \sum_{i=1}^{n} r_i t_i \in T$ and $t(1_H) = 1_R$.

(4) By (2), there exists $\chi \in T^{coH}$ such that $T^{coH} = R\chi \simeq R$. By the Fundamental Theorem 3.1, the map

$$T^{coH} \otimes_R H \to T, \quad \chi \otimes h \mapsto \chi \leftharpoondown_s h,$$

is an isomorphism in $\mathcal{M}_H^H$. Thus there exists $\bar{h} \in H$ such that $\chi \leftharpoondown_s \bar{h} = t$. $\square$

It was shown in Radford [9, Proposition 2] that for semiperfect Hopf algebras over fields the antipode is bijective and his proof was simplified in Calinescu [5]. Applying our previous results we can essentially follow these ideas to prove the corresponding result for Hopf algebras over QF rings.

3.8 Proposition. *Let H be a (right) semiperfect Hopf algebra over a QF ring R. Then the antipode S of H is bijective.*

Proof. Put $T := \mathrm{Rat}({}_{H^*}H^*)$ and let $T^{coH} = R\chi$ (by 3.7).

To prove that S is injective assume $S(a) = 0$ for some $a \in H$. Then

$$R \otimes a \simeq T^{coH} \otimes a \simeq T^{coH} \leftharpoondown_s a = S(a) \rightharpoondown T^{coH} = 0,$$

this implies $a = 0$ and hence S is injective.

Now assume $S(H) \neq H$. Since $S(H)$ is a subcoalgebra we may consider it as left subcomodule of H. Then $0 \neq H/S(H) \in {}^H\mathcal{M}$ and hence there is a non-zero morphism

$$\omega : H/S(H) \to E(U) \text{ in } {}^H\mathcal{M},$$

for some simple object U with injective hull $E(U)$ in ${}^H\mathcal{M}$.

R being a cogenerator in R-Mod, we have an R-morphism $\alpha : E(U) \to R$ with $\alpha \circ \omega \neq 0$. Composing this with the canonical projection $\pi : H \to H/S(H)$, we have a non-zero R-morphism

$$\lambda := \alpha \circ \omega \circ \pi : H \to R,$$

and $\mathrm{Ke}\,\lambda \supset N \supset S(H)$, where $\mathrm{Ke}\,\omega = N/S(H)$. By definition $N \subset H$ is a left subcomodule and H/N is finitely R-generated (since $E(U)$ is). So by 1.4, $\lambda \in T$ and there exists some $\tilde{h} \in H$ such that $\lambda = \chi \leftharpoondown_s \tilde{h}$.

By construction, $\lambda(S(H)) = \chi \leftharpoondown_s \tilde{h}(S(H)) = 0$. So for any $h \in H$,

$$0 = \chi \leftharpoondown_s \tilde{h}(S(h)) = \chi(S(h)S(\tilde{h})) = \chi(S(\tilde{h}h)) = \chi \circ S(\tilde{h}h),$$

and so $\chi \circ S(\tilde{h}H) = 0$.

It is straightforward to prove that for the left integral χ, the composition $\chi \circ S$ is a right integral and hence $\chi \circ S(H^* \rightharpoonup \tilde{h}H) = 0$.

Since H is a progenerator in $\mathcal{M}_H^H$ (see 3.1(2)), the subbimodule $H^* \rightharpoonup \tilde{h}H \subset H$ is of the form IH, for some ideal $I \subset R$, and we have

$$0 = \chi \circ S(H^* \rightharpoonup \tilde{h}H) = \chi \circ S(IH) = I\chi \circ S(H).$$

As we have seen in 3.7, there exists $\bar{h} \in H$ with $\chi \circ S(\bar{h}) = 1_R$. This implies $I = 0$ and $\tilde{h}H \subset IH = 0$, i.e., $\tilde{h} = 0$. This contradicts the fact that by construction $0 \neq \lambda = \chi \leftharpoondown_s \tilde{h}$. $\square$

We are now in a position to characterize semiperfect Hopf algebras in various ways.

3.9 Theorem. *Let H be a Hopf algebra over a QF ring R and $T := \mathrm{Rat}({}_{H^*}H^*)$. Then the following are equivalent:*

(a) H is a right semiperfect coalgebra;

(b) H is an s-rational object in ${}^H\mathcal{M}$;

(c) T is a faithful and flat R-module;

(d) T^{coH} is a faithful and flat R-module;

(e) $T^{coH} = R\chi \simeq R$, for some $\chi \in T^{coH}$;

(f) T is a projective generator in $\mathcal{M}^H$;

(g) T is a flat R-module and the injective hull of R in ${}^H\mathcal{M}$ is finitely generated as R-module;

(h) H is cogenerated by H^* as left H^*-module (left QcF);

(i) H is left co-Frobenius;

(j) H is projective in $\mathcal{M}^H$;

(k) H is a projective generator in $\mathcal{M}^H$;

(l) H is a left semiperfect coalgebra.

Of course the left side versions of (a)*-*(i) *are also equivalent.*

Proof. $(a) \Leftrightarrow (b)$ is shown in 2.6.

$(a) \Rightarrow (c)$ We know from 2.6(l) that $\mathcal{M}^H$ has a generator which is projective in H^*-Mod. So the assertion follows from 3.5.

$(a) \Rightarrow (g)$ As mentioned above, 3.5 applies and so T_R is projective, and by 2.6, the injective hull of any simple comodule in ${}^H\mathcal{M}$ is finitely generated.

$(c) \Leftrightarrow (d)$ Recall that H_R is faithful and projective, and R is QF. So H_R is a generator in R-Mod and hence is faithfully flat. By the Fundamental Theorem we have $T \simeq T^{coH} \otimes H$. Hence T_R is faithfully flat if and only if T^{coH} is so.

$(d) \Rightarrow (h)$ For any $t \in T^{coH}$, the map $H \to H^*$, $b \mapsto t \leftharpoondown_s b$, is a left H^*-morphism. Since $T^{coH} \otimes b \neq 0$, we know by the isomorphism

$$T^{coH} \otimes_R H \to T,\ t \otimes b \mapsto t \leftharpoondown_s b,$$

that for any $b \in H$ there exists some $t \in T^{coH}$ such that $t \leftharpoondown_s b \neq 0$. Hence H is cogenerated by H^*.

$(a) \Rightarrow (e)$ is shown in 3.7(2).

$(e) \Rightarrow (i)$ This follows from the fact that $H \to H^*$, $b \mapsto \chi \leftharpoondown_s b$, is a monomorphism.

$(i) \Rightarrow (h)$ is trivial and $(h) \Rightarrow (j) \Rightarrow (l)$ are clear by 3.6 and 2.9.

$(g) \Rightarrow (c)$ Let $E(R)$ denote the injective hull of R in ${}^H\mathcal{M}$. Assume it is finitely generated as R-module. Then $E(R)^*$ is projective and cogenerated by T; since $R \subset E(R)^*$ we have that R is cogenerated by T. Hence T is a faithful R-module.

$(f) \Rightarrow (a)$ and $(k) \Rightarrow (a)$ are clear in view of 2.6.

$(a) \Rightarrow (f)$ Since $(a) \Rightarrow (l)$ we obtain from 2.7 that T is a ring with enough idempotents. From 2.6 and [16, 49.1] we know that T is a projective generator in $\mathcal{M}^H = T$-Mod.

$(a) \Rightarrow (k)$ The implications $(a) \Rightarrow (j)$ and $(a) \Rightarrow (l)$ imply that H is projective as left and right comodule. So by 2.9 and 2.11, H is a projective generator in $\mathcal{M}^H$ (and ${}^H\mathcal{M}$).

$(l) \Rightarrow (a)$ is clear by symmetry. □

Over a field every nonzero vector space is faithfully flat and so we obtain from 3.9:

3.10 Corollary. *For a Hopf algebra H over a field R and $T := \mathrm{Rat}({}_{H^*}H^*)$, the following are equivalent:*

(a) *H is a right semiperfect coalgebra;*

(b) $T \neq 0$;

(c) $T^{coH} \neq 0$;

(d) T^{coH} *is one-dimensional over* R;

(e) R *is an s-rational object in* $\mathcal{M}^H$;

(f) *there exists a (simple) s-rational object in* $\mathcal{M}^H$;

(g) *the injective hull of* R *in* ${}^H\mathcal{M}$ *is finite dimensional;*

(h) H *is cogenerated by* H^* *as left* H^**-module;*

(i) H *is projective in* $\mathcal{M}^H$ *(left QcF);*

(j) H *is left co-Frobenius;*

(k) H *is a projective generator in* $\mathcal{M}^H$;

(l) H *is a left semiperfect coalgebra.*

The left side versions of (a)-(k) are also equivalent.

If these conditions are satisfied the antipode of H *is bijective.*

Some of these equivalences appear in Lin [7, Theorem 3]. The characterization of these algebras by (g) is given in Sullivan [12, Theorem 1] and for *affine group schemes* it is shown in Donkin [6]. The one-dimensionality of T^{coH} (property (d)) was first proved in [12, Theorem 2] and another proof is given in [4].

Prof. Masuoka drew our attention to the following consequences of the preceding corollary (see also [4, Corollary 2.2]).

3.11 Corollary. *The coradical of an infinite dimensional co-Frobenius Hopf algebra* H *over a field is infinite dimensional. In particular, for a non-zero Lie algebra* $\mathcal{G}$*, the enveloping algebra* $U(\mathcal{G})$ *is not co-Frobenius.*

3.12 Corollary. *Let* H *be a co-commutative Hopf algebra over a field* R*.*

(i) H *is co-Frobenius if and only if the irreducible component* H_1 *of* H *(containing* 1_H*) is finite dimensional over* R*.*

(ii) *If* $char(R) = 0$ *then* H *is co-Frobenius if and only if* H *is co-semisimple.*

Proof. (i) By 1.6, the injective envelope of $R1_H$ in $\mathcal{M}^H$ coincides with H_1.

(ii) Clearly any cosemisimple Hopf algebra is semiperfect and hence co-Frobenius.

Now assume $char(R) = 0$ and H to be co-Frobenius. Let $\pi \in T \subset H^*$ denote the idempotent which corresponds to the canonical projection $H \to H_1$.

Let $\lambda \in T^{coH}$ and $h \in H$ such that $\lambda \leftharpoondown_s h = \pi$. Then for every $\gamma \in H^*$ we have (by 3.4 (e))

$$\lambda \leftharpoondown_s (\gamma \rightharpoonup h) = \gamma * (\lambda \leftharpoondown_s h) = \gamma * \pi.$$

In particular we get $\lambda \leftharpoondown_s (\pi \rightharpoonup h) = \pi * \pi = \pi$ so that $h = \pi \rightharpoonup h \in H_1$. Hence we can write

$$\Delta(h) = \sum_{i=1}^{n} h_i \otimes \tilde{h}_i \in H_1 \otimes_R H_1,$$

where the $\tilde{h}_1, \ldots, \tilde{h}_n \in H_1$ are linearly independent over R. By definition, for any $x \in H$,

$$\pi \rightharpoonup x = (\lambda \leftharpoondown_s h) \rightharpoonup x = \sum_{i=1}^{n} \lambda(xS(h_i))\tilde{h}_i,$$

and for $a \in H_1$ the equality $a = \pi \rightharpoonup a$ implies

$$a = \sum_{i=1}^{n} \lambda(aS(h_i))\tilde{h}_i.$$

From this we see that the $\tilde{h}_1, \ldots, \tilde{h}_n$ form a basis in H_1, and for any $j \leq n$,

$$\tilde{h}_j = \sum_{i=1}^{n} \lambda(\tilde{h}_j S(h_i))\tilde{h}_i.$$

So we have $\lambda(\tilde{h}_j S(h_i)) = \delta_{ij}$ and finally

$$0 \neq n1_R = \sum_{i=1}^{n} \lambda(\tilde{h}_i S(h_i)) = \lambda(\sum_{i=1}^{n} \tilde{h}_i S(h_i)) = \lambda(\varepsilon(h)) = \lambda(1_H)\varepsilon(h).$$

Therefore $\lambda(1_H) \neq 0$ and hence H is cosemisimple (e.g., [14, Lemma 14.0.2]). □

Let us mention that assertion (ii) also holds in case H is commutative (instead of co-commutative, see Abe [1, Theorem 3.3.11]).

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