Semiperfect coalgebras over rings

Robert Wisbauer
University of Düsseldorf, Germany
e-mail: wisbauer@math.uni-duesseldorf.de

Abstract

Our investigation of coalgebras over commutative rings R is based on the close relationship between comodules over a coalgebra C and modules over the dual algebra C^*. If C is projective as an R-module the category of right C-comodules can be identified with the category $\sigma[_{C^*}C]$ of left C^*-modules which are subgenerated by C. In this context semiperfect coalgebras are described by results from module theory. Over QF rings semiperfect coalgebras are characterized by the exactness of the trace functor $\mathrm{Tr}(\sigma[_{C^*}C], -)$.

1 Introduction

Although there are many interesting examples of coalgebras over rings R, a large part of literature on the structure theory is restricted to coalgebras over fields. This is mainly due to the fact that for certain basic proofs the existence of an R-basis is needed.

Here we work with coalgebras C over any commutative ring R. A good deal of the basic definitions and properties carry over from base fields to rings nearly verbatim. For such situations we do not repeat proofs. Of course, the properties of C as an R-module will be of importance. Without any restriction on the coalgebra C we observe that the category of right C-comodules Comod-C is subgenerated by C, i.e., every right C-comodule is a subcomodule of some comodule which is generated by C. It turns out that Comod-C is a Grothendieck category if and only if $_RC$ is flat.

In the classical structure theory of C the dual algebra C^* plays an important part. To make sure that C^* is not trivial we will need the condition that $_RC$ is projective. Then there exists a dual basis for $_RC$ and this will allow to transfer proofs known for base fields. In particular we will show that in this case Comod-C can be identified with $\sigma[_{C^*}C]$, the category of left C^*-modules which are subgenerated by C.

A coalgebra C is called *right semiperfect* if every simple right comodule has a projective cover in Comod-C. If $_RC$ is projective this corresponds to the condition that in $\sigma[_{C^*}C]$ every simple module has a projective cover in $\sigma[_{C^*}C]$, a situation which was well studied in module theory. Over QF rings R we have relationships between right C-comodules and left C-comodules (via finitely presented modules). Based on this we characterize right semiperfect coalgebras C by the exactness of the trace functor $\mathrm{Tr}(\sigma[_{C^*}C], -) : C^*\text{-Mod} \to \sigma[_{C^*}C]$. In fact most of the propositions known for coalgebras over fields carry over to coalgebras over QF rings R provided $_RC$ is projective.

For introductory texts on coalgebras the reader is referred to Abe [1], Beidar [6], Kaplansky [11], Montgomery [14] and Sweedler [15]. The main references for module theory are [17] and [18].

The author is indebted to K.I. Beidar, S. Dăscălescu and J. Gómez Torrecillas for inspiring discussions on the topic.

2 Modules over algebras

Throughout this text R will denote an associative commutative ring with unit. To clarify notation we recall some basic definitions and properties of modules.

2.1 Tensor products. Let K, L, M, N be R-modules. We have the isomorphism

$$\tau: \quad M \otimes_R N \quad \to \quad N \otimes_R M, \quad m \otimes n \longmapsto n \otimes m.$$

For R-homomorphisms $f : M \to N$, $g : K \to L$, there is a unique R-linear map

$$f \underline{\otimes} g : M \otimes_R K \to N \otimes_R L, \quad m \otimes k \longmapsto f(m) \otimes g(k),$$

called the *tensor product of* f *and* g.

For a submodule $M' \subset M$ let $i : M' \to M$ denote the inclusion. Then the map

$$i \underline{\otimes} id_K : M' \otimes_R K \to M \otimes K$$

need not be injective. We call $M' \subset M$ is a K*-pure submodule* if $i \underline{\otimes} id_K$ is injective. In this case we identify $M' \otimes_R K$ with its canonical image in $M \otimes_R K$.

For example, any pure submodule $M' \subset M$ (in the sense of Cohn) is K-pure (for every R-module K), and if K is a flat R-module, then every submodule $M' \subset M$ is K-pure.

If f and g are surjective then $f \underline{\otimes} g$ is surjective and $\mathrm{Ke}\, f \underline{\otimes} g$ is the sum of the canonical images of $\mathrm{Ke}\, f \otimes K$ and $M \otimes \mathrm{Ke}\, g$.

In case $\mathrm{Ke}\, f \subset M$ is K-pure and $\mathrm{Ke}\, g \subset K$ is M-pure, we have

$$\mathrm{Ke}\, f \underline{\otimes} g = \mathrm{Ke}\, f \otimes K + M \otimes \mathrm{Ke}\, g.$$

If $M' \subset M$ and $K' \subset K$ are pure submodules, then

$$M' \otimes_R K' = M' \otimes_R K \cap M \otimes_R K'.$$

2.2 s-unital T-modules. Let T be any associative ring T (without unit). A left T-module N is called *s-unital* if $u \in Tu$ for every $u \in N$. T itself is called *left s-unital* if it is s-unital as a left T-module. From [16] (or [7]) we have the basic properties:

For any left T-module N the following are equivalent:

(a) N is an s-unital T-module;

(b) for any $n_1, \ldots, n_k \in N$, there exists $t \in T$ with $n_i = tn_i$ for all $i \le k$;

(c) for any set Λ, $N^{(\Lambda)}$ is an s-unital T-module.

2.3 Finite topology. For nonempty sets X, Y, by the *finite topology* on $\mathrm{Map}(X, Y)$ we mean the product topology, where Y is endowed with the discrete topology.

For $f \in \mathrm{Map}(X, Y)$ a basis of open neighbourhoods is given by the sets

$$\{\, g \in \mathrm{Map}(X, Y) \,|\, g(x_i) = f(x_i) \quad \text{for } i = 1, \ldots, n \,\},$$

where $\{x_1, \ldots, x_n\}$ ranges over the finite subsets of X.

For subsets $U \subset V$ of $\mathrm{Map}(X, X)$ we say that U is X*-dense in* V if it is dense in the finite topology in $\mathrm{Map}(X, X)$, i.e., for any $v \in V$ and $x_1, \ldots, x_n \in X$ there exists $u \in U$ such that $u(x_i) = v(x_i)$ for $i = 1, \ldots, n$.

Let A be an associative R-algebra. For an A-module M, the finite topology on $\mathrm{Map}(M,M)$ induces the finite topology on $\mathrm{End}({}_RM)$. We may characterize s-unital modules over an ideal in the following way:

2.4 Dense ideals. *Let N be a faithful left A-module. For an ideal $T \subset A$ the following are equivalent:*

(a) N is an s-unital T-module;

(b) T is N-dense in A.

By $\sigma[M]$ we denote the full subcategory of A-Mod whose objects are submodules of M-generated modules. $N \in \sigma[M]$ is called a *subgenerator* if $\sigma[M] = \sigma[N]$.

2.5 The trace functor. For any $N, M \in A$-Mod the *trace of M in N* is defined as

$$\mathrm{Tr}(M,N) = \sum \{\mathrm{Im}\, f \mid f \in \mathrm{Hom}_A(M,N)\},$$

and we define the *trace of $\sigma[M]$ in N* by

$$\mathcal{T}^M(N) := \mathrm{Tr}(\sigma[M],N) = \sum \{\mathrm{Im}\, f \mid f \in \mathrm{Hom}_A(K,N),\ K \in \sigma[M]\}.$$

If G is a generator in $\sigma[M]$ then obviously $\mathcal{T}^M(N) = \mathrm{Tr}(G,N)$.

Since $\sigma[M]$ is a subclass of A-Mod which is closed under direct sums, factor modules and submodules (*hereditary pretorsion class*), $\mathcal{T}^M(N)$ is the largest submodule of N which belongs to $\sigma[M]$ and we have a left exact functor

$$\mathcal{T}^M : A\text{-Mod} \to \sigma[M], \quad N \mapsto \mathcal{T}^M(N),$$

which is right adjoint to the inclusion functor $\sigma[M] \to A$-Mod (see [17, 45.11]).

The trace of $\sigma[M]$ in A, $\mathcal{T}^M(A) \subset A$, is an ideal of A called the *trace ideal*. It can be used to describe conditions on the class $\sigma[M]$ (see [19]):

2.6 $\mathcal{T}^M$ as exact functor. *Putting $T := \mathcal{T}^M(A)$ the following are equivalent:*

(a) The functor $\mathcal{T}^M : A\text{-Mod} \to \sigma[M]$ is exact;

(b) $\sigma[M]$ is closed under extensions and the class of A-modules X with $\mathcal{T}^M(X) = 0$ is closed under factor modules;

(c) for every $N \in \sigma[M]$, $TN = N$;

(d) M is an s-unital T-module;

(e) for every $N \in \sigma[M]$, the canonical map $\varphi_N : T \otimes_A N \to N$ is an isomorphism;

(f) T is idempotent and a generator in $\sigma[M]$;

(g) $TM = M$ and A/T is flat as a right A-module;

(h) $T/\mathrm{An}(M)$ is an M-dense subring of $A/\mathrm{An}(M)$.

2.7 Corollary. *Suppose that $\sigma[M]$ has a generator which is projective in A-Mod. Then $\mathcal{T}^M : A\text{-Mod} \to \sigma[M]$ is an exact functor.*

The importance of the M-density determined by the finite topology in $\mathrm{End}_A(M)$ (see 2.3) for our investigations is derived from the following facts (see [17, 15.7, 15.8]).

2.8 Density properties. *Let M be a left A-module and $S = \mathrm{End}_A(M)$.*

(1) *For any subring $A \subset B \subset \mathrm{End}_R(M)$, $\sigma[_AM] = \sigma[_BM]$ if and only if A is M-dense in B.*

(2) *If M is a generator or a (weak) cogenerator in $\sigma[M]$, then A is M-dense in $\mathrm{End}(M_S)$.*

Any module M over an R-algebra A is also an R-module. We are interested in the interplay between the properties of M as an R- and an A-module.

2.9 (A,R)-finite modules. A module M over the R-algebra A is said to be (A,R)-*finite* if every finitely generated A-submodule of M is finitely generated as R-module. $\sigma[M]$ is said to be (A,R)-*finite* if every module in $\sigma[M]$ is (A,R)-finite.

(1) *The following are equivalent:*
  (a) *$\sigma[M]$ is (A,R)-finite;*
  (b) *every finitely generated A-module in $\sigma[M]$ is a finitely generated R-module;*
  (c) *$M^{(\mathbb{N})}$ is (A,R)-finite.*

(2) *If R is noetherian and M is (A,R)-finite, then $\sigma[M]$ is (A,R)-finite.*

(3) *Assume $\sigma[M]$ to be (A,R)-finite. Then $\sigma[M] = A/\mathrm{An}(M)$-Mod if and only if $A/\mathrm{An}(M)$ is a finitely generated R-module.*

(4) *Let $\sigma[M]$ be (A,R)-finite.*
  (i) *If R is a perfect ring, then every module in $\sigma[M]$ has dcc on finitely generated A-submodules.*
  (ii) *If R is noetherian then every module in $\sigma[M]$ is locally noetherian.*
  (iii) *If R is artinian then every finitely generated module in $\sigma[M]$ has finite length.*

2.10 Relative notions. A sequence in A-Mod

$$(*) \quad 0 \longrightarrow K \xrightarrow{f} L \xrightarrow{g} N \longrightarrow 0$$

is said to be (A,R)-*exact* if it is exact in A-Mod and splitting in R-Mod. In this case f is called an *R-split A-monomorphism* and g is an *R-split A-epimorphism.*

Let M, P, Q be A-modules. P is called (M,R)-*projective* if $\mathrm{Hom}_A(P,-)$ is exact with respect to all (A,R)-exact sequences in $\sigma[M]$. Q is called (M,R)-*injective* if $\mathrm{Hom}_A(-,Q)$ is exact with respect to all (A,R)-exact sequences in $\sigma[M]$.

Let U be a submodule of the A-module M. A submodule $V \subset M$ is called a *supplement* of U in M if V is minimal with respect to the property $U + V = M$.

A module $N \in \sigma[M]$ is said to be *semiperfect in* $\sigma[M]$ if every factor module of N has a projective cover in $\sigma[M]$. By [17, 42.5, 42.12], a projective module $P \in \sigma[M]$ is supplemented if and only if it is semiperfect in $\sigma[M]$.

For a module M, we call $\sigma[M]$ a *semiperfect category* if every simple module in $\sigma[M]$ has a projective cover in $\sigma[M]$.

2.11 Semiperfect categories. *For an A-module M the following are equivalent:*

(a) *$\sigma[M]$ is semiperfect;*

(b) *$\sigma[M]$ has a generating set of supplemented (local) projective modules;*

(c) *in $\sigma[M]$ every finitely generated module has a projective cover;*

If $\sigma[M]$ is (A,R)-finite and R is perfect then (a)-(c) are equivalent to:

(d) $\sigma[M]$ has a generating set of finitely generated projective modules.

Proof. $(a) \Rightarrow (b)$ The projective covers of all simples in $\sigma[M]$ are local and form a generating set of $\sigma[M]$ (by [17, 18.5]). Notice that local modules are supplemented.

$(b) \Rightarrow (c)$ Any finite direct sum of supplemented modules is supplemented. Hence for every finitely generated $N \in \sigma[M]$, there exists an epimorphism $P \to N$ with some supplemented projective module $P \in \sigma[M]$. Then every factor module of P has a projective cover in $\sigma[M]$ and so does N.

$(c) \Rightarrow (d)$ is obvious.

$(d) \Rightarrow (c)$ Let P be a finitely generated projective module in $\sigma[M]$. By 2.9, P has dcc on finitely generated submodules and hence it is supplemented. □

3 Coalgebras and comodules

In this section we recall some basic definitions for coalgebras and comodules.

3.1 Coalgebras. An R-module C is an *R-coalgebra* if there is an R-linear map

$$\Delta : C \to C \otimes_R C,$$

called the *comultiplication* of C.

C is *coassociative*, resp. *cocommutative*, if the following diagrams commute:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes_R C \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle id\otimes\Delta} \\ C \otimes_R C & \xrightarrow{\Delta\otimes id} & C \otimes_R C \otimes_R C. \end{array} \quad , \quad \begin{array}{ccc} & & C \otimes_R C \\ & {\scriptstyle\Delta}\nearrow & \\ C & & \downarrow \tau \\ & {\scriptstyle\Delta}\searrow & \\ & & C \otimes_R C\,. \end{array}$$

An R-linear map $\varepsilon : C \to R$ is called *counit* if it yields commutative diagrams

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes_R C \\ & \simeq\searrow & \downarrow{\scriptstyle id\otimes\varepsilon} \\ & & C \otimes_R R \end{array} \quad \text{and} \quad \begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes_R C \\ & \simeq\searrow & \downarrow{\scriptstyle \varepsilon\otimes id} \\ & & R \otimes_R C\,. \end{array}$$

3.2 Duals of coalgebras. Let C be an R-coalgebra and put $C^* = \mathrm{Hom}_R(C, R)$. Then there exists an R-linear map

$$C^* \otimes_R C^* \longrightarrow C^*\,, \quad f \otimes g \longmapsto [C \xrightarrow{\Delta} C \otimes_R C \xrightarrow{f\otimes g} R \otimes_R R \simeq R],$$

making C^* an R-algebra. If C is coassociative (cocommutative) then C^* is associative (commutative). A counit ε of C is the unit of C^*.

Henceforth we will always assume C to be a coassociative R-algebra with counit and so C^* will be an associative R-algebra with unit.

3.3 Coideals and sub-coalgebras. A C-pure R-submodule $D \subset C$ is called a

left (right) coideal if $\quad \Delta(D) \subset C \otimes_R D$ (resp., $\subset D \otimes_R C$),

coideal if $\quad \Delta(D) \subset C \otimes_R D + D \otimes_R C$.

If C has a counit $\varepsilon : C \to R$, for D to be a coideal we also demand $\varepsilon(D) = 0$.
A pure R-submodule $D \subset C$ is called a *sub-coalgebra* if $\Delta(D) \subset D \otimes_R D$.

3.4 Comodules. An R-module M is called a *right C-comodule* if there exists an R-linear map (the *comodule structure map*) $\varrho : M \to M \otimes_R C$ such that the following diagrams commute,

$$\begin{array}{ccc} M & \xrightarrow{\varrho} & M \otimes_R C \\ \downarrow \varrho & & \downarrow id\underline{\otimes}\Delta \\ M \otimes_R C & \xrightarrow{\varrho\underline{\otimes} id} & M \otimes_R C \otimes_R C \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\varrho} & M \otimes_R C \\ & \simeq \searrow & \downarrow id\underline{\otimes}\varepsilon \\ & & M \otimes_R R \end{array}$$

A C-pure R-submodule $N \subset M$ is called *C-sub-comodule* if $\varrho(N) \subset N \otimes_R C$ which means that

$$\varrho|_N : N \to N \otimes_R C$$

makes N a right C-comodule.

Left C-comodules are defined in a symmetric way. Clearly C is a right and left C-comodule, and right (left) sub-comodules of C are right (left) coideals.

3.5 Comodule morphisms. Let $\varrho : M \to M \otimes_R C$ and $\varrho' : M' \to M' \otimes_R C$ be right C-comodules. An R-linear map $f : M \to M'$ is called a *comodule morphism* if the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{f} & M' \\ \downarrow\varrho & & \downarrow\varrho' \\ M \otimes_R C & \xrightarrow{f\underline{\otimes} id} & M' \otimes_R C. \end{array}$$

The set of all comodule morphisms $M \to M'$ is an abelian group which we denote by $\mathrm{Com}_C(M, M')$.

3.6 Category of comodules. The right (left) C-comodules as objects and the comodule morphisms as morphisms form a category which we denote by Comod-C (C-Comod). These are additive categories.

The following observations are easily verified.

3.7 Factor comodules.

(1) *Let $K \subset M$ be a sub-comodule of the right C-comodule M. Then there is a unique right C-comodule structure on the factor module M/K which makes the canonical projection $p : M \to M/K$ a comodule morphism.*

(2) *Let $f : M \to M'$ be a comodule morphism.*
If $\mathrm{Ke}\, f \subset M$ *is C-pure as R-submodule, then it is a C-sub-comodule of M.*
If $\mathrm{Im}\, f \subset M'$ *is C-pure as R-submodule, then it is a C-sub-comodule of M'.*

(3) *For any sub-comodule $K \subset M$ with $K \subset$* $\mathrm{Ke}\, f$, *there exists a comodule morphism $\bar{f} : M/K \to M'$ such that $\bar{f} \circ p = f$.*

Notice that by (2) the kernel of a comodule morphism is only a (sub-) comodule under certain conditions. In particular these are satisfied if C_R is flat.

3.8 Coproduct of comodules. Let $\{\varrho_\lambda : M_\lambda \to M_\lambda \otimes C\}_\Lambda$ be a family of C-comodules. Put $M := \bigoplus_\Lambda M_\lambda$, $i_\lambda : M_\lambda \to M$ the canonical inclusions, and consider the linear maps

$$M_\lambda \xrightarrow{\varrho_\lambda} M_\lambda \otimes_R C \subset M \otimes_R C.$$

By the properties of coproducts of R-modules there exists a unique map

$$\varrho : M \to M \otimes_R C \text{ such that } \varrho \circ i_\lambda = \varrho_\lambda.$$

It is straightforward to verify that this makes M a C-comodule and the $i_\lambda : M_\lambda \to M$ are comodule morphisms. (M, i_λ) is the *coproduct (direct sum)* of the comodules M_λ in Comod-C.

3.9 Free comodules. A coproduct $C^{(\Lambda)}$, Λ any set, is called a *free C-comodule.* Factor comodules of free C-comodules are called *C-generated comodules.*

For an R-algebra A and any R-module N, $A \otimes_R N$ is an A-module. It is easily shown that the corresponding properties hold true for comodules.

3.10 Comodules and tensor products. *Let M be a right C-comodule.*

(1) For any R-module X, $X \otimes_R M$ is a right C-comodule by

$$id \underline{\otimes} \varrho : X \otimes_R M \longrightarrow X \otimes_R M \otimes_R C \,,$$

and for any R-module morphism $f : X \to_R X'$, $f \underline{\otimes} id : X \otimes_R M \to X' \otimes_R M$ is a C-comodule morphism.

(2) In particular, $X \otimes_R C$ is a right C-comodule by

$$id \underline{\otimes} \Delta : X \otimes_R C \longrightarrow X \otimes_R C \otimes_R C \,,$$

and $f \underline{\otimes} id : X \otimes_R C \to X' \otimes_R C$ is a C-comodule morphism.

(3) For any index set Λ, the right C-comodule $R^{(\Lambda)} \otimes C$ is isomorphic to the free right C-comodule $C^{(\Lambda)}$.

The close connection between the category of comodules to the categories of type $\sigma[M]$ is based on the following observation.

3.11 Subgenerator for comodules. *Let M be a right C-comodule.*

(1) The structure map $\varrho : M \to M \otimes_R C$ is a comodule morphism, where $M \otimes_R C$ is considered as a C-comodule (by 3.10(2)).

(2) M is a sub-comodule of a C-generated right C-comodule.

By (2), every C-comodule is a submodule of a C-generated module and hence C is a subgenerator in Comod-C.

Proof. (1) This follows from properties of the structure map ϱ (in 3.4).

(2) There is an R-epimorphism $h : R^{(\Lambda)} \to M$ (Λ some set) and by 3.10(2),

$$h \underline{\otimes} id : R^{(\Lambda)} \otimes_R C \to M \otimes_R C$$

is a surjective C-comodule morphism.

By definition, ϱ splits as R-linear map and hence $\mathrm{Im}\, \varrho \simeq M$ is a pure R-submodule of $M \otimes_R C$. So by 3.7, M is isomorphic to a sub-comodule of the C-generated comodule $M \otimes_R C$. □

By standard arguments we obtain the important

3.12 Hom-Com **relations.** *Let M be a right C-comodule and X an R-module.*

(1) The R-linear map

$$\mathrm{Com}_C(M, X \otimes_R C) \to \mathrm{Hom}_R(M, X), \quad f \mapsto (id\underline{\otimes}\varepsilon) \circ f,$$

is an isomorphism with inverse map $h \mapsto (h\underline{\otimes}id) \circ \varrho$.

(2) For every right C-comodule M', the R-linear map

$$\mathrm{Com}_C(X \otimes_R M, M') \to \mathrm{Hom}_R(X, \mathrm{Com}_C(M, M')), \quad g \mapsto [x \mapsto g \circ (x \otimes -)],$$

is an isomorphism with inverse map $h \mapsto [x \otimes m \mapsto h(x)(m)]$.

A special case of the R-linear in 3.12(1) yields the

3.13 Endomorphism ring of C. *The map*

$$\mathrm{Com}_C(C, C) \to C^*, \quad f \mapsto \varepsilon \circ f,$$

is an R-algebra anti-isomorphism with inverse map $h \mapsto (h\underline{\otimes}id) \circ \Delta$.

Hence the algebra of left comodule endomorphisms of C is anti-isomorphic to the dual algebra C^*.

Writing endomorphisms on the right, C becomes a right C^*-module.

The isomorphisms considered in 3.12 are related to properties of

3.14 Functors between R-Mod and Comod-C.

(1) The functor $-\otimes_R C : R\text{-Mod} \to \text{Comod-}C$
is right adjoint to the forgetful functor Comod-$C \to R$-Mod *by the functorial isomorphisms (for $M \in$ Comod-C, $X \in R$-Mod),*

$$\mathrm{Com}_C(M, X \otimes_R C) \to \mathrm{Hom}_R(M, X).$$

(2) For any $M \in$ Comod-C, the functor $-\otimes_R M : R\text{-Mod} \to \text{Comod-}C$
is left adjoint to the functor

$$\mathrm{Com}_C(M, -) : \text{Comod-}C \to R\text{-Mod}$$

by the isomorphism (for $M' \in$ Comod-C, $X \in R$-Mod),

$$\mathrm{Com}_C(X \otimes_R M, M') \to \mathrm{Hom}_R(X, \mathrm{Com}_C(M, M'))\,.$$

We state some immediate consequences of the existence of these functors.

3.15 Corollary.

(1) For any $X \in R$-Mod, $X \otimes_R C$ is (C, R)-injective.
In particular, C is (C, R)-injective.

(2) If X is injective in R-Mod, then $X \otimes_R C$ is injective in Comod-C.

(3) If M_R is flat and M' is injective in Comod-C, *then* $\mathrm{Com}_C(M, M')$ *is injective in* R-Mod.

(4) Comod-C *is a Grothendieck category if and only if C is flat as an R-module.*

Proof. (1)-(3) are direct consequences of (1) and (2) in 3.14, respectively.

(4) Assume ${}_RC$ is flat. Then every R-submodule of any $M \in$ Comod-C is C-pure and hence by 3.7 the category Comod-C has kernels and cokernels. It also has arbitrary coproducts (by 3.8) and hence it is a Grothendieck category.

Now suppose that Comod-C is a Grothendieck category. Any functor between abelian categories which is right adjoint to a covariant functor is left exact (e.g., [17, 45.6]). Hence $-\otimes_R C$ is an exact functor and so ${}_RC$ is flat. □

4 C-comodules and C^*-modules

Let C be a coalgebra with counit and $C^* = \mathrm{Hom}_R(C,R)$. We investigate the C^*-module structure of C-comodules over C. In particular if ${}_RC$ is projective we have a close correspondence between these structures and we can apply ordinary module theory to describe properties of comodules.

4.1 C-comodules and C^*-modules. *Let $\varrho : M \to M \otimes_R C$ be any R-linear map. Put*

$$\psi : C^* \otimes_R M \to M, \quad f \otimes m \mapsto \vartheta \circ (id\underline{\otimes} f) \circ \varrho(m).$$

(1) If ϱ is a right C-comodule structure map, then M is a left C^*-module by ψ.

(2) Assume C is projective as R-module and M is a left C^*-module by ψ. Then
– M is a right C-comodule by ϱ and
– every C^*-submodule of M is a C-sub-comodule.

Proof. Put $f \cdot m := \psi(f \otimes m)$. A condition for ϱ to be a comodule structure map is

$$(*) \qquad (\varrho\underline{\otimes} id) \circ \varrho = (id\underline{\otimes}\Delta) \circ \varrho.$$

(1) This is shown by straightforward computation.

(2) Let $\{p_i, d_i \,|\, p_i \in C^*,\, d_i \in C\}_I$ a dual basis of C. Assume M is a C^*-module by ψ, i.e., $f{*}g \cdot m = f \cdot (g \cdot m)$. By definition,

$$(id\underline{\otimes} f\underline{\otimes} g) \circ (id\underline{\otimes}\Delta) \circ \varrho(m) = (id\underline{\otimes} f\underline{\otimes} g) \circ (\varrho\underline{\otimes} id) \circ \varrho(m),$$

which can be written as

$$(**) \quad (id\underline{\otimes} f\underline{\otimes} g) \circ [(id\underline{\otimes}\Delta) \circ \varrho(m) - (\varrho\underline{\otimes} id) \circ \varrho(m)] = 0\,.$$

Put $u := (id\underline{\otimes}\Delta) \circ \varrho(m) - (\varrho\underline{\otimes} id) \circ \varrho(m)$. Using the dual basis we can write

$$u = \sum_{i,j} u_{ij} \otimes d_i \otimes d_j, \qquad \text{where } u_{ij} \in M.$$

By $(**)$ we have for each $k, l \in I$,

$$0 = (id \otimes p_k \otimes p_l)u = \sum_{i,j} u_{ij} p_k(d_i) p_l(d_j)\,.$$

Inserting the sums $d_i = \sum_k p_k(d_i) d_k$, and $d_j = \sum_l p_k(d_j) d_l$, we have

$$u = \sum_{i,j} u_{ij} \otimes \sum_k p_k(d_i) d_k \otimes \sum_l p_k(d_j) d_l = \sum_{k,l} \left(\sum_{i,j} u_{ij}\, p_k(d_i) p_l(j) \right) \otimes d_k \otimes d_l = 0\,.$$

From this we see that $(*)$ holds.

Now consider any C^*-submodule $X \subset M$. We have to show that $\varrho(X) \subset X \otimes C$. For $x \in X$ we can write

$$\varrho(x) = \sum_i x_i \otimes d_i, \text{ where } x_i \in M .$$

Since $C^* \cdot x \subset X$ we have for each $k \in I$, $p_k \cdot x = \sum_i x_i p_k(d_i) \in X$,

Inserting $d_i = \sum_k p_k(d_i) d_k$, we have

$$\varrho(x) = \textstyle\sum_i x_i \otimes (\sum_k p_k(d_i) d_k) = \sum_k (\sum_i x_i p_k(d_i)) \otimes d_k \in X \otimes_R C . \qquad \square$$

Not only C-comodules and C^*-modules are closely related but also their morphisms.

4.2 C-comodule and C^*-module morphisms. *Let $\varrho : M \to M \otimes_R C$ and $\varrho' : M' \to M' \otimes_R C$ be right C-comodules and $h : M \to M'$ an R-linear map.*

(1) If h is a comodule morphism, then h is a C^*-module morphism.

(2) If ${}_RC$ is projective and h is a C^*-module morphism, then h is a comodule morphism.

Proof. Recall that h is a comodule map if $\quad (*) \quad \varrho' \circ h = (h \underline{\otimes} id) \circ \varrho$.

(1) This is shown by direct computation.

(2) Suppose that h is a C^*-module morphism. Then for $f \in C^*$, $m \in M$,

$$(id \underline{\otimes} f) \circ \varrho'(h(m)) = (id \underline{\otimes} f) \circ (h \underline{\otimes} id) \varrho(m) ,$$

which we can write as

$$(id \underline{\otimes} f) \circ (\varrho'(h(m)) - (h \underline{\otimes} id) \varrho(m)) = 0 , \text{ for all } f \in C^* .$$

Let $\{p_i, d_i \,|\, p_i \in C^*, \, d_i \in C\}_I$ be a dual basis of C. Then for

$$u := \varrho'(h(m)) - (h \underline{\otimes} id) \varrho(m) = \textstyle\sum_i m_i \otimes d_i, \text{ where } m_i \in M' ,$$

we have for any $k \in I$,

$$0 = id \underline{\otimes} p_k(u) = \sum_i m_i \, p_k(d_i) .$$

Using $d_i = \sum_k p_k(d_i) d_k$, we obtain

$$u := \sum_i m_i \otimes (\sum_k p_k(d_i) d_k) = \sum_k \sum_i m_i \, p_k(d_i) \otimes d_k = 0 .$$

This proves that h is a C-comodule map. $\square$

By 4.1 and 4.2 we have an intimate relationship between the categories of C-comodules and $\sigma[{}_{C^*}C]$, the full subcategory of C^*-Mod which is subgenerated by C.

4.3 The categories Comod-C **and** $\sigma[{}_{C^*}C]$**.**

(1) Comod-C *is a subcategory of* $\sigma[{}_{C^*}C]$.

(2) If C is projective as R-module then Comod-$C = \sigma[{}_{C^*}C]$.

The above observations allow to use any notion from categories of type $\sigma[M]$ for the category of comodules, provided ${}_RC$ is a projective R-module.

A C^*-module U is *C-injective* if $\mathrm{Hom}_{C^*}(-,U)$ turns monomorphisms $0 \to K \to C$ in C^*-Mod into epimorphisms and this is equivalent to the fact that

$$\mathrm{Hom}_{C^*}(-,U) : \sigma[{}_{C^*}C] \to R\text{-Mod}$$

is an exact functor. There are enough injectives in $\sigma[{}_{C^*}C]$, i.e., every object in $\sigma[{}_{C^*}C]$ is contained in a C-injective object of $\sigma[{}_{C^*}C]$.

A C^*-module P is *C-projective* if $\mathrm{Hom}_{C^*}(P,-)$ respects epimorphism $C \to N$ in C^*-Mod. $P \in \sigma[{}_{C^*}C]$ is *projective in* $\sigma[{}_{C^*}C]$ if

$$\mathrm{Hom}_{C^*}(P,-) : \sigma[{}_{C^*}C] \to R\text{-Mod}$$

is an exact functor. A finitely generated C-projective module P is projective in $\sigma[{}_{C^*}C]$. There need not be projectives in $\sigma[{}_{C^*}C]$ even if R is a field.

As mentioned before, any coalgebra C is a left and right C-comodule. It is also a left and right C^*-module and we collect the information about these structures which are verified in an obvious way.

4.4 C^*-module structure of C.

(1) C is a faithful left and right C^*-module and a (C^*,C^*)-bimodule by the operations

$$\begin{aligned} \rightharpoonup : C^* \otimes_R C \to C, &\quad f \otimes c \mapsto f \rightharpoonup c := \vartheta \circ (id\underline{\otimes} f) \circ \Delta(c), \\ \leftharpoonup : C \otimes_R C^* \to C, &\quad c \otimes g \mapsto c \leftharpoonup g := \vartheta \circ (g\underline{\otimes} id) \circ \Delta(c)\,. \end{aligned}$$

(2) Assume ${}_RC$ is projective. Then:

(i) An R-submodule $D \subset C$ is a right (left) coideal if and only if D is a left (right) C^*-submodule.

(ii) A pure R-submodule $D \subset C$ is a sub-coalgebra if and only if D is a (C^*,C^*)-submodule.

Some properties of coalgebras can be described by certain bilinear forms and we give some basic observations which can be proved by standard methods (using dual basis arguments).

4.5 Balanced bilinear forms. *Let C be any coalgebra and $\beta : C \times C \to R$ a bilinear form. Associated to β there are R-linear maps*

$$\begin{array}{rcclcl} \bar\beta : & C \otimes_R C & \to & R, & c \otimes d & \mapsto & \beta(c,d), \\ \beta^l : & C & \to & C^*, & d & \mapsto & \beta(-,d), \\ \beta^r : & C & \to & C^*, & c & \mapsto & \beta(c,-)\,. \end{array}$$

*β is said to be C-*balanced *if*

$$\beta(c \leftharpoonup f, d) = \beta(c, f \rightharpoonup d), \textit{ for all } c,d \in C,\ f \in C^*.$$

If ${}_RC$ is projective the following are equivalent:

(a) β is C-balanced;

(b) $\beta^l : C \to C^*$ is a left C^*-homomorphismus;

(c) $\beta^r : C \to C^*$ is a right C^*-homomorphismus.

A bilinear form $\beta : C \times C \to R$ is *left non-degenerated* if β^l is injective. A family of bilinear forms $\{\beta_\lambda : C \times C \to R\}_\Lambda$ is *left non-degenerated* if $\bigcap_\Lambda \operatorname{Ke} \beta^l_\lambda = 0$.

4.6 Non-degenerated bilinear forms. *Let C be any coalgebra with $_RC$ projective.*

(1) The following are equivalent:

(a) There exists a left C^*-monomorphism $\alpha : C \to C^*$;

(b) there exists a left non-degenerated C-balanced bilinear form $\beta : C \times C \to R$.

(2) The following are equivalent:

(a) There exists a left C^*-monomorphism $\alpha : C \to (C^*)^\Lambda$;

(b) there exists a left non-degenerated family of C-balanced bilinear forms $\{\beta_\lambda : C \times C \to R\}_\Lambda$.

(3) Assume the conditions in (2) hold and R is noetherian. Then essential extensions of simple C^*-submodules of C are finitely generated as R-modules.

The next proposition considers density properties in our context.

4.7 Density in C^*. *For an R-submodule $U \subset C^*$ the following are equivalent:*

(a) U is dense in C^* in the finite topology (of R^C);

(b) U is a C-dense subset of C^* (in the finite topology of $\operatorname{End}_{\mathbb{Z}}(C)$).

If R is a cogenerator in R-Mod, then $(a), (b)$ are equivalent to:

(c) $\operatorname{Ke} U = \{x \in C \,|\, u(x) = 0$ for all $u \in U\} = 0$.

Proof. $(a) \Leftrightarrow (b)$ Let $f \in C^*$, $x_1, \ldots, x_n \in C$ and $\Delta x_k = \sum_i x_{k,i} \otimes \tilde{x}_{k,i}$. Then

$$f \rightharpoonup x_k = \sum_i x_{k,i} f(\tilde{x}_{k,i}), \text{ for } k = 1, \ldots, n.$$

Assume (a). Then there exists $u \in U$ such that $f(\tilde{x}_{k,i}) = u(\tilde{x}_{k,i})$, for all i, k, and clearly

$$f \rightharpoonup x_k = u \rightharpoonup x_k, \text{ for } k = 1, \ldots, n.$$

Now assume (b). Then there exists $u \in U$ such that $f \rightharpoonup x_k = u \rightharpoonup x_k$ for all k and this implies

$$f(x_k) = \varepsilon(f \rightharpoonup x_k) = \varepsilon(u \rightharpoonup x_k) = u(x_k), \text{ for } k = 1, \ldots, n.$$

Let R be a cogenerator in R-Mod.

$(a) \Rightarrow (c)$ For any $0 \neq x \in C$, there exists $f \in C^*$ such that $f(x) \neq 0$. Then for some $u \in U$, $u(x) = f(x) \neq 0$, i.e., $x \notin \operatorname{Ke} U$ and hence $\operatorname{Ke} U = 0$.

$(c) \Rightarrow (b)$ Let $f \in C^*$ and $x_1, \ldots, x_n \in C$. Assume

$$f \rightharpoonup (x_1, \ldots, x_n) \notin U \rightharpoonup (x_1, \ldots, x_n) \subset C^n.$$

Then there exists an R-linear map $g : C^n \to R$ such that

$$g(f \rightharpoonup (x_1, \ldots, x_n)) \neq 0 \text{ and } g(U \rightharpoonup (x_1, \ldots, x_n)) = 0.$$

For each $u \in U$,

$$\sum_i g_i(u \rightharpoonup x_i) = \sum_i u(x_i \leftharpoonup g_i) = u(\sum_i x_i \leftharpoonup g_i),$$

where $g_i : C \to C^n \xrightarrow{g} R$, and this implies $\sum_i x_i \leftharpoonup g_i = 0$ and

$$\sum_i g_i(f \rightharpoonup x_i) = \sum_i f(x_i \leftharpoonup g_i) = f(\sum_i x_i \leftharpoonup g_i) = 0,$$

contradicting the choice of g. □

Even if C is not finitely generated as an R-module it is (C^*, R)-finite (see 2.9):

4.8 Finiteness Theorem. *Let C be a coalgebra with ${}_RC$ projective.*

(1) Let $\varrho : M \to M \otimes_R C$ be a right C-comodule. Every finite subset of M is contained in a sub-comodule of M which is finitely generated as an R-module.

(2) Any finite subset of C is contained in a left (right) C^*-submodule and (C^*, C^*)-submodule which are finitely generated as an R-module.

(3) The following are equivalent:
(a) C is finitely generated as R-module;
(b) C is finitely generated as left (right) C^*-module;
(c) $\sigma[{}_{C^*}C] = C^*$-Mod (=$\sigma[C_{C^*}]$).

Proof. (1) Since any sum of sub-comodules is again a sub-comodule it is enough to show that each $m \in M$ lies in a sub-comodule which is finitely generated as an R-module. Moreover, by the correspondence of sub-comodules and C^*-submodules this amounts to proving that $C^* \cdot m$ is a finitely generated R-module.

Let $\{p_i, d_i \,|\, p_i \in C^*,\, d_i \in C\}_I$ be a dual basis of C and write

$$\varrho(m) = \sum_i m_i \otimes d_i, \text{ where } m_i \in M\,.$$

For each $i \in I$, $d_i = \sum_k p_k(d_i)d_k$, and for each $k \in I$ we have

$$u_k := p_k \cdot m = \sum_i m_i p_k(d_i) \in C^* \cdot m\,,$$

with only finitely many u_k's non-zero. Putting this in the sum for $\varrho(m)$ we obtain

$$\varrho(m) = \sum_i m_i \otimes \sum_k p_k(d_i)d_k = \sum_k (\sum_i m_i p_k(d_i)) \otimes d_k = \sum_k u_k \otimes d_k\,.$$

Hence for every $f \in C^*$, $\quad f \cdot m = (id \underline{\otimes} f)\varrho(m) = \sum_k u_k\, f(d_k)\,,$
showing that $C^* \cdot m$ is generated as an R-module by the u_k.

(2) and (3) are immediate consequences of (1). □

Remark. Notice that without the coassociativity of C we do not get that finitely generated (left, right) coideals are finitely generated as R-modules (see 4.4(2)(iii)). In general this need not be true but it still holds for alternative and Jordan coalgebras over fields (see [4, 5.3, 5.6]).

By the Finiteness Theorem 4.8 and the Hom-Com relations 3.12, properties of the base ring R have a strong influence on the module properties of C. From 2.9 and 3.12(1) we have:

4.9 Coalgebras over special rings. *Let ${}_RC$ be projective.*

(1) If R is noetherian, then C is locally noetherian and direct sums of injectives in $\sigma[{}_{C^*}C]$ are injective.

(2) If R is perfect, then every finitely generated module in $\sigma[{}_{C^*}C]$ has dcc on finitely generated submodules.

(3) If R is artinian, then every finitely generated module in $\sigma[{}_{C^*}C]$ has finite length.

(4) If R is injective, then C is injective in $\sigma[{}_{C^*}C]$.

A C-comodule N is said to be a *simple comodule* if it does not contain a non-trivial proper sub-comodule. N is a *semisimple comodule* if it is a direct sum of simple comodules.

The coalgebra C is said to be *left (right) semisimple* if it is semisimple as a left (right) comodule. C is called a *simple coalgebra* if it has no non-trivial proper sub-coalgebra.

Referring to the structure theory of semisimple C^*-modules and the fact that under the given conditions fully invariant C^*-submodules of C are sub-coalgebras we obtain the following characterization of

4.10 Simple and right semisimple coalgebras. *Let ${}_RC$ be projective.*

(1) The following are equivalent:

- *(a) C is a simple right (left) semisimple coalgebra;*
- *(b) C is a simple (C^*, C^*)-module with a minimal left C^*-submodule;*
- *(c) C is a simple coalgebra and C is a finite dimensional vector space over R/m, for some maximal ideal $m \subset R$.*

(2) The following are equivalent:

- *(a) C is a semisimple right C-comodule;*
- *(b) C is a semisimple left C^*-module;*
- *(c) every module in $\sigma[{}_{C^*}C]$ is semisimple;*
- *(d) every module in $\sigma[{}_{C^*}C]$ is injective (projective);*
- *(e) C is a direct sum of simple coalgebras which are right (left) semisimple;*
- *(f) C is a direct sum of simple coalgebras which are finite dimensional over some factor field of R;*
- *(g) C is a semisimple left C-comodule.*

A coalgebra C is called *right semiperfect* if every simple right comodule has a projective cover in Comod-C. If ${}_RC$ is projective as an R-module this is obviously equivalent to the condition that every simple module in $\sigma[{}_{C^*}C]$ has a projective cover in $\sigma[{}_{C^*}C]$ (by 4.3). More precisely we have by 2.11:

4.11 Right semiperfect coalgebras. *Let ${}_RC$ be projective. The following are equivalent:*

(a) C is a right semiperfect coalgebra;

(b) every simple module in $\sigma[{}_{C^*}C]$ has a projective cover;

(c) $\sigma[{}_{C^*}C]$ has a generating set of local projective modules;

(d) every finitely generated module in $\sigma[{}_{C^*}C]$ has a projective cover.

If R is a perfect ring then (a)-(c) are equivalent to:

(e) $\sigma[{}_{C^*}C]$ has a generating set of finitely generated C-projective modules.

Over QF rings further characterizations of these coalgebras will be given in 6.3.

5 Rational C^*-modules and the trace ideal

Again C denotes a coalgebra over R. By 4.1 we may consider each right C-comodule as a left C^*-module. Starting with any left C^*-module M we may ask for right C-comodules associated with it.

5.1 Rational C^*-modules. Let ${}_RC$ be projective, $\varphi : C^* \otimes_R M \to M$ a left C^*-module and put $\varphi(f \otimes m) = f \cdot m$. We have inclusions

$$\begin{aligned} \rho : M \simeq \mathrm{Hom}_{C^*}(C^*, M) &\to \mathrm{Hom}_R(C^*, M), \quad m \mapsto [f \mapsto f \cdot m]\,, \\ \alpha : M \otimes_R C &\to \mathrm{Hom}_R(C^*, M), \quad m \otimes c \mapsto [f \mapsto f(c)m]\,, \end{aligned}$$

and we identify $M \otimes_R C$ with its image under α.

M is called a *rational C^*-module* if $\rho(M) \subset M \otimes_R C$. In this case,

$$\text{for } \rho(m) = \sum m_i \otimes c_i \text{ and } f \in C^*, \text{ we have } f \cdot m = \sum m_i f(c_i).$$

(1) A left C^*-module is rational if and only if it is subgenerated by ${}_{C^*}C$.

(2) If ${}_RC$ is a finitely generated R-module then every C^*-module is rational.

Remark. By 5.1, $\sigma[{}_{C^*}C]$ coincides with all rational left C^*-modules. This fact was also proved in [9]. If R is a field, $(b) \Leftrightarrow (f)$ in 4.10(2) says that all rational C^*-modules are semsimple if and only if C is a direct sum of simple sub-coalgebras. This was shown in [15, Lemma 14.0.1].

5.2 Rational functor. Let ${}_RC$ be projective. For any left C^*-module M, define the *rational submodule*

$$\mathcal{T}^C(M) = \mathrm{Tr}_{C^*}(\sigma[{}_{C^*}C], M) = \sum \{\mathrm{Im}\, f \mid f \in \mathrm{Hom}_{C^*}(U, M),\, U \in \sigma[{}_{C^*}C]\}.$$

We have $M = \mathcal{T}^C(M)$ if and only if M is a rational C^*-module.

By the *rational functor* we mean the left exact functor

$$\mathcal{T}^C : C^*\text{-Mod} \to \sigma[{}_{C^*}C].$$

The corestriction $\Phi_C : C \to \mathcal{T}^C(C^{**})$ is an isomorphism.

Proof. Clearly Φ_C is injective. Let $\varrho : \mathcal{T}^C(C^{**}) \to \mathcal{T}^C(C^{**}) \otimes_R C$ denote the comodule structure map, $\gamma \in \mathcal{T}^C(C^{**})$ and $\varrho(\gamma) = \sum_i u_i \otimes c_i$. Then for any $f \in C^*$,

$$\gamma(f) = f \cdot \gamma(\varepsilon) = \sum_i u_i f(c_i)(\varepsilon) = f(\sum_i u_i(\varepsilon)c_i),$$

where $\sum_i u_i(\varepsilon)c_i \in C$. This proves that Φ_C is surjective. $\square$

Another proof for this isomorphism is given in [9].

It is obvious that properties of the rational functor $\mathcal{T}^C$ depend on properties of the coalgebra C. Transferring 2.5 we obtain:

5.3 The rational functor exact. *Let ${}_RC$ be projective and put $T := \mathcal{T}^C({}_{C^*}C^*)$ (the* left trace ideal*). Then the following are equivalent:*

(a) $\sigma[{}_{C^*}C]$ is closed under extensions in C^*-Mod *and the class of C^*-modules X with $\mathcal{T}^C(X) = 0$ is closed under factor modules;*

(b) *the functor* $\mathcal{T}^C : C^*\text{-Mod} \to \sigma[_{C^*}C]$ *is exact;*

(c) *for every* $N \in \sigma[_{C^*}C]$ *(or* $N \subset C$*),* $TN = N$*;*

(d) *for every* $N \in \sigma[_{C^*}C]$*, the canonical map* $T \otimes_{C^*} N \to N$ *is an isomorphism;*

(e) C *is an s-unital* T*-module;*

(f) $TC = C$ *and* C^*/T *is flat as a right* C^**-module;*

(g) $T^2 = T$ *and* T *is a generator in* $\sigma[_{C^*}C]$*;*

(h) T *is a* C*-dense subring of* C^**.*

If these conditions are satisfied the right trace $\mathcal{T}^C(C^*{}_{C^*}) \subset T$*.*

In particular, $\mathcal{T}^C$ is exact if $\sigma[_{C^*}C]$ contains a generator which is projective in C^*-Mod. We will see in 6.3 that over QF rings R, $\mathcal{T}^C$ exact is equivalent to $\sigma[_{C^*}C]$ being semiperfect.

Under certain finiteness conditions we have an interplay between left and right C-comodules. Recall that for a finitely presented R-module M, a flat R-module Q and any R-module M', there is an isomorphism (e.g., [18, 15.7])

$$\nu_M : Q \otimes_R \mathrm{Hom}_R(M, M') \to \mathrm{Hom}_R(M, Q \otimes_R M'), \quad q \otimes h \mapsto q \otimes h(-)\,.$$

Applying this isomorphism we obtain by standard arguments:

5.4 Comodules finitely presented as R-modules. *Let* $\varrho : M \to M \otimes_R C$ *be a right* C*-comodule. Assume that* $_RC$ *is flat and* $_RM$ *is finitely presented.*

Then $M^* = \mathrm{Hom}_R(M, R)$ *is a left* C*-comodule by the structure map*

$$\bar{\varrho} : M^* \to \mathrm{Hom}_R(M, C) \simeq C \otimes M^*, \quad g \mapsto (g \otimes id) \circ \varrho\,,$$

and hence is a rational right C^**-module by the map*

$$M^* \otimes_R C^* \to M^*, \quad g \otimes f \mapsto (g \underline{\otimes} f) \circ \varrho\,.$$

(1) *If* M *is injective as right* C*-comodule and is contained in a free* R*-module, then* M^* *is projective in* Mod-C^**.*

(2) *If* M *is projective in* Comod-C *and* R*-reflexive, then* M^* *is* (N, R)*-injective for all* $N \in$ C-Comod *which are finitely presented and* R*-reflexive as* R*-modules.*

(3) *Suppose that* R *is self-injective. If* M *is projective in* Comod-C*, then* M^* *is* N*-injective for all* $N \in C$-Comod *which are finitely presented as* R*-modules.*

As a first application we prove the following:

5.5 Characterization of the trace ideal. *Let* $_RC$ *be projective and denote by* $T := \mathcal{T}^C(_{C^*}C^*)$ *the left trace ideal.*

(1) *Let* $f \in C^*$ *and assume* $f \rightharpoonup C$ *is a finitely presented* R*-module. Then* $f \in T$*.*

(2) *Assume* R *to be noetherian. Then* T *can be described as*

$$\begin{aligned} T_1 &= \{f \in C^* \mid C^* {*} f \ \text{ is a finitely generated } R\text{-module}\}, \\ T_2 &= \{f \in C^* \mid \ \mathrm{Ke}\, f \text{ contains a left coideal } K, \text{ such that } C/K \\ &\qquad \text{is a finitely generated } R\text{-module}\}, \\ T_3 &= \{f \in C^* \mid f \rightharpoonup C \ \text{ is a finitely generated } R\text{-module}\}. \end{aligned}$$

Proof. (1) Assume the rational right C^*-module $f \rightharpoonup C$ to be a finitely presented R-module. Then by 5.4, $(f \rightharpoonup C)^*$ is a rational left C^*-module. Since $\varepsilon(f \rightharpoonup c) = f(c)$ for all $c \in C$, we have $f \in (f \rightharpoonup C)^*$ and hence $f \in T$.

(2) By the Finiteness Theorem, $T \subset T_1$. Let $f \in T_1$ and let $C^* * f$ be finitely R-generated by $g_1, \ldots, g_k \in C^*$. Consider the kernel of $C^* * f$,

$$K := \bigcap \{\mathrm{Ke}\, h \mid h \in C^* * f\} = \bigcap_{i=1}^{k} \mathrm{Ke}\, g_i\,.$$

Clearly K is a right C^*-submodule of C and hence a left coideal. Moreover all the $C/\mathrm{Ke}\, g_i$ are finitely generated R-modules and hence

$$C/K \subset \bigoplus_{i=1}^{k} C/\mathrm{Ke}\, g_i$$

is a finitely generated R-module. This proves $T_1 \subset T_2$.

Now let $f \in T_2$. Since $\Delta(K) \subset C \otimes_R K$, $f \rightharpoonup K = 0$ and $f \rightharpoonup C = f \rightharpoonup C/K$ is finitely generated R-module, i.e., $f \in T_3$.

Finally $T_3 \subset T$ follows from (1). □

Except when ${}_RC$ is finitely generated (which means $\mathcal{T}^C(C^*) = C^*$) the trace ideal does not contain a unit element. However if C is a direct sum of finitely generated left and right coideals the trace ideal has enough idempotents:

5.6 Trace ideal and decompositions. *Let ${}_RC$ be projective, $T := \mathcal{T}^C({}_{C^*}C^*)$ and $T' := \mathcal{T}^C(C^*{}_{C^*})$ (left and right trace ideals). Assume $C = \bigoplus_\Lambda C \leftharpoonup e_\lambda$, with a family of orthogonal idempotents $e_\lambda \in C^*$.*

(1) There is a monomorphism

$$\gamma : T \to \bigoplus_\Lambda e_\lambda * T, \quad t \mapsto \sum_\Lambda e_\lambda * t.$$

(2) If all the $C \leftharpoonup e_\lambda$ are finitely generated C^*-modules, then all $e_\lambda \in T'$ and γ yields an embedding $T \to T'$.
Moreover $C = CT'$ and C is an s-unital T'-module (i.e., $T' \subset C^*$ is C-dense).

(3) If all the $C \leftharpoonup e_\lambda$ and the $e_\lambda \rightharpoonup C$ are finitely generated C^*-modules, then

$$T' = T = \bigoplus_\Lambda e_\lambda * T = \bigoplus_\Lambda T * e_\lambda,$$

i.e., T is a ring with enough idempotents. Moreover the left and right rational functors are exact.

Proof. (1) For any $t \in T$, $C^* \leftharpoonup t$ is a finitely generated R-module and hence $e_\lambda * t = 0$ for almost all $\lambda \in \Lambda$. Hence the map γ is well-defined.

Assume $\gamma(t) = 0$. Then for any $c \in C$, $0 = e_\lambda * t(c) = t(c \leftharpoonup e_\lambda)$, for all $\lambda \in \Lambda$ implying $t = 0$.

(2) By the Finiteness Theorem the $C \leftharpoonup e_\lambda$ are finitely generated as R-modules and they are R-projective as direct summands of C. Now it follows from 5.5 that $e_\lambda \in T$. Clearly for any $c \in C$ there is an (idempotent) $t \in T'$ satisfying $c \leftharpoonup t = c$. Hence C is an s-unital T'-module.

(3) By symmetry we conclude from (2) that $T' = T = \bigoplus_\Lambda e_\lambda * T$.

Now consider $f \in e_\lambda * T$. Then $C \leftharpoonup f$ is finitely generated as R-module and we may choose some $e = e_{\lambda_1} + \ldots + e_{\lambda_k}$, such that

$$c \leftharpoonup f \leftharpoonup e = c \leftharpoonup f \quad \text{for all } c \in C.$$

This implies $f = f * e$ and $T \subset \bigoplus_\Lambda T * e_\lambda$,

By (2), C is left and right s-unital over T. □

6 Coalgebras over QF rings

For a QF ring R, the functor $\mathrm{Hom}_R(-, R)$ is a duality for the finitely generated R-modules and this provides a close relationship between the left and right properties of C as a C^*-module. In this section we will present the consequences and it will turn out that most of the properties of coalgebras over fields remain true over QF rings.

Recall that a QF ring R is artinian and injective and a cogenerator in R-Mod.

6.1 Coalgebras over QF rings. *Let ${}_RC$ be projective and R a QF ring. Then:*

(1) C is an injective cogenerator in $\sigma[{}_{C^*}C]$.

(2) Every module in $\sigma[{}_{C^*}C]$ is a submodule of a free module $C^{(\Lambda)}$.

(3) $K := \mathrm{Soc}_{C^*} C \trianglelefteq C$ (essential submodule) and $\mathrm{Jac}(C^*) = \mathrm{Hom}_R(C/K, R)$.

Proof. By 4.9, C is injective in $\sigma[{}_{C^*}C]$ and $K \trianglelefteq C$.

Over a QF ring R every R-module M is contained in a free R-module $R^{(\Lambda)}$. This implies for any right C-comodule

$$\varrho : M \to M \otimes_R C \subset R^{(\Lambda)} \otimes_R C \simeq C^{(\Lambda)}.$$

By the characterization of endomorphism rings of self-injective modules (see [17, 22.1]) we obtain by 3.12 (writing endomorphisms on the right)

$$\mathrm{Jac}(C^*) = \mathrm{Hom}_{C^*}(C/K, C) \simeq \mathrm{Hom}_R(C/K, R).$$ □

The most remarkable feature of our next result is that over QF rings, for some modules injectivity and projectivity in $\sigma[{}_{C^*}C]$ extend to injectivity resp. projectivity in C^*-Mod.

6.2 Finitely presented modules over QF rings. *Let ${}_RC$ be projective, R a QF ring and and M a right C-comodule.*

(1) If M is projective in Comod-C *then M^* is C-injective as right C^*-module and $\mathcal{T}^C(M^*)$ is injective in* C-Comod.

(2) If M is finitely presented as an R-module then:

(i) M is injective in Comod-C *if and only if M is injective in* C^*-Mod.

(ii) M is projective in Comod-C *if and only if M is projective in* C^*-Mod.

Proof. (1) Consider any diagram with exact row in C-Comod,

$$\begin{array}{ccccc} 0 & \to & K & \to & N \\ & & \downarrow f & & \\ & & M^*, & & \end{array}$$

where N is finitely generated as R-module. Applying $(-)^* = \mathrm{Hom}_R(-,R)$ we obtain - with the canonical map $\Phi_M : M \to M^{**}$ - the diagram

$$\begin{array}{ccccc} M & \to & M^{**}, & & \\ & & \downarrow f^* & & \\ N^* & \to & K^* & \to & 0 \end{array}$$

where the lower row is in Comod-C and hence can be extended commutatively by some right comodule morphism $g : M \to N^*$. Again applying $(-)^*$ - and recalling that the composition $M^* \xrightarrow{\Phi_{M^*}} M^{***} \xrightarrow{(\Phi_M)^*} M^*$ yields the identity (e.g., [17, 45.10]) - we see that g^* extends f to N. This proves that M^* is N-injective for all modules $N \in C$-Comod which are finitely presented as R-modules.

In particular, by the Finiteness Theorem 4.8, every finitely generated C^*-submodule of C is finitely generated - hence finitely presented - as an R-module. So M^* is N-injective for all these modules and hence it is C-injective as left C^*-module (see [17, 16.3]).

Notice that M^* need not be in $\sigma[_{C^*}C]$ (not rational). It is straightforward to show that $\mathcal{T}^C(M^*)$ is an injective object in $\sigma[_{C^*}C]$.

(2)(i) Let M be injective in Comod-C. Since R is QF, M is contained in a free R-module and so M^* is projective in Mod-C^* (by 5.4(1)). Consider any monomorphism in C^*-Mod, $0 \to M \to X$. Then $X^* \to M^* \to 0$ is exact and splits in Mod-C^* and in the diagram

$$\begin{array}{ccccc} 0 & \to & M & \to & X \\ & & \downarrow\simeq & & \downarrow f^* \\ 0 & \to & M^{**} & \to & X^{**}, \end{array}$$

the lower row splits in C^*-Mod and so does the upper row proving that M is injective in C^*-Mod.

(ii) Let M be projective in Comod-C. Since M^* is in $\sigma[C_{C^*}]$ (by 5.4) we know from (1) that it is injective in C-Comod. Now we conclude by the right hand version of 5.4 that $M \simeq M^{**}$ is projective in Mod-C^*. $\square$

Over QF rings semiperfect coalgebras are characterized by the fact that the left trace functor is exact:

6.3 Right semiperfect coalgebras over QF rings. *Let $_RC$ be projective, R a QF ring and put $T := \mathcal{T}^C(_{C^*}C^*)$.*

(1) The following are equivalent:

(a) C is a right semiperfect coalgebra;

*(b) $\sigma[_{C^*}C]$ has a generating set of finitely generated modules which are projective in C^*-*Mod*;*

(c) injective hulls of simple left *C-comodules are finitely generated R-modules;*

*(d) the functor $\mathcal{T}^C : C^*$-*Mod *$\to \sigma[_{C^*}C]$ is exact;*

(e) T is left C-dense in C^*;

(f) $\mathrm{Ke}\, T = \{x \in C \,|\, T(x) = 0\} = 0$.

(2) Assume C to be a right semiperfect coalgebra. Then:

(i) The right trace ideal $\mathcal{T}^C(C^*{}_{C^*}) \subset T$.

(ii) For every $M \in \sigma[C_{C^*}]$, the trace of $\sigma[_{C^*}C]$ in M^* is non-zero.

(iii) Every module in $\sigma[C_{C^*}]$ has a maximal submodule and has small radical.

Proof. (1) $(a) \Rightarrow (b) \Rightarrow (c)$ Let U be a simple left C-comodule, i.e., a simple right C^*-module in $\sigma[C_{C^*}]$, with injective hull $U \to \widehat{U}$ in $\sigma[C_{C^*}]$. Applying $\mathrm{Hom}_R(-,R)$ we have an epimorphism in C^*-Mod,

$$\widehat{U}^* \to U^* \to 0,$$

where U^* is a simple module. Moreover since R is QF we may assume that $\widehat{U}$ is a direct summand of C_{C^*} and so $\widehat{U}^*$ is direct summand of C^* and hence is projective in C^*-Mod.

By assumption there exists a projective cover $P \to U^*$ in $\sigma[_{C^*}C]$. Being finitely generated as R-module and projective in $\sigma[_{C^*}C]$, P is also projective in C^*-Mod (by 6.2). As easily seen this implies $\widehat{U}^* \simeq P$, hence it is finitely generated as an R-module and so is $\widehat{U}$.

$(c) \Rightarrow (a)$ Let $V \subset C$ be a simple left C^*-submodule. Then V^* is a simple right C^*-module in $\sigma[C_{C^*}]$. Let $V^* \to K$ be its injective hull in $\sigma[C_{C^*}]$. By assumption, K is a finitely generated R-module and so K^* is a projective C^*-module (by 5.4) and $K^* \to V^{**} \simeq V$ is a projective cover in $\sigma[_{C^*}C]$.

$(b) \Rightarrow (d)$ Since the projectives in $\sigma[_{C^*}C]$ are projective in C^*-Mod (by 6.2) the assertion follows from 2.6.

$(d) \Leftrightarrow (e) \Leftrightarrow (f)$ The first equivalence follows from 5.3, the second from 4.7.

$(d) \Rightarrow (c)$ Let $U \subset C$ be a simple right C^*-submodule with injective hull $\widehat{U}$ in $\sigma[C_{C^*}]$. We may assume $U \subset Q \subset C$, where Q is the socle of C_{C^*}. By an obvious construction we obtain the commutative exact diagram

$$\begin{array}{ccccccccc}
0 & \to & (C/Q)^* & \to & C^* & \to & Q^* & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (\widehat{U}/U)^* & \to & \widehat{U}^* & \to & U^* & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & .
\end{array}$$

Since $(C/Q)^* = \mathrm{Jac}(C^*)$ by 4.9, we have that $(\widehat{U}/U)^*$ is superfluous in $\widehat{U}^*$.

By (d) the lower row yields an epimorphism $\mathcal{T}^C(\widehat{U}^*) \to U^*$ and hence

$$\widehat{U}^* = (C/U)^* + \mathcal{T}^C(\widehat{U}^*) = \mathcal{T}^C(\widehat{U}^*) \in \sigma[_{C^*}C].$$

Since $\widehat{U}^*$ is a cyclic C^*-module this implies that $\widehat{U}^*$ is a finitely generated R-module and so is $\widehat{U}$.

(2) (i) This was already observed in 5.3.

(ii) For every simple submodule $S \subset M$ with injective hull $\widehat{S}$ in $\sigma[C_{C^*}]$, we have commutative diagrams

$$\begin{array}{cccc}
0 \to & S & \to M & \\
 & i\downarrow & \swarrow j & \\
 & \widehat{S} & & ,
\end{array}
\qquad
\begin{array}{cccc}
 & M^* & \to S^* & \\
 & j^*\uparrow & \nearrow i^* & \\
 & \widehat{S}^* & & ,
\end{array}$$

where i is injective and j is non-zero. By 5.4, $\widehat{S}^*$ belongs to $\sigma[_{C^*}C]$ and so does its image under j.

(iii) Let $M \in \sigma[C_{C^*}]$. By (ii), there exists a simple $T \subset M^*$ with $T \in \sigma[_{C^*}C]$. Then $\mathrm{Ke}\,T = \{m \in M \mid T(m) = 0\}$ is a maximal C^*-submodule of M. This implies that all modules in $\sigma[C_{C^*}]$ have small radical. □

In contrast to associative algebras, for coalgebras semiperfectness is a strictly one-sided property: right semiperfect need not imply left semiperfect. Next we describe coalgebras which are both right and left semiperfect.

6.4 Left and right semiperfect coalgebras. *Let ${}_RC$ be projective and R a QF ring. Put $T := \mathcal{T}^C({}_{C^*}C^*)$ and $T' := \mathcal{T}^C(C^*{}_{C^*})$. Then the following are equivalent:*

(a) C is a left and right semiperfect coalgebra;

(b) all left C-comodules and all right C-comodules have projective covers;

(c) $T = T'$ and is dense in C^*;

(d) ${}_{C^*}C$ and C_{C^*} are direct sums of finitely generated C^*-modules.

Under these conditions T is a ring with enough idempotents.

Proof. $(b) \Rightarrow (a)$ is obvious.

$(a) \Rightarrow (b)$ By 2.11, all finitely generated projective modules in $\sigma[{}_{C^*}C]$ are semiperfect in $\sigma[{}_{C^*}C]$. According to [17, 42.4], a direct sum of projective semiperfect modules in $\sigma[{}_{C^*}C]$ is semiperfect provided it has small radical. However this is true by 6.3 and we conclude that every module in $\sigma[{}_{C^*}C]$ has a projective cover. The same arguments hold for the $\sigma[C_{C^*}]$.

$(a) \Leftrightarrow (c)$ This is obvious by the characterization of exactness of the rational functor in 5.3 and 6.3.

$(c) \Leftrightarrow (d)$ and the final assertion follow from 5.6. $\square$

To have enough projectives in $\sigma[{}_{C^*}C]$ does not mean that C itself is projective. Since over a QF ring C always is self-injective the latter condition has strong consequences.

6.5 Self-projective coalgebras over QF rings. *Let R be a QF ring and ${}_RC$ projective. Then the following are equivalent:*

(a) C is a submodule of a free left C^*-module;

(b) C is cogenerated by C^* as left C^*-module;

(c) there exists a left non-degenerated family of C-balanced bilinear forms on C;

(d) in $\sigma[{}_{C^*}C]$ every (indecomposable) injective object is projective;

(e) C is a projective right C-comodule;

(f) C is a projective object in $\sigma[{}_{C^*}C]$;

(g) C is projective in C^*-Mod.

If these conditions are satisfied, then C is a left semiperfect coalgebra and C is a generator in $\sigma[C_{C^*}]$.

Proof. $(a) \Leftrightarrow (b)$ By 4.9, C is a direct sum of injective hulls of simple modules in $\sigma[{}_{C^*}C]$. If C is cogenerated by C^*, then each of these modules is contained in a copy of C^* and hence C is contained in a free C^*-module.

$(b) \Leftrightarrow (c)$ This is shown in 4.6.

$(c) \Rightarrow (g)$ Let U be a simple left C^*-submodule of C with injective hull $\widehat{U} \subset C$ in $\sigma[{}_{C^*}C]$. Then $\widehat{U}$ is a finitely generated R-module by 4.6(3). Now we conclude from 6.2 that $\widehat{U}$ is injective in C^*-Mod. Being cogenerated by C^*, we have in fact that $\widehat{U}$ is a direct summand of C^* and hence it is projective in C^*-Mod.

This implies that C is projective in C^*-Mod.

$(g) \Rightarrow (a)$ and $(g) \Rightarrow (f) \Rightarrow (d)$ are obvious.

$(d) \Rightarrow (g)$ C is a direct sum of injective hulls $\widehat{U} \subset C$ of simple submodules $U \subset C$. By (e), $\widehat{U}$ is projective in $\sigma[{}_{C^*}C]$. Since it is projective with local endomorphism ring it is finitely generated as C^*-module and hence finitely generated as R-module (by 4.8). Now we conclude from 6.2 that $\widehat{U}$ is projective in C^*-Mod and so is C.

Finally assume the conditions hold. By the proof of 6.5 the injective hulls of simple modules in $\sigma[{}_{C^*}C]$ (right C-comodules) are finitely generated R-modules. By 4.11, this characterizes left semiperfect coalgebras implying that the right trace ideal $T' := \mathcal{T}^C(C^*_{C^*})$ is a generator in $\sigma[C_{C^*}]$. Now we learn from 6.2 that T' is injective in $\sigma[C_{C^*}]$ and hence it is generated by C. This makes C a generator in $\sigma[C_{C^*}]$. □

From 6.1 we know that over a QF ring C is always an injective cogenerator in $\sigma[{}_{C^*}C]$. Which additional properties make C a projective generator?

6.6 C **as a projective generator in** $\sigma[{}_{C^*}C]$**.** *Let R be a QF ring, ${}_RC$ projective and $T := \mathcal{T}^C({}_{C^*}C^*)$. The following are equivalent:*

(a) C is projective in Comod-C *and* C-Comod*;*

(b) C is a projective generator in Comod-C*;*

(c) C is a projective generator in C-Comod*;*

(d) $C = TC$, T is a ring with enough idempotents and an injective cogenerator in Comod-C.

Proof. $(a) \Rightarrow (b)$ This is obtained from 6.5 and 6.4.

$(b) \Rightarrow (a)$ To prove that C is projective in C-Comod we show that C^* cogenerates C as right C^*-module (see 6.5). For this it suffices to prove that each simple submodule $U \subset C_{C^*}$ can be embedded in C^*. By 4.8, U is a finitely generated R-module. Clearly U^* is a simple module in $\sigma[{}_{C^*}C]$ and hence there is a C^*-epimorphism $C \to U^*$. From this we obtain an embedding $U \simeq U^{**} \subset C^*$ which proves our assertion.

$(a) \Leftrightarrow (c)$ is clear by symmetry.

$(a) \Rightarrow (d)$ From above we know that C is left and right semiperfect. Hence T is a ring with enough idempotents and $\sigma[{}_{C^*}C] = \sigma[{}_{C^*}T]$ (by 6.4). C being projective, $C \subset T^\Lambda$ and hence T is a cogenerator in $\sigma[{}_{C^*}C]$. T is injective in $\sigma[{}_{C^*}C]$ by 6.3.

$(d) \Rightarrow (b)$ Since T is projective in $\sigma[{}_{C^*}C]$ injective hulls of simple modules in $\sigma[{}_{C^*}C]$ are projective and so C is projective in $\sigma[{}_{C^*}C]$. T being injective it is generated by C. By our assumptions T is a generator in $\sigma[{}_{C^*}C]$ and so is C. □

If ${}_RC$ is finitely generated and projective, then $\sigma[{}_{C^*}C] = C^*$-Mod and we have:

6.7 C **as projective generator in** C^*-Mod**.** *Let R be a QF ring and ${}_RC$ projective. Then the following are equivalent:*

(a) C is a projective generator in C^*-Mod*;*

(b) C is a generator in C^*-Mod*;*

(c) C is a generator in $\sigma[{}_{C^*}C]$ and ${}_RC$ is finitely generated;

(d) C^* is a QF algebra and ${}_RC$ is finitely generated.

Remarks. Several of the results of this section were obtained in the literature for coalgebras over fields. For R a field, 6.2(1) was shown in Lin [13, Lemma 11] and (2) is proved in Doi [8, Proposition 4]. The characterizations of semiperfect coalgebras in 6.3 are partly shown in Lin [13, Theorem 10] and Gómez-Năstăsescu [10, Theorem 3.3]. The equivalence of (a) and (b) from 6.4 can be found in Lin [13, Corollary 18]. Related results also appear in Beattie-Dăscălescu-Grünfelder-Năstăsescu [5, Theorem 2.4].

The coalgebras considered in 6.4 are called *coproper* in [2] where it was shown that for them $T = \bigoplus_\Lambda T{*}e_\lambda$.

The self-projective coalgebras described in 6.5 are called *Quasi-co-Frobenius* in Gómez-Năstăsescu [10] and some of the characterizations are in [10, Theorem 1.3].

Moreover, characterizations of C as a projective generator as given in 6.6 are contained in Gómez-Năstăsescu [10, Theorem 2.6].

References

[1] Abe, E., *Hopf Algebras*, Cambridge Univ. Press (1977)

[2] Allen, H.P., Trushin, D., *Coproper Coalgebras*, J. Algebra 54, 203-215 (1978)

[3] Allen, H.P., Trushin, D., *A generalized Frobenius structure for coalgebras with applications to character theory*, J. Algebra 62, 430-449 (1980)

[4] Anquela, J.A., Cortés, T., Montaner, F., *Nonassociative Coalgebras*, Comm. Algebra 22, 4693-4716 (1994)

[5] Beattie, M., Dăscălescu, S., Grünfelder, L., Năstăsescu, C., *Finiteness conditions, Co-Frobenius Hopf Algebras and Quantum Groups*, J. Algebra 200, 312-333 (1998)

[6] Beidar, K.I., *Hopf Algebras*, Lecture Notes, National Cheng-Kung University, Taiwan (1995)

[7] Berning, J., *Beziehungen zwischen links-linearen Toplogien und Modulkategorien*, Dissertation, Universität Düsseldorf (1994)

[8] Doi, Y., *Homological coalgebra*, J. Math. Soc. Japan 33, 31-50 (1981)

[9] Gómez Torrecillas, J., *Coalgebras and comodules over a commutative ring*, manuscript (1996)

[10] Gómez Torrecillas, J., Năstăsescu, C., *Quasi-co-Frobenius Coalgebras*, J. Algebra 174, 909-923 (1995)

[11] Kaplansky, I., *Bialgebras*, Lecture Notes in Math., University of Chicago 1975

[12] Lin, I.-P., *Products of torsion theories and applications to coalgebras*, Osaka J. Math. 12, 433-439 (1975)

[13] Lin, I.-P., *Semiperfect Coalgebras*, J. Algebra 49, 357-373 (1977)

[14] Montgomery, S., *Hopf Algebras and Their Actions on Rings*, Reg. Conf. Series in Math. (CBMS), No.82, AMS (1993)

[15] Sweedler, M.E., *Hopf Algebras* , Benjamin, New York 1969

[16] Tominaga, H., *On s-unital rings*, Math. J. Okayama Univ. 18, 117-134 (1976)

[17] Wisbauer, R., *Foundations of Module and Ring Theory*, Gordon and Breach, Reading, Paris (1991)

[18] Wisbauer, R., *Modules and Algebras: Bimodule Structure and Group Actions on Algebras*, Pitman Mono. PAM 81, Addison Wesley Longman, Essex (1996)

[19] Wisbauer, R., *On module classes closed under extensions*, Rings and Radicals, Gardner e.a. (ed), Pitman RN 346, 73-97 (1996)