General Distributivity and Thickness of Modules

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Abstract

Let $\omega \geq 2$ be a cardinal and $\mathcal{S}$ a class of semisimple left R-modules (closed under isomorphisms). A module ${}_RM$ is called *ω-thick relative to $\mathcal{S}$*, if $\dim S < \omega$ for each subfactor S of M with $S \in \mathcal{S}$. This notion allows to study from a unified point of view *ω-distributive* modules, i.e., modules satisfying

$$A + \bigcap_{\lambda\in\Lambda} B_\lambda = \bigcap_{\lambda\in\Lambda}(A + \bigcap_{\mu\in\Lambda\setminus\{\lambda\}} B_\mu)$$

for all submodules A and families $\{B_\lambda\}_\Lambda$ of submodules with cardinality $|\Lambda| = \omega$, and *ω-thick* modules, i.e., modules which are ω-thick relative to the class of all semisimple left R-modules. In particular 2-distributive modules coincide with distributive modules, 2-thick modules coincide with uniserial modules, $\aleph_0$-thick modules coincide with q.f.d. modules, i.e., modules whose factor modules have finite uniform dimension.

We also consider *relative ω-quasi-invariant, relative ω-noetherian* and *relative ω-Bézout* modules. Properties of modules from these classes are investigated including the relationship between them. Moreover, for modules ${}_RM$ and ${}_RU$, the relationship between ω-distributivity of M and properties of the left $\mathrm{End}_R(U)$-module $\mathrm{Hom}_R(U, M)$ and the right $\mathrm{End}_R(U)$-module $\mathrm{Hom}_R(M, U)$ are studied.

1 Introduction and preliminaries

In this paper associative rings with unit and unital modules will be considered, and homomorphisms will be written on the opposite side to the scalars. Terminology and general notations will be taken from [1] without reference.

Throughout ω will denote a cardinal and n a finite cardinal. We write ω^+ for the smallest cardinal larger than ω. The notation $|X|$ is used for the cardinality of any set X, and L^* for the partially ordered set which is dual to a partially ordered set L.

R will always be an associative ring with unit, *R-Mod* (resp. *Mod-R*) denote the category of unital left (resp. right) R-modules, and M will be a left or right R-module depending on the situation, and we will write ${}_RM$ or M_R if it is appropriate to indicate the side of the module.

$J(M)$, $\mathcal{L}(M)$, End(M) and $\max(M)$ will stand for the Jacobson radical, the lattice of submodules, the endomorphism ring and the set of all maximal submodules of M. $M^{(\omega)}$ and M^ω denote the direct sum and the direct product of ω copies of M, and $\dim V$ denotes the dimension of any semisimple module V (the cardinal number of simple summands of V, e.g. [1, 20.5]).

Moreover, for any element x and subset A of ${}_RM$, we put

$$(A:x) = \{r \in R \mid rx \in A\} \text{ and } \ell(x) = (0:x).$$

Submodules of factor modules are called *subfactors* of M. By $\mathrm{crs}(M)$ we denote the cardinality of a representing set of all simple subfactors of M. In particular, $\mathrm{crs}(R)$ is the cardinality of all (non-isomorphic) simple left R-modules, and for any semisimple module V, $\mathrm{crs}(V)$ coincides with the cardinality of the set of all homogeneous components of V.

Let us recall some definitions from set theory. For $\omega \geq 2$ the *cofinal character* $\mathrm{cf}(\omega)$ is defined as the smallest cardinal ϱ, such that there exists a family $\{\alpha_\xi\}_\Xi$ of cardinals, where $|\Xi| = \varrho$, $\alpha_\xi < \omega$ for all $\xi \in \Xi$, and $\sum_\Xi \alpha_\xi = \omega$. The cardinal ω is called *regular* provided $\mathrm{cf}(\omega) = \omega$, and ω is said to be *singular* if ω is not regular.

Usually these definitions are only applied to an infinite ω. However it will be convenient for us to use them also for the case of finite ω. Hereby, obviously, $\mathrm{cf}(n) = 2$ for any finite cardinal $n \geq 2$, and hence the cardinal 2 is regular whereas any finite cardinal $n \geq 3$ is singular.

In module theory theorems are of importance which describe the structure of modules by modules with a relatively simple structure, for example uniserial modules. Natural generalizations of uniserial modules are first of all distributive modules, i.e., modules with a distributive lattice of submodules, and, secondly, $AB5^*$ modules, i.e., modules satisfying

$$A + \bigcap_\Lambda B_\lambda = \bigcap_\Lambda (A + B_\lambda)$$

for all submodules A and inverse families of submodules $\{B_\lambda\}_\Lambda$.

In contrast to distributivity no convenient criterion is known for the $AB5^*$ condition and this makes the investigation of $AB5^*$ modules more difficult. To overcome this problem in [2] a weaker condition was introduced - countably distributive modules. In [3] it was suggested to treat this as a special case of the notion of ω-distributive modules, where $\omega \geq 2$ is any cardinal, corresponding to $\omega = \aleph_0$.

Recall some definitions and propositions from [3]. Let $\omega \geq 2$ and $n \geq 2$ a natural number. A lattice is called *ω-distributive*, if any non-empty subset of cardinality not greater than ω has a greatest lower bound, and

$$a \vee \bigwedge_{\lambda \in \Lambda} b_\lambda = \bigwedge_{\lambda \in \Lambda} (a \vee \bigwedge_{\mu \in \Lambda \setminus \{\lambda\}} b_\mu),$$

for all elements $a \in L$ and families $\{b_\lambda\}_\Lambda$ of elements, where $|\Lambda| = \omega$.

A module M is called *ω-distributive* if the lattice $\mathcal{L}(M)$ is ω-distributive. Hereby 2-distributivity of a lattice (module) coincides with the usual distributivity of a lattice (module). The investigation of n-distributive lattices was initiated in [4], [5]. Some differences in the terminology should be pointed out: an n-distributive lattice in the sense defined above corresponds exactly to an $(n-1)$-distributive lattice in the sense of [5].

As usual, for any property $\mathbb{P}$ of a module we say that the ring R has this *property on the left (right)* provided the module ${}_RR$ (respectively R_R) has property $\mathbb{P}$. Left noetherian, local left n-distributive rings were studied in [6]. Left countably distributive rings appeared in connection with the study of weakly injective modules in [7, Theorem 3.2] and [8, Corollary 9]. In [3] the following generalization of a well-known criterion of distributivity for modules (see [9, Theorem 1.6], [10, Lemma 1.3]) is shown.

1.1 Lemma. *For ${}_RM$ and $\omega \geq 2$, the following are equivalent:*

(a) *M is ω-distributive;*

(b) $\sum_{\lambda \in \Lambda}((\sum_{\mu \in \Lambda \setminus \{\lambda\}} Ra_\mu) : a_\lambda) = R$, *for any family $\{a_\lambda\}_\Lambda$ of elements of M, where $|\Lambda| = \omega$;*

(c) $\sum_\Lambda \ell(a_\lambda) = R$, *for each independent family $\{Ra_\lambda\}_\Lambda$ of cyclic submodules of any factor module of M, where $|\Lambda| = \omega$.*

A number of characterizations of distributive modules is known involving uniserial modules (see, for example, [10 - 16]).

In view of extending suitable results to ω-distributive modules, a generalization of uniserial modules will be introduced - *ω-thick modules*. Moreover we will show that ω-distributive and ω-thick modules may be considered from a unified point of view by introducing *relative ω-thick* modules.

1.2 Abstract classes of modules. A class of left R-modules is called *abstract* if it is closed under isomorphic images. In what follows $\mathcal{S}$ will denote an abstract class of semisimple left modules, $\mathcal{K}$ a class of simple left R-modules and $\mathcal{K}^{'}$ the *abstract closure* of the class $\mathcal{K}$, i.e.

$$\mathcal{K}^{'} = \{{}_RQ \mid Q \simeq P \text{ for some } P \in \mathcal{K}\}.$$

The modules in $\mathcal{K}^{'}$ will be called *$\mathcal{K}$-simple.* A submodule A of ${}_RM$ is said to be *$\mathcal{K}$-maximal* if the factor module M/A is $\mathcal{K}$-simple. By $\mathcal{S}_1(\mathcal{K})$ we will denote the class of all semisimple modules ${}_RM$, for which $Q \in \mathcal{K}^{'}$ for each simple submodule Q of M. In case $\mathcal{K}$ is determined by a single module K we write $\mathcal{S}_1(\mathcal{K}) = \mathcal{S}_1(K)$.

Clearly $\mathcal{S}_1(\mathcal{K}) = \mathcal{S}_1(\mathcal{K}^{'})$, and we have the obvious characterization of classes of this type:

1.3 Lemma. *The following properties of an abstract class of semisimple left R-modules are equivalent:*

(a) *$\mathcal{S}$ is closed with respect to submodules and direct sums;*

(b) *$\mathcal{S} = \mathcal{S}_1(\mathcal{K})$ for some class $\mathcal{K}$ of simple left R-modules.*

For any ${}_RM$ we put

$$\mathrm{Soc}_{\mathcal{S}}(M) = \sum\{N \subseteq M \mid N \in \mathcal{S}\}.$$

If the abstract class $\mathcal{S}$ of semisimple left R-modules is closed with respect to submodules and direct sums, then $\mathcal{S}$ is also closed with respect to factor modules and hence $\mathrm{Soc}_{\mathcal{S}}(M)$ is the largest submodule of M belonging to $\mathcal{S}$. Hereby

$$\mathrm{Soc}_{\mathcal{S}}(M) = \sum\{Q \subseteq M \mid Q \in \mathcal{K}^{'}\},$$

where $\mathcal{K}$ is any class of simple left R-modules with $\mathcal{S}_1(\mathcal{K}) = \mathcal{S}$.

A module ${}_RM$ is called *ω-thick relative to $\mathcal{S}$* provided $\dim S < \omega$ for any subfactor S of M with $S \in \mathcal{S}$.

By $\mathcal{T}$ we denote the class of all semisimple modules. Given a property $I\!\!P$ of modules relative to any class $\mathcal{S}$ we will simply say that a module M *has property $I\!\!P$* if M has property $I\!\!P$ relative to $\mathcal{T}$.

Modules which are $\aleph_0$-thick relative to $\mathcal{S}_1(\mathcal{K})$ were studied in [8] under the name *countably thick relative to* $\mathcal{K}$. The smallest cardinality ω for which a module M is ω-thick is called *thickness* of the module M. A notion similar to our thickness is considered in [17].

In Section 2 suitable techniques are developed to handle independent and coindependent families of submodules.

In Section 3 these techniques are applied to the investigation of modules M over arbitrary rings, which are ω-thick relative to $\mathcal{S}$. Theorem 3.1 contains several characterizations of such modules in terms of families of submodules of M or of any subfactor of M. A very transparent form of these characterizations is obtained for ω-thick modules. They are collected in Corollary 3.5.

A module is called *q.f.d.* (*quotient finite dimensional*) if all its factor modules have finite uniform dimension. By Corollary 3.5, 2-thick modules coincide with uniserial modules, and $\aleph_0$-thick modules are just q.f.d. modules.

With any abstract class $\mathcal{S}$ of semisimple left R-modules we associate the class $\widetilde{\mathcal{S}}$ consisting of all modules $N \in \mathcal{S}$ which are not square free, i.e., which have at least one homogeneous component of dimension > 1, and put $\mathcal{S}_2(\mathcal{K}) = \widetilde{\mathcal{S}_1(\mathcal{K})}$. We call M *ω-hyperdistributive*, if M is ω-thick relative to $\widetilde{\mathcal{T}}$.

By a well-known criterion for distributivity of modules [18, Theorem 1], 2-hyperdistributivity of a module is equivalent to its distributivity. In [3] a generalization of this criterion for distributivity is proved which we formulate using the definition given above:

1.4 Lemma. *For ${}_RM$ and $\omega \geq 2$, the following are equivalent:*

(a) *M is ω-distributive;*

(b) *for any simple module ${}_RP$, M is ω-thick relative to $\mathcal{S}_1(P)$;*

(c) *for any simple module ${}_RP$, M is ω-thick relative to $\mathcal{S}_2(P)$.*

The lemma allows to deduce several characterizations of ω-distributivity of modules from Theorem 3.1. This is done in Corollary 3.7.

For any cardinality $\omega \geq 2$ we define a cardinality $\omega - 1$ in the following way:

If $\omega < \aleph_0$, i.e. $\omega = n$, then we put $\omega - 1 = n - 1$.

If $\omega \geq \aleph_0$, then we put $\omega - 1 = \omega$.

1.5 Semi-minimal (-maximal) submodules. We call a submodule A of M ω-*semi-minimal* relative to $\mathcal{S}$, if $A \in \mathcal{S}$ and $\dim A = \omega - 1$. Dually, a submodule A of M is called ω-*semi-maximal* relative to $\mathcal{S}$, provided the factor module $M/A \in \mathcal{S}$ and $\dim M/A = \omega - 1$. A submodule A of M will be called ω-*semi-minimal* (ω-*semi-maximal*) if A is ω-semi-minimal (resp., ω-semi-maximal) relative to $\mathcal{T}$.

1.6 (Co-) quasi-invariant submodules. Recall that a module M is said to be *quasi-invariant* (see [16]) provided every maximal submodule is fully invariant in M. Dually, we say that M is *co-quasi-invariant*, if every minimal submodule is fully invariant in M. Generalizing these definitions we call M ω-*quasi-invariant* relative to $\mathcal{S}$, provided any submodule, which is ω-semi-maximal relative to $\mathcal{S}$, is fully invariant in M. Similarly, M is said to be ω-*co-quasi-invariant* relative to $\mathcal{S}$ if any submodule, which is ω-semi-minimal relative to $\mathcal{S}$, is fully invariant in M. By definition quasi-invariance (co-quasi-invariance) of a module M is equivalent to its 2-quasi-invariance (2-co-quasi-invariance).

It is well-known that distributivity of a module is equivalent to quasi-invariance of all its subfactors. One direction of this is proved in [9], the converse implication is shown in [16]. In Section 4 a generalization of these facts is obtained (see 4.4). In particular, ω-hyperdistributive modules are characterized by ω-quasi-invariant and ω-co-quasi-invariant modules (see 4.5).

Consider abstract classes $\mathcal{L}$ and $\mathcal{S}$ of semisimple left R-modules. Clearly, if $\mathcal{L} \subseteq \mathcal{S}$ then any module which is ω-thick relative to $\mathcal{S}$ is also ω-thick relative to $\mathcal{L}$. From this and Lemma 1.4 we deduce that

(1) *any ω-thick module is ω-hyperdistributive;*

(2) *any ω-hyperdistributive module is ω-distributive.*

In Section 5 we ask how to describe rings for which, for a given $\omega \geq 2$,

(1) all ω-(hyper-)distributive left modules are ω-thick;

(2) all ω-distributive left modules are ω-hyperdistributive.

The answers are given in Lemma 5.2, 5.6 and 5.7, and Theorem 5.9, 5.10.

In Corollary 5.11 and 5.12, rings R with only finitely many non-isomorphic simple modules are characterized in terms of ω-thick, ω-hyperdistributive, and ω-distributive R-modules. Corollary 5.13 generalizes the well-known characterization of distributive commutative rings by localization with respect to all maximal ideals [13] to the case of n-distributive commutative rings.

A module M is said to be *fully cyclic* [15] if all its submodules are cyclic, and M is a *Bézout module* if all its finitely generated submodules are cyclic. It is well-known (see, for example, [9], [15] - [16]) that fully cyclic and Bézout modules are closely related to distributive and uniserial modules. To establish analogous relationships for relative ω-thick modules we define the notions of relative ω-noetherian and relative ω-Bézout modules.

1.7 Relative ω-noetherian modules. A module M is called *ω-noetherian relative to $\mathcal{S}$*, provided for any of its subfactors S, whose semisimple subfactors belong to $\mathcal{S}$, there exists a cardinality $\varrho < \omega$ such that S is ϱ-generated (i.e., has a generating set of cardinality ϱ). Considering the special case $\mathcal{S} = \mathcal{T}$ it is easy to see the equivalence of the following properties of ${}_RM$:

(a) *M is ω-noetherian;*

(b) *for any submodule A of M, there exists a cardinality $\varrho < \omega$ such that A is ϱ-generated;*

If $\omega = n$ is finite then these assertions are obviously equivalent to:

(c) *all submodules of M are $(n-1)$-generated.*

Obviously $\aleph_0$-noetherian modules coincide with noetherian modules, and 2-noetherian modules coincide with fully cyclic modules. Notice that ω-noetherian modules were introduced in Osofsky [19] in a slightly different way: as modules for which all submodules are ω-generated. So the ω-noetherian modules in the sense of Osofsky correspond exactly to ω^+-noetherian modules in our sense.

1.8 Relative ω-Bézout modules. We call a module ${}_RM$ *ω-Bézout relative to $\mathcal{S}$*, provided for any ω-generated subfactor S, whose semisimple subfactors belong to $\mathcal{S}$, there exists a cardinal $\varrho < \omega$ such that S is ϱ-generated. Putting $\mathcal{S} = \mathcal{T}$ we easily obtain the equivalence of the following properties of a module ${}_RM$:

(a) *M is ω-Bézout;*

(b) *for every ω-generated submodule A of M, there exists a cardinality $\varrho < \omega$ such that A is ϱ-generated;*

If $\omega = n$ is a finite cardinality then these assertions are equivalent to:

(c) *all finitely generated submodules of M are $(n-1)$-generated.*

${}_RM$ will be called *homogeneously ω-noetherian* provided M is ω-noetherian relative to $\mathcal{S}_1(P)$, for every simple R-module ${}_RP$. M will be called *homogeneously fully cyclic (homogeneously Bézout)* if M is homogeneously 2-noetherian (respectivly, homogeneously 2-Bézout).

If $\mathcal{L} \subseteq \mathcal{S}$, where $\mathcal{L}$ and $\mathcal{S}$ are abstract classes of semisimple left R-modules, then - as in the case of relative ω-thick modules - we have:

(1) *any module which is ω-noetherian to $\mathcal{S}$ is ω-noetherian relative to $\mathcal{L}$;*

(2) *any module which is ω-Bézout relative to $\mathcal{S}$ is ω-Bézout relative to $\mathcal{L}$.*

In particular, all ω-noetherian modules are homogeneously ω-noetherian, and all ω-Bézout modules are homogeneously ω-Bézout.

It is easy to see that a ring R is left quasi-invariant if and only if all cyclic left R-modules are quasi-invariant. We call a ring R *generalized left quasi-invariant* provided all cyclic semisimple left R-modules are quasi-invariant.

For any module M and cardinality $\omega \geq 2$ consider the conditions:

(i)	M is ω-noetherian;	(v)	M is ω-distributive;
(ii)	M is homogeneously ω-noetherian;	(vi)	M is ω-hyperdistributive;
(iii)	M is ω-Bézout;	(vii)	M is ω-thick.
(iv)	M is homogeneously ω-Bézout;		

In Section 6 the study of relationships between the conditions (i)-(vii) will be continued, which was begun in Section 5 for the properties (v)-(vii). The implications (i)$\Rightarrow$(iii) and (ii)$\Rightarrow$(iv) are obvious. The implications (i)$\Rightarrow$(ii), (iii)$\Rightarrow$(iv), (vii)$\Rightarrow$(vi)$\Rightarrow$(v) were observed above.

In Theorem 6.11 rings are characterized for which all Bézout left R-modules are distributive. From the other results of Section 6 we mention the following:

(1) *If ω is infinite then* (iii)⇒(vii) *and* (iv)⇒(v) *hold (Corollary 6.5);*

(2) *if ω is finite then* (vii)⇒(iii) *(Lemma 6.7);*

(3) *if R is a left perfect ring then* (v)⇒(i) *(Lemma 6.14*(2)*);*

(4) *if R is a semilocal ring and ω finite, then* (v)⇒(iii) *(Lemma 6.14*(1)*).*

A detailled list of the interdependence of the above mentioned properties will be given in Theorem 6.17.

As corollaries to these assertions we obtain well-known facts about the relationships between fully cyclic, Bézout, distributive and uniserial modules (see [9], [15]-[16]), results about connections between $\aleph_0$-distributive and noetherian modules [3], and also a theorem about left noetherian local rings proved in [6].

Let ${}_RU$ and ${}_RM$ denote left R-modules and $E = \mathrm{End}_R(U)$. Considering U as a bimodule ${}_RU_E$ in the usual way we have the functors

$$\begin{aligned} h^U &= \mathrm{Hom}_R(U,-): \quad R\text{-}Mod \to E\text{-}Mod, \text{ and} \\ h_U &= \mathrm{Hom}_R(-,U): \quad R\text{-}Mod \to Mod\text{-}E. \end{aligned}$$

Section 7 is devoted to the study of the relationships between ω-distributivity of the module M and properties of the E-modules $h^U(M)$ and $h_U(M)$. Notice that for $\omega = 2$ such connections were already investigated (see, for example, [11], [12], [14], [16]). The interest in this kind of questions stems, in particular, from their application to the study of endomorphism rings (for $M = U$ we have ${}_EE = h^U(U)$ and $E_E = h_U(U)$), and also from the application to the study of U as a module over its endomorphims ring (for $M = R$ we have $U_E \simeq h_U(R)$).

Lemma 7.6 gives a sufficient, and Lemma 7.13 gives a necessary condition for the ω-distributivity of the left E-module $h^U(M)$. The dual results on ω-distributivity of the right E-module $\mathrm{Hom}_R(M,U)$ are considered in Lemma 7.7 and 7.14. Moreover, Theorem 7.16 and 7.17 contain characterizations of ω-distributive modules M by properties of the left E-module $h^U(M)$ and the right E-module $h_U(M)$.

In Corollary 7.19 we collect applications of these results to endomorphism rings and modules over their endomorphism rings. Our observations generalize many known results about distributivity of modules (see [11], [12], [16], a.o.).

2 (Co-)independent families of submodules

In this section techniques suggested in [3] for handling independent and coindependent families of submodules are further developed. Our considerations will be summed up by Lemma 2.2 improving Lemma 4 in [3].

For any module ${}_RX$ and set Λ of cardinality $|\Lambda| \geq 2$, we denote by $\mathcal{F}_0(X,\Lambda)$ the set of all families $\mathbf{Y} = \{Y_\lambda\}_\Lambda$ of submodules of X. Define maps

$$\begin{array}{lll} \Sigma: & \mathcal{F}_0(X,\Lambda) \to \mathcal{L}(X), & \mathbf{Y} \mapsto \sum_\Lambda Y_\lambda, \\ \Gamma: & \mathcal{F}_0(X,\Lambda) \to \mathcal{L}(X), & \mathbf{Y} \mapsto \bigcap_\Lambda Y_\lambda. \end{array}$$

Moreover, we consider the maps

$$\begin{array}{lll} \sigma: & \mathcal{F}_0(X,\Lambda) \to \mathcal{F}_0(X,\Lambda), & \mathbf{Y} \mapsto \{\sigma(\mathbf{Y})_\lambda\}_\Lambda, \text{ where } \sigma(\mathbf{Y})_\lambda = \sum_{\Lambda\setminus\{\lambda\}} Y_\mu, \\ \gamma: & \mathcal{F}_0(X,\Lambda) \to \mathcal{F}_0(X,\Lambda), & \mathbf{Y} \mapsto \{\gamma(\mathbf{Y})_\lambda\}_\Lambda, \text{ where } \gamma(\mathbf{Y})_\lambda = \bigcap_{\Lambda\setminus\{\lambda\}} Y_\mu. \end{array}$$

To any $\mathbf{Y} = \{Y_\lambda\}_\Lambda \in \mathcal{F}_0(X,\Lambda)$ and submodule Z of $\Gamma(\mathbf{Y})$ we associate a *factor family*

$$\mathbf{Y}/Z = \{(Y/Z)_\lambda\}_\Lambda \in \mathcal{F}_0(X/Z,\Lambda), \text{ setting } (Y/Z)_\lambda = Y_\lambda/Z.$$

$\mathbf{Y}$ is called a *correct* family if $\Sigma(\mathbf{Y}) = X$, and is called *co-correct* if $\Gamma(\mathbf{Y}) = 0$.

We define an order relation on the set $\mathcal{F}_0(X,\Lambda)$, by putting

$$\mathbf{Y} = \{Y_\lambda\}_\Lambda \leq \mathbf{Y}' = \{Y'_\lambda\}_\Lambda \quad \text{provided} \quad Y_\lambda \subseteq Y'_\lambda \text{ for each } \lambda \in \Lambda.$$

For a family $\mathbf{Y} = \{Y_\lambda\}_\Lambda \in \mathcal{F}_0(X, \Lambda)$ consider the conditions:

(i1) $Y_\lambda \cap \sum_{\Theta \setminus \{\lambda\}} Y_\mu = 0$ *for any finite subset* $\emptyset \neq \Theta \subseteq \Lambda$ *and* $\lambda \in \Theta$*;*

(i2) $Y_\lambda \cap \sigma(\mathbf{Y})_\lambda = 0$ *for each* $\lambda \in \Lambda$*;*

(i3) $\Gamma(\sigma(\mathbf{Y})) = 0$*;*

(i4) $\gamma(\sigma(\mathbf{Y})) = \mathbf{Y}$ *and* $\Gamma(\sigma(\mathbf{Y})) = 0$*;*

and dually:

(c1) $Y_\lambda + \bigcap_{\Theta \setminus \{\lambda\}} Y_\lambda = X$*, for any finite subset* $\emptyset \neq \Theta \subseteq \Lambda$ *and* $\lambda \in \Theta$*;*

(c2) $Y_\lambda + \gamma(\mathbf{Y})_\lambda = X$ *for each* $\lambda \in \Lambda$*;*

(c3) $\Sigma(\gamma(\mathbf{Y})) = X$*;*

(c4) $\sigma(\gamma(\mathbf{Y})) = \mathbf{Y}$ *and* $\Sigma(\gamma(\mathbf{Y})) = X$*.*

It is easy to see that the conditions (i1)-(i4) are equivalent and characterize independent families $\mathbf{Y}$ of submodules of X. Concerning the conditions (c1)-(c4), we know by [3, Lemma 2],

(1) *the implications* (c4)$\Rightarrow$(c3)$\Rightarrow$(c2)$\Rightarrow$(c1) *hold;*

(2) *if* $\mathbf{Y}$ *is cocorrect then* (c4)$\Leftrightarrow$(c3)*;*

(3) *if* $\sigma(\gamma(\mathbf{Y})) = \mathbf{Y}$*, then* (c4)$\Leftrightarrow$(c3)$\Leftrightarrow$(c2)*;*

(4) *if* X *is an AB5** *module then* (c4)$\Leftrightarrow$(c3)$\Leftrightarrow$(c2)$\Leftrightarrow$(c1)*.*

A family $\mathbf{Y} = \{Y_\lambda\}_\Lambda$ of submodules of a module X is called *coindependent in the sense of Takeuchi* [21] (*weakly coindependent, coindependent, strongly coindependent*), if $\mathbf{Y}$ satisfies the conditions (c1) (respectively, (c2), (c3), (c4)). Notice that weakly coindependent families of submodules are considered in [22] under the name "coindependent".

In the partially ordered set $\mathcal{F}_0(X, \Lambda)$ we introduce two subsets:

$\mathbf{I}_0(X, \Lambda)$, consisting of all correct independent families $\mathbf{Y} = \{Y_\lambda\}_\Lambda$, and

$\mathbf{C}_0(X, \Lambda)$, consisting of all cocorrect strongly coindependent families $\mathbf{Y} = \{Y_\lambda\}_\Lambda$.

We immediately obtain:

2.1 Lemma. *For any* ${}_RX$ *and set* Λ *with* $|\Lambda| \geq 2$*, the following are equivalent:*

(a) σ *and* γ *establish a Galois correspondence between* $\mathcal{F}_0(X, \Lambda)$ *and* $\mathcal{F}_0(X, \Lambda)^*$*;*

(b) σ *and* γ *induce an anti-isomorphism of partially ordered sets between* $\mathbf{I}_0(X, \Lambda)$ *and* $\mathbf{C}_0(X, \Lambda)$*;*

(c) *the following diagrams are commutative:*

$$\begin{tikzcd} \mathcal{F}_0(X,\Lambda) \arrow[r, "\sigma"] \arrow[dr, "\Sigma"'] & \mathcal{F}_0(X,\Lambda) \arrow[r, "\gamma"] & \mathcal{F}_0(X,\Lambda) \arrow[dl, "\Sigma"] \\ & \mathcal{L}(X) & \end{tikzcd} , \quad \begin{tikzcd} \mathcal{F}_0(X,\Lambda) \arrow[r, "\gamma"] \arrow[dr, "\Gamma"'] & \mathcal{F}_0(X,\Lambda) \arrow[r, "\sigma"] & \mathcal{F}_0(X,\Lambda) \arrow[dl, "\Gamma"] \\ & \mathcal{L}(X) & \end{tikzcd} .$$

We call a family $\mathbf{Z} = \{Z_\lambda\}_\Lambda$ of left R-modules *$\mathcal{S}$-suitable* provided there exist simple subfactors P_λ of Z_λ ($\lambda \in \Lambda$) such that $\bigoplus_\Lambda P_\lambda \in \mathcal{S}$. A family $\mathbf{Y} = \{Y_\lambda\}_\Lambda$ of submodules of a module ${}_RX$ is called *$\mathcal{S}$-cosuitable* if the family $\{X/Y_\lambda\}_\Lambda$ is $\mathcal{S}$-suitable.

Submodules A and B of a module ${}_RX$ are called *coisomorphic* provided the factor modules X/A and X/B are isomorphic. A submodule A of X is called *$\mathcal{K}$-specific* if A has a $\mathcal{K}$-simple subfactor. Dually A is said to be *$\mathcal{K}$-cospecific* if the factor module X/A has a $\mathcal{K}$-simple subfactor.

For any ${}_RM$ and set Λ with $|\Lambda| \geq 2$, a pair $(S, \mathbf{K})$, consisting of a subfactor S of M and a family $\mathbf{K} = \{K_\lambda\}_\Lambda$ of submodules of S is called an (M, Λ)-*family*.

Let $I\!\!P$ be any property of families of submodules of some module. We say that an (M, Λ)-family $(S, \mathbf{K})$ *has property* $I\!\!P$ if the family $\mathbf{K}$ of submodules of S satisfies property $I\!\!P$.

By $\mathbf{F}(M, \Lambda)$ (respectively, $\mathbf{W}(M, \Lambda)$, $\mathbf{D}(M, \Lambda)$, $\mathbf{I}(M, \Lambda)$, $\mathbf{C}(M, \Lambda)$) we denote the set of all (respectively, all correct, all cocorrect, all correct independent, all cocorrect strongly coindependent) (M, Λ)-families. In view of the condition $|\Lambda| \geq 2$ it is easy to see that

$$\mathbf{I}(M, \Lambda) \subseteq \mathbf{D}(M, \Lambda) \quad \text{and} \quad \mathbf{C}(M, \Lambda) \subseteq \mathbf{W}(M, \Lambda).$$

Let $\mathcal{A} = (V/B, \mathbf{A}/B)$, where $B \subseteq A_\lambda \subseteq V \subseteq M$ for each $\lambda \in \Lambda$, be any (M, Λ)-family. We define mappings

$$\alpha_{M,\Lambda} : \mathbf{F}(M, \Lambda) \rightleftharpoons \mathbf{F}(M, \Lambda) : \beta_{M,\Lambda},$$

$$\varphi_{M,\Lambda} : \mathbf{W}(M, \Lambda) \rightleftharpoons \mathbf{D}(M, \Lambda) : \psi_{M,\Lambda},$$

putting

$$\begin{aligned} \alpha_{M,\Lambda}(\mathcal{A}) &= (\Sigma(\mathbf{A})/B, \mathbf{A}/B); \\ \beta_{M,\Lambda}(\mathcal{A}) &= (V/\Gamma(\mathbf{A}), \mathbf{A}/\Gamma(\mathbf{A})); \\ \varphi_{M,\Lambda}(\alpha_{M,\Lambda}(\mathcal{A})) &= (\Sigma(\mathbf{A})/\Gamma(\sigma(\mathbf{A})), \sigma(\mathbf{A})/\Gamma(\sigma(\mathbf{A}))); \\ \psi_{M,\Lambda}(\beta_{M,\Lambda}(\mathcal{A})) &= (\Sigma(\gamma(\mathbf{A}))/\Gamma(\mathbf{A}), \gamma(\mathbf{A})/\Gamma(\mathbf{A})). \end{aligned}$$

On the set of all subfactors of M, we introduce an order relation by the condition

$$V/B \leq V^{'}/B^{'}, \text{ provided } V \subset V^{'} \text{ and } B \subset B^{'},$$

where $B \subset V$ and $B^{'} \subset V^{'}$ are any submodules of M. Then $\mathbf{F}(M, \Lambda)$ (and its subsets $\mathbf{W}(M, \Lambda)$, $\mathbf{D}(M, \Lambda)$, $\mathbf{I}(M, \Lambda)$, $\mathbf{C}(M, \Lambda)$) is turned into a partially ordered set by putting for any $\mathcal{A} = (S, \mathbf{A})$, $\mathcal{A}^{'} = (S^{'}, \mathbf{A}^{'})$ in $\mathbf{F}(M, \Lambda)$,

$$\mathcal{A} \leq \mathcal{A}^{'} \quad \text{provided} \quad S \leq S^{'} \text{ and } A_\lambda \leq A^{'}_\lambda, \text{ for each } \lambda \in \Lambda.$$

2.2 Lemma. *Let ${}_RM$ be a module, Λ any set of cardinality $|\Lambda| \geq 2$, $\mathcal{S}$ an abstract class of semisimple left R-modules and $\mathcal{A} = (V/B, \mathbf{A}/B)$ an (M, Λ)-family, where $B \subseteq A_\lambda \subseteq V \subseteq M$ for all $\lambda \in \Lambda$. Then:*

(1) *$\alpha_{M,\Lambda}$ and $\beta_{M,\Lambda}$ are coclosure operations on the partially ordered sets $\mathbf{F}(M, \Lambda)$ and $\mathbf{F}(M, \Lambda)^*$, respectively, where*
$\alpha_{M,\Lambda}(\mathbf{F}(M, \Lambda)) = \mathbf{W}(M, \Lambda)$ *and* $\beta_{M,\Lambda}(\mathbf{F}(M, \Lambda)^*) = \mathbf{D}(M, \Lambda)^*$.

(2) *$\varphi_{M,\Lambda}$ and $\psi_{M,\Lambda}$ yield a Galois correspondence between $\mathbf{W}(M, \Lambda)$ and $\mathbf{D}(M, \Lambda)^*$, where*
$\varphi_{M,\Lambda}(\mathbf{W}(M, \Lambda)) = \mathbf{C}(M, \Lambda)^*$ *and* $\psi_{M,\Lambda}(\mathbf{D}(M, \Lambda)^*) = \mathbf{I}(M, \Lambda)$.

(3) *If $\mathbf{A}/B$ is an $\mathcal{S}$-suitable independent family of submodules of V/B, then $\sigma(\mathbf{A})/\Gamma(\sigma(\mathbf{A}))$ is an $\mathcal{S}$-cosuitable, cocorrect and strongly coindependent family of submodules of $\Sigma(\mathbf{A})/\Gamma(\sigma(\mathbf{A}))$.*

(4) *If $\mathbf{A}/B$ is an independent family of non-zero ($\mathcal{K}$-simple, pairwise isomorphic, $\mathcal{K}$-cospecific) submodules of V/B, then $\sigma(\mathbf{A})/\Gamma(\sigma(\mathbf{A}))$ is a cocorrect, strongly coindependent family of proper ($\mathcal{K}$-maximal, pairwise coisomorphic, $\mathcal{K}$-cospecific) submodules of $\Sigma(\mathbf{A})/\Gamma(\sigma(\mathbf{A}))$.*

(5) *If $\mathbf{A}/B$ is an $\mathcal{S}$-cosuitable weakly coindependent family of submodules of V/B, then $\gamma(\mathbf{A})/\Gamma(\mathbf{A})$ is an $\mathcal{S}$-suitable independent family of submodules of*
$\Sigma(\gamma(\mathbf{A}))/\Gamma(\mathbf{A})$.

(6) *If $\mathbf{A}/B$ is a weakly coindependent family of proper ($\mathcal{K}$-maximal, pairwise coisomorphic, $\mathcal{K}$-cospecific) submodules of V/B, then $\gamma(\mathbf{A})/\Gamma(\mathbf{A})$ is an independent family of non-zero ($\mathcal{K}$-simple, isomorphic copies, $\mathcal{K}$-cospecific) submodules of $\Sigma(\gamma(\mathbf{A}))/\Gamma(\mathbf{A})$.*

Proof. (1) is obvious.

(2) Clearly the maps $\varphi_{M,\Lambda} : \mathbf{W}(M,\Lambda) \rightleftharpoons \mathbf{D}(M,\Lambda)^* : \psi_{M,\Lambda}$ are antitone.

We want to show that

$$\mathcal{A} \le \psi_{M,\Lambda}(\varphi_{M,\Lambda}(\mathcal{A})) \quad \text{and} \quad \mathcal{A}' \ge \varphi_{M,\Lambda}(\psi_{M,\Lambda}(\mathcal{A}')),$$

for any $\mathcal{A} = (\Sigma(\mathbf{A})/B, \mathbf{A}/B) \in \mathbf{W}(M,\Lambda)$, $\mathcal{A}' = (V'/\Gamma(\mathbf{A}'), \mathbf{A}'/\Gamma(\mathbf{A}')) \in \mathbf{D}(M,\Lambda)^*$. For this notice that by Lemma 2.1(c),

$$\begin{aligned}\psi_{M,\Lambda}(\varphi_{M,\Lambda}(\mathcal{A})) &= (\Sigma(\mathbf{A})/\Gamma(\sigma(\mathbf{A})), \gamma(\sigma(\mathbf{A}))/\Gamma(\sigma(\mathbf{A}))),\\ \varphi_{M,\Lambda}(\psi_{M,\Lambda}(\mathcal{A}')) &= (\Sigma(\gamma(\mathbf{A}'))/\Gamma(\mathbf{A}'), \sigma(\gamma(\mathbf{A}'))/\Gamma(\mathbf{A}')).\end{aligned}$$

Applying Lemma 2.1(a) we obtain our assertion.

It remains to prove that $\varphi_{M,\Lambda}(\mathbf{W}(M,\Lambda)) = \mathbf{C}(M,\Lambda)^*$ and $\psi_{M,\Lambda}(\mathbf{D}(M,\Lambda)^*) = \mathbf{I}(M,\Lambda)$:
Indeed, $\mathcal{A}' \in C(M,\Lambda)^*$ means that $B' = \Gamma(\mathbf{A}')$, $\sigma(\gamma(\mathbf{A}')) = \mathbf{A}'$ and $\Sigma(\gamma(\mathbf{A}')) = V'$, implying the first equality. Similarly the second equality is obtained.

(3),(4) By (2), the direct decomposition

$$\Sigma(\mathbf{A})/B = (A_\lambda/B) \oplus (\sigma(\mathbf{A})_\lambda/B),$$

and the isomorphisms

$$(\Sigma(\mathbf{A})/\Gamma(\sigma(\mathbf{A})))/(\sigma(\mathbf{A})_\lambda/\Gamma(\sigma(\mathbf{A}))) \simeq A_\lambda/B,$$

hold for any $\lambda \in \Lambda$.

(5),(6) Since weak coindependence of the family $\mathbf{A}/B$ of submodules of V/B implies weak coindependence of the correct family $\mathbf{A}/\Gamma(\mathbf{A})$ of submodules of $V/\Gamma(\mathbf{A})$, in view of (2) the direct decompositions

$$V/\Gamma(\mathbf{A}) = (A_\lambda/\Gamma(\mathbf{A})) \oplus (\gamma(\mathbf{A})_\lambda/\Gamma(\mathbf{A})),$$

and isomorphisms

$$(V/B)/(A_\lambda/B) \simeq \gamma(\mathbf{A})_\lambda/\Gamma(\mathbf{A}),$$

hold for each $\lambda \in \Lambda$. □

3 Characterizations of relative ω-thick modules

Now the techniques developed in Section 2 will be applied to study modules which are ω-thick relative to $\mathcal{S}$ over abitrary rings.

A class $\mathcal{S}$ of semisimple left R-modules will be called *weakly hereditary* if for any module $X \in \mathcal{S}$ and cardinal ω with $2 \le \omega \le \dim X$, there exists a submodule $Y \subseteq X$ such that $Y \in \mathcal{S}$ and $\dim Y = \omega$. In particular, every *hereditary* (i.e. closed under submodules) class of semisimple modules is weakly hereditary. As an example of a weakly hereditary class of semisimple left R-modules which is not hereditary, one may consider $\widetilde{\mathcal{S}}$, where $\mathcal{S}$ is a hereditary class of semisimple R-modules.

3.1 Theorem. *For any module ${}_RM$, $\omega \ge 2$, and any abstract class $\mathcal{S}$ of semisimple left R-modules, the following are equivalent:*

(a) *M is ω-thick relative to $\mathcal{S}$;*

(b) *any $\mathcal{S}$-suitable independent family $\{K_\lambda\}_\Lambda$ of submodules of any subfactor S of M has $|\Lambda| < 2$;*

(c) *any cocorrect strongly coindependent family $\{K_\lambda\}_\Lambda$ of maximal submodules of any subfactor S of M, for which $\bigoplus_\Lambda(S/K_\lambda) \in \mathcal{S}$, has $|\Lambda| < \omega$;*

(d) *any $\mathcal{S}$-cosuitable weakly coindependent family $\{K_\lambda\}_\Lambda$ of submodules of any subfactor S of M has $|\Lambda| < \omega$;*

(e) *for any family* $\mathbf{A} = \{A_\lambda\}_\Lambda$ *of submodules of* M *with* $|\Lambda| \geq \omega$, *the family* $\gamma(\mathbf{A})/\Gamma(\mathbf{A})$ *of submodules of* $\Sigma(\Gamma(\mathbf{A}))/\Gamma(\mathbf{A})$ *is not* $\mathcal{S}$*-suitable;*

(f) *for any family* $\mathbf{A} = \{A_\lambda\}_\Lambda$ *of submodules of* M *with* $|\Lambda| \geq \omega$, *the family* $\sigma(\mathbf{A})/\Gamma(\sigma(\mathbf{A}))$ *of submodules of* $\Sigma(\mathbf{A})/\Gamma(\sigma(\mathbf{A}))$ *is not* $\mathcal{S}$*-cosuitable.*

If the class $\mathcal{S}$ *is weakly hereditary the following are equivalent to (a)-(f):*

(g) *for any family* $\mathbf{A} = \{A_\lambda\}_\Lambda$ *of submodules of* M *with* $|\Lambda| = \omega$, *the family* $\gamma(\mathbf{A})/\Gamma(\mathbf{A})$ *of submodules of* $\Sigma(\gamma(\mathbf{A}))/\Gamma(\mathbf{A})$ *is not* $\mathcal{S}$*-suitable;*

(h) *for any family* $\mathbf{A} = \{A_\lambda\}_\Lambda$ *of submodules of* M *with cardinality* $|\Lambda| = \omega$, *the family* $\sigma(\mathbf{A})/\Gamma(\sigma(\mathbf{A}))$ *of submodules of* $\Sigma(\mathbf{A})/\Gamma(\sigma(\mathbf{A}))$ *is not* $\mathcal{S}$*-cosuitable.*

The proof of Theorem 3.1 follows immediately from Lemma 3.2 and 3.3 below. To formulate these we need one more definition. We call a module ${}_RM$ *ω-pseudo-thick relative to $\mathcal{S}$*, if $S \notin \mathcal{S}$ for any semisimple subfactor of M with $\dim S = \omega$.

3.2 Lemma. *For any module* ${}_RM$, $\omega \geq 2$, *and any abstract class* $\mathcal{S}$ *of semisimple left* R*-modules, the following are equivalent:*

(a) M *is* ω*-thick relative to* $\mathcal{S}$*;*

(b) M *is* ϱ*-pseudo-thick relative to* $\mathcal{S}$, *for any cardinality* $\varrho \geq \omega$.

If the class $\mathcal{S}$ *is weakly hereditary, the following is equivalent to (a)-(b):*

(c) M *is* ω*-pseudo-thick relative to* $\mathcal{S}$.

The *proof* of the lemma is obvious.

3.3 Lemma. *For any module* ${}_RM$, $\omega \geq 2$, *and any abstract class* $\mathcal{S}$ *of semisimple left* R*-modules, the following are equivalent:*

(a) M *is* ω*-pseudo-thick relative to* $\mathcal{S}$*;*

(b) *any independent family* $\{K_\lambda\}_\Lambda$ *of submodules of any subfactor* S *of* M, *with* $|\Lambda| = \omega$, *is not* $\mathcal{S}$*-suitable;*

(c) $\bigoplus_\Lambda (S/K_\lambda) \notin \mathcal{S}$ *for any cocorrect strongly coindependent family* $\{K_\lambda\}_\Lambda$ *of maximal submodules of any subfactor* S *of* M *with* $|\Lambda| = \omega$*;*

(d) *any weakly coindependent family* $\{K_\lambda\}_\Lambda$ *of submodules of any subfactor* S *of* M, *with* $|\Lambda| = \omega$, *is not* $\mathcal{S}$*-cosuitable;*

(e) *for any family* $\mathbf{A} = \{A_\lambda\}_\Lambda$ *of submodules of* M *with* $|\Lambda| = \omega$, *the family* $\gamma(\mathbf{A})/\Gamma(\mathbf{A})$ *of submodules of* $\Sigma(\gamma(\mathbf{A}))/\Gamma(\mathbf{A})$ *is not* $\mathcal{S}$*-suitable;*

(f) *for any family* $\mathbf{A} = \{A_\lambda\}_\Lambda$ *of submodules of* M *with* $|\Lambda| = \omega$, *the family* $\sigma(\mathbf{A})/\Gamma(\sigma(\mathbf{A}))$ *of submodules of* $\Sigma(\mathbf{A})/\Gamma(\sigma(\mathbf{A}))$ *is not* $\mathcal{S}$*-cosuitable.*

Proof. (a)$\Rightarrow$(b) Assume $\{K_\lambda\}_\Lambda$ to be an independent $\mathcal{S}$-suitable family of submodules of a subfactor S of M and cardinality $|\Lambda| = \omega$. Then there exist simple subfactors P_λ of K_λ ($\lambda \in \Lambda$) with $\bigoplus_\Lambda P_\lambda \in \mathcal{S}$. This is a contradiction since $\bigoplus_\Lambda P_\lambda$ is a subfactor of M and $\dim(\bigoplus_\Lambda P_\lambda) = \omega$.

(b)$\Rightarrow$(d) By Lemma 2.2(5), the assumption of the existence of an $\mathcal{S}$-cosuitable weakly coindependent family $\mathbf{A}/B$ of submodules of a subfactor V/B of M, where $\{A_\lambda\}_\Lambda$, $|\Lambda| = \omega$, and $B \subseteq A_\lambda \subseteq V \subseteq M$ for all $\lambda \in \Lambda$, leads to an $\mathcal{S}$-suitable independent family $\gamma(\mathbf{A})/\Gamma(\mathbf{A})$ of submodules of $\Sigma(\gamma(\mathbf{A}))/\Gamma(\mathbf{A})$. This yields a contradicition to condition (b).

(d)⇒(c) is obvious.

(c)⇒(a) In contrast to (a), assume there exists an independent family $\mathbf{A}/B$ of simple submodules of the subfactor V/B of M, $\mathbf{A} = \{A_\lambda\}_\Lambda$, $|\Lambda| = \omega$, $\bigoplus_\Lambda (A_\lambda/B) \in \mathcal{S}$ and $B \subseteq A_\lambda \subseteq V \subseteq M$ for all $\lambda \in \Lambda$. In view of Lemma 2.2(4), $\sigma(\mathbf{A})/\Gamma(\sigma(\mathbf{A}))$ is a cocorrect strongly coindependent family of maximal submodules of $\Sigma(\mathbf{A})/\Gamma(\sigma(\mathbf{A}))$ where $\Sigma(\mathbf{A})/\sigma(\mathbf{A})_\lambda \simeq A_\lambda/B$, for all $\lambda \in \Lambda$. This contradicts condition (c).

(b)⇒(e) It suffices to recall that, by Lemma 2.2(2), the family $\gamma(\mathbf{A})/\Gamma(\mathbf{A})$ of submodules of $\Sigma(\gamma(\mathbf{A}))/\Gamma(\mathbf{A})$ are independent.

(e)⇒(d) Arguing by contradiction, let us assume the existence of an $\mathcal{S}$-cosuitable weakly coindependent family $\mathbf{A}/B$ of submodules of the factor module V/B of M, where $\mathbf{A} = \{A_\lambda\}_\Lambda$, $|\Lambda| = \omega$ and $B \subseteq A_\lambda \subseteq V \subseteq M$ for all $\lambda \in \Lambda$. By Lemma 2.2(5) we obtain an $\mathcal{S}$-suitable independent family $\gamma(\mathbf{A})/\Gamma(\mathbf{A})$ of submodules of $\Sigma(\gamma(\mathbf{A}))/\Gamma(\mathbf{A})$, which contradicts condition (e).

(d)⇒(f) It suffices to recall that, by Lemma 2.2(2), the family $\sigma(\mathbf{A})/\Gamma((\sigma(\mathbf{A}))$ of submodules of $\Sigma(\mathbf{A})/\Gamma(\sigma(\mathbf{A}))$ are strongly (and hence, in particular, weakly) coindependent.

(f)⇒(b) In contrast to (b), assume that there exists an $\mathcal{S}$-suitable independent family $\mathbf{A}/B$ of submodules of a subfactor V/B of M, where $\mathbf{A} = \{A_\lambda\}_\Lambda$, $|\Lambda| = \omega$ and $B \subseteq A_\lambda \subseteq V \subseteq M$ for all $\lambda \in \Lambda$. By Lemma 2.2(3), we have an $\mathcal{S}$-cosuitable family $\sigma(\mathbf{A})/\Gamma(\sigma(\mathbf{A}))$ of submodules of $\Sigma(\mathbf{A})/\Gamma(\sigma(\mathbf{A}))$, which is impossible by (f). This completes the proof of the Lemma. □

Putting $\mathcal{S} = \mathcal{S}_1(\mathcal{K})$, we obtain from Theorem 3.1:

3.4 Corollary. *For any module ${}_RM$, $\omega \geq 2$, and class $\mathcal{K}$ of simple left R-modules the following are equivalent:*

(a) *M is ω-thick relative to $\mathcal{S}_1(\mathcal{K})$;*

(b) *any independent family $\{K_\lambda\}_\Lambda$ of $\mathcal{K}$-specific submodules of any subfactor S of M has $|\Lambda| < \omega$;*

(c) *any cocorrect strongly coindependent family $\{K_\lambda\}_\Lambda$ of $\mathcal{K}$-maximal submodules of any subfactor S of M has $|\Lambda| < \omega$;*

(d) *any weakly coindependent family $\{K_\lambda\}_\Lambda$ of $\mathcal{K}$-cospecific submodules of any subfactor S of M has $|\Lambda| < \omega$;*

(e) *for any family $\mathbf{A} = \{A_\lambda\}_\Lambda$ of submodules of M with $|\Lambda| = \omega$, there exists $\lambda \in \Lambda$, such that $\gamma(\mathbf{A})_\lambda/\Gamma(\mathbf{A})$ is not a $\mathcal{K}$-specific submodule in $\Sigma(\gamma(\mathbf{A}))/\Gamma(\mathbf{A})$;*

(f) *for any family $\mathbf{A} = \{A_\lambda\}_\Lambda$ of submodules of M with $|\Lambda| = \omega$, there exists some $\lambda \in \Lambda$, such that $\sigma(\mathbf{A})_\lambda/\Gamma(\sigma(\mathbf{A}))$ is not a $\mathcal{K}$-cospecific submodule in $\Sigma(\mathbf{A})/\Gamma(\sigma(\mathbf{A}))$.*

Continuing the specialization of Theorem 3.1, we obtain by Corollary 3.4 for $\mathcal{K} = \mathcal{P}$, the class of all simple left R-modules:

3.5 Corollary. *For any module ${}_RM$ and $\omega \geq 2$, the following are equivalent:*

(a) *M is ω-thick;*

(b) *any independent family $\{K_\lambda\}_\Lambda$ of non-zero submodules of any subfactor S of M has $|\Lambda| < \omega$;*

(c) *any cocorrect strongly coindependent family $\{K_\lambda\}_\Lambda$ of maximal submodules of any subfactor S of M has $|\Lambda| < \omega$;*

(d) *any weakly coindependent family $\{K_\lambda\}_\Lambda$ of proper submodules of any subfactor S of M has $|\Lambda| < \omega$;*

(e) *for any family $\{A_\lambda\}_\Lambda$ of submodules of M and $|\Lambda| = \omega$, there exists $\lambda \in \Lambda$, such that*

$$A_\lambda \supseteq \bigcap_{\Lambda\setminus\{\lambda\}} A_\mu;$$

(f) *for any family $\{A_\lambda\}_\Lambda$ of submodules of M and $|\Lambda| = \omega$, there exists $\lambda \in \Lambda$, such that*

$$A_\lambda \subseteq \sum_{\Lambda\setminus\{\lambda\}} A_\mu.$$

3.6 Remark. The equivalence of the condition (a) and (b) in Corollary 3.5 shows, that $\aleph_0$-thick modules coincide with q.f.d. modules generalizing a well-known characterization of q.f.d. modules [20, Lemma]. For $\omega = \aleph_0$ the conditions (e) and (f) provide new characterizations of q.f.d. modules. The equivalence of the conditions (a) and (e) (or (a) and (f)) shows that 2-thick modules coincide with uniserial modules.

3.7 Corollary. *For any module $_RM$ and $\omega \geq 2$, the following are equivalent:*

(a) *M is ω-distributive;*

(b) *any independent family $\{K_\lambda\}_\Lambda$ of isomorphic simple submodules of any subfactor S of M has $|\Lambda| < \omega$;*

(c) *any independent family $\{K_\lambda\}_\Lambda$ of isomorphic non-zero submodules of any subfactor S of M has $|\Lambda| < \omega$;*

(d) *any cocorrect strongly coindependent family $\{K_\lambda\}_\Lambda$ of coisomorphic maximal submodules of any subfactor S of M has $|\Lambda| < \omega$;*

(e) *any weakly coindependent family $\{K_\lambda\}_\Lambda$ of coisomorphic proper submodules of any subfactor S of M has $|\Lambda| < \omega$;*

(f) *for any family $\mathbf{A} = \{A_\lambda\}_\Lambda$ of submodules of M with $|\Lambda| = \omega$, and any simple module $_RP$, there exists some $\lambda \in \Lambda$, such that $\gamma(\mathbf{A})_\lambda/\Gamma(\mathbf{A})$ has no subfactor isomorphic to P;*

(g) *for any family $\mathbf{A} = \{A_\lambda\}_\Lambda$ of submodules of M with $|\Lambda| = \omega$, and any simple module $_RP$, there exists some $\lambda \in \Lambda$, such that $\Sigma(\mathbf{A})/\sigma(\mathbf{A})_\lambda$ has no subfactor isomorphic to P.*

Proof. The conditions (a) and (b) are equivalent by Lemma 1.4. Moreover, notice that any family $\{K_\lambda\}_\Lambda$ of pairwise isomorphic non-zero submodules in S form a family of $\{P\}$-specific submodules of S, for some simple module $_RP$ (depending on $\{K_\lambda\}_\Lambda$). Similarly, any family of coisomorphic proper submodules of S form a family of $\{P\}$-cospecific submodules of S, for some simple module $_RP$. Therefore the equivalence of the conditions (b)-(g) follows from Corollary 3.4. □

3.8 Remark. The equivalence of the conditions (a)-(e) in Corollary 3.7 were proved in [3, Theorem 1]. Corollary 3.7 generalizes many known results about distributive modules ([23, Proposition 4.1.1], [9, p. 293, Corollary 1], [18, Theorem 1] and others).

4 ω-thick and ω-(co-)quasi-invariant modules

The following observation is obvious.

4.1 Lemma. *For any module $_RM$, $\omega \geq 2$, and any abstract class $\mathcal{S}$ of semisimple left R-modules, the following are equivalent:*

(a) *M is ω-thick relative to $\mathcal{S}$;*

(b) *all subfactors of M are ω-thick relative to $\mathcal{S}$;*

(c) *all semisimple subfactors of M are ω-thick relative to $\mathcal{S}$.*

Putting some conditions on the class of semisimple modules we have:

4.2 Lemma. *Let $\omega \geq 2$ and let $\mathcal{S}$ be an abstract class of semisimple left R-modules which is closed under submodules and finite direct sums. Then every module M which is ω-thick relative to $\widetilde{\mathcal{S}}$ is ω-quasi-invariant relative to $\mathcal{S}$ and ω-co-quasi-invariant relative to $\mathcal{S}$.*

Proof. Let ${}_RM$ be ω-thick relative to $\widetilde{\mathcal{S}}$. We prove that M is ω-quasi-invariant relative to $\mathcal{S}$. Assuming the contrary, consider a submodule $A \subseteq M$ which is ω-semi-maximal relative to $\mathcal{S}$, but is not fully invariant in M. Then $S = M/A \in \mathcal{S}$ and $\dim S = \omega - 1$, and $(A)\alpha \nsubseteq A$, for some $\alpha \in \mathrm{End}_R(M)$.

We have a monomorphism $\quad M/(A)\alpha^{-1} \to S, \quad x + (A)\alpha^{-1} \mapsto (x)\alpha + A.$

Moreover, the module $N = M/((A)\alpha^{-1} \cap A)$ embeds into $(M/(A)\alpha^{-1}) \oplus S$, and hence there exists a monomorphism $N \to S^2$. Since $S^2 \in \mathcal{S}$, we have $N \in \mathcal{S}$. Noticing that $(A)\alpha \nsubseteq A$, we deduce that the natural epimorphism $\pi : N \to S$ has kernel $\mathrm{Ke}\,\pi = A/((A)\alpha^{-1} \cap A) \neq 0$. So $N \simeq \mathrm{Ke}\,\pi \oplus S \in \widetilde{\mathcal{S}}$ and $\dim N \geq \omega$. This contradicts the fact that M is ω-thick relative to $\mathcal{S}$.

Dually the co-quasi-invariance of M relative to $\mathcal{S}$ is established. $\square$

4.3 Lemma. *For any semisimple module ${}_RT$, $\omega \geq 2$, and any abstract class $\mathcal{S}$ of semisimple left R-modules, which is closed under submodules and finite direct sums, the following are equivalent:*

(a) *T is ω-thick relative to $\widetilde{\mathcal{S}}$;*

(b) *T is ω-quasi-invariant relative to $\mathcal{S}$;*

(c) *T is ω-co-quasi-invariant relative to $\mathcal{S}$.*

Proof. (a)⇒(b) and (a)⇒(c) follow from Lemma 4.2.

(b)⇒(a) Assume that, in contrast to (a), the module T is not ω-thick relative to $\widetilde{\mathcal{S}}$. Then by the weak hereditariness of the class $\widetilde{\mathcal{S}}$, there exists a submodule S of T, such that $S \in \widetilde{\mathcal{S}}$ and $\dim S = \omega$. Then $T = S \oplus V$ for some submodule V of T. For a decomposition $S = \bigoplus_\Lambda P_\lambda$, where P_λ are simple modules, there exist $\xi, \eta \in \Lambda$, such that $\xi \neq \eta$ and $P_\xi \simeq P_\eta$. Since $T/(P_\xi \oplus V) \simeq \bigoplus_{\Lambda \setminus \{\xi\}} P_\lambda$, the submodule $P_\xi \oplus V$ of T is semi-maximal relative to $\mathcal{S}$. However, $P_\xi \oplus V$ is not a fully invariant submodule in T, since there exists $\alpha \in \mathrm{End}_R(M)$, for which $(P_\xi \oplus V)\alpha \subseteq P_\eta \oplus V$, a contradiction.

(c)⇒(a) This is shown with dual arguments. $\square$

4.4 Theorem. *For any module ${}_RM$, $\omega \geq 2$, and any abstract class $\mathcal{S}$ of semisimple left R-modules, which is closed under submodules and finite direct sums, the following are equivalent:*

(a) *M is ω-thick relative to $\widetilde{\mathcal{S}}$;*

(b) *each subfactor of M is ω-quasi-invariant relative to $\mathcal{S}$;*

(c) *each semisimple subfactor of M is ω-quasi-invariant relative to $\mathcal{S}$;*

(d) *each subfactor of M is ω-co-quasi-invariant relative to $\mathcal{S}$;*

(e) *each semisimple subfactor of M is ω-co-quasi-invariant relative to $\mathcal{S}$.*

Proof. (a)⇒(b) and (a)⇒(d) follow from Lemma 4.1 and 4.2.

(b)⇒(c) and (d)⇒(e) are obvious.

(c)⇒(a) and (e)⇒(a) follow from Lemma 4.1 and 4.3. $\square$

Putting $\mathcal{S} = \mathcal{T}$, we obtain from Theorem 4.4:

4.5 Corollary. *For a module ${}_RM$ and $\omega \geq 2$, the following are equivalent:*

(a) *M is ω-hyperdistributive;*

(b) *each subfactor of M is ω-quasi-invariant;*

(c) *each semisimple subfactor of M is ω-quasi-invariant;*

(d) *each subfactor of M is ω-co-quasi-invariant;*

(e) *each semisimple subfactor of M is ω-co-quasi-invariant.*

4.6 Remark. Putting $\omega = 2$, we obtain from Corollary 4.5 the well-known equivalence of the following properties of a module ${}_RM$:

(a) *M is distributive;*

(b) *each subfactor of M is quasi-invariant;*

(c) *each subfactor of M is co-quasi-invariant.*

The implications (a)⇒(b), (a)⇒(c) are shown in [9, Page 293, Corollary 4], and (b)⇒(a) is proved in [16, Lemma 12].

5 ω-thick and ω-(hyper-)distributive modules

As already mentioned in Section 1, any ω-thick module is ω-hyperdistributive and any ω-hyperdistributive module is ω-distributive. We are going to investigate conditions under which the converse implications hold.

For convenient reference we formulate the obvious

5.1 Lemma. *Let ${}_RT$ be a semisimple module, $\omega \geq 2$, and $\mathcal{S}$ an abstract class of semisimple left R-modules closed under submodules and direct sums. Then:*

(1) *T is ω-thick relative to $\mathcal{S}$ if and only if* $\dim \mathrm{Soc}_{\mathcal{S}}(T) < \omega$.

(2) *T is ω-thick relative to $\widetilde{\mathcal{S}}$ if and only if one of the following conditions holds:*

  (i) $\dim \mathrm{Soc}_{\mathcal{S}}(T) < \omega$*;*

  (ii) $\dim \mathrm{Soc}_{\mathcal{S}}(T) \geq \omega$ *and* $\dim \mathrm{Soc}_{\mathcal{S}}(T)$ *is square free.*

(3) *T is ω-thick if and only if* $\dim (T) < \omega$.

(4) *T is ω-hyperdistributive if and only if one of the following conditions holds:*

  (i) $\dim (T) < \omega$*;*

  (ii) $\dim (T) \geq \omega$ *and T is square free.*

(5) *T is ω-distributive if and only if each of its homogeneous components has dimension less than ω.*

5.2 Lemma. *For any module ${}_RM$ and $\omega \geq 2$, the following are equivalent:*

(a) *M is ω-thick;*

(b) *M is ω-hyperdistributive and* $\mathrm{crs}(S) < \omega$*, for any semisimple subfactor S of M.*

Proof. (a)⇒(b) It suffices to notice that by condition (a), for any semisimple subfactor S of M we have $\mathrm{crs}(S) \leq \dim S < \omega$.

(b)⇒(a) By Lemma 4.1 and 5.1(4), for any semisimple subfactor S of M one of the following condition is satisfied:

$$\text{(i) } \dim S < \omega; \quad \text{(ii) } \dim S \geq \omega \text{ and } S \text{ is square free.}$$

Since condition (ii) contradicts (b), we conclude that (i) holds. □

5.3 Corollary. *For $\omega \geq 2$ and a module ${}_RM$, satisfying* $\mathrm{crs}(M) < \omega$*, the following are equivalent:*

(a) *M is ω-thick;*

(b) *M is ω-hyperdistributive.*

Bad and good modules. Let ${}_RT$ be a semisimple module and $\omega \geq 2$. T is called *ω-bad* if $\dim T \geq \omega$ and every homogeneous component of T has dimension less than ω. We call T *ω-quasi-bad*, if T is ω-bad and is not square free. T is called *ω-good (ω-quasi-good)*, if T is not ω-bad (respectively, ω-quasi-bad).

If the module T is ω-bad, then we say that a cardinal ϱ is *ω-bad for T*, if $\varrho = \operatorname{crs}(V)$ for some submodule V of T of dimension $\dim V = \omega$ (such a submodule V is necessarily ω-bad). If T is ω-quasi-bad, we say that a cardinal ϱ is *ω-quasi-bad for T*, if $\varrho = \operatorname{crs}(V)$ for some ω-quasi-bad submodule V of T of dimension $\dim V = \omega$.

Moreover, we define cardinals $\operatorname{cf}_T(\omega)$ and $\operatorname{qcf}_T(\omega)$, which we will call, respectively, *cofinal character* of the cardinal ω relative to T, and *quasi-cofinal character* of ω relative to T:

If T is ω-good, we put $\operatorname{cf}_T(\omega) = \operatorname{qcf}_T(\omega) = \omega$. If T is ω-bad we define $\operatorname{cf}_T(\omega)$ as smallest ω-bad cardinality of T. If T is ω-bad and square free, we put $\operatorname{qcf}_T(\omega) = (\dim T)^+$. Finally, if T is ω-quasi-bad we define $\operatorname{qcf}_T(\omega)$ as smallest ω-quasi-bad cardinality of T. Obviously:

5.4 Lemma. *For any semisimple module ${}_RT$ and $\omega \geq 2$ we have*

$$2 \leq \operatorname{cf}(\omega) \leq \operatorname{cf}_T(\omega) \leq \omega, \quad \operatorname{cf}_T(\omega) \leq \operatorname{qcf}_T(\omega) \leq (\dim T)^+.$$

In particular, if ω is regular then $\operatorname{cf}_T(\omega) = \omega$.

5.5 Lemma. *If $\omega \geq 2$ and R satisfies $\operatorname{crs}({}_RR) \geq \operatorname{cf}(\omega)$, then there exists some ω-distributive semisimple module ${}_RT$, for which $\operatorname{crs}(T) = \operatorname{cf}(\omega)$, and moreover,*

(1) *$\operatorname{cf}_T(\omega) = \operatorname{cf}(\omega)$;*

(2) *if $\omega \geq 3$ then $\operatorname{qcf}_T(\omega) = \operatorname{cf}(\omega)$.*

Proof. By definition of $\operatorname{cf}(\omega)$, there exists a family of cardinalities $\{\alpha_\xi\}_\Xi$, such that $\alpha_\xi < \omega$ for each $\xi \in \Xi$, $\sum_\Xi \alpha_\xi = \omega$ and $|\Xi| = \operatorname{cf}(\omega)$. Let $\{P_\xi\}_\Xi$ be a family of pairwise non-isomorphic simple left R-modules. Putting $T = \bigoplus_\Xi P_\xi^{(\alpha_\xi)}$ and recalling the definition of $\operatorname{cf}_T(\omega)$ and $\operatorname{cf}(\omega)$, it is easily seen that (1) holds.

We procede to the proof of (2). If $\omega \geq 3$ is regular, then ω is infinite. This gives the possibility to assume that $\alpha_\xi \geq 2$ for some $\xi \in \Xi$ (otherwise we may replace T by $T \oplus P_\xi$). If $\omega \geq 3$ is singular, then again $\alpha_\xi \geq 2$ for some $\xi \in \Xi$ (otherwise $\sum_\Xi \alpha_\xi = \operatorname{cf}(\omega) \neq \omega$). So in both cases the module T is ω-quasi-bad. Now referring to the definition of $\operatorname{qcf}_T(\omega)$ and $\operatorname{cf}(\omega)$, we easily derive that $\operatorname{qcf}_T(\omega) = \operatorname{cf}(\omega)$. $\square$

5.6 Lemma. *For any module ${}_RM$ and $\omega \geq 2$, the following are equivalent:*

(a) *M is ω-thick;*

(b) *M is ω-distributive and each semisimple subfactor of M is ω-good;*

(c) *M is ω-distributive and $\operatorname{crs}(S) < \operatorname{cf}_S(\omega)$, for any semisimple subfactor S of M.*

Proof. (a)$\Rightarrow$(b) is obvious.

(b)$\Rightarrow$(a) It suffices to notice that, by Lemma 4.1 and 5.1(5), $\dim S < \omega$ for any ω-good semisimple subfactor S of M.

(b)$\Leftrightarrow$(c) This follows from the fact that for any ω-distributive semisimple module S, the condition $\operatorname{crs}(S) < \operatorname{cf}_S(\omega)$ is equivalent to S being ω-good. $\square$

Now we turn to the study of the relations between ω-hyperdistributivity and ω-distributivity. First recall that for any module ${}_RM$ the following are equivalent:

(a) *M is 2-hyperdistributive;* (b) *M is 2-distributive;* (c) *M is distributive.*

5.7 Lemma. *For any module ${}_RM$ and $\omega \geq 2$, the following are equivalent:*

(a) *M is ω-hyperdistributive;*

(b) *M is ω-distributive and every semisimple subfactor of M is ω-quasi-good;*

(c) *M is ω-distributive and* $\mathrm{crs}(S) < \mathrm{qcf}_S(\omega)$, *for any semisimple subfactor S of M.*

The proof is similar to the proof of Lemma 5.6 which also implies:

5.8 Corollary. *For $\omega \geq 2$ and a module ${}_RM$ with* $\mathrm{crs}(M) < \mathrm{cf}(\omega)$, *the following are equivalent:*

(a) *M is ω-thick;*

(b) *M is ω-hyperdistributive;*

(c) *M is ω-distributive.*

The next results show the influence of the number of simple modules on the relation between thickness and distributivity.

5.9 Theorem. *For any ring R and $\omega \geq 2$, the following are equivalent:*

(a) $\mathrm{crs}({}_RR) < \omega$*;*

(b) *every ω-hyperdistributive left R-module is ω-thick;*

(c) *every ω-hyperdistributive semisimple left R-module is ω-thick.*

Proof. (a)$\Rightarrow$(b) follows from Corollary 5.3; (b)$\Rightarrow$(c) is obvious.

(c)$\Rightarrow$(a) Let $\{P_\xi\}_\Xi$ be a representing family of all simple left R-modules. Assuming, contrary to (a), that $|\Xi| \geq \omega$, we consider the module $T = \bigoplus_\Xi P_\xi$. By Lemma 5.1(3),(4), T is ω-hyperdistributive and is not ω-thick contradicting (c). $\square$

5.10 Theorem. *For any ring R and $\omega \geq 2$, the following are equivalent:*

(a) $\mathrm{crs}({}_RR) < \mathrm{cf}(\omega)$*;*

(b) *every ω-distributive left R-module is ω-thick;*

(c) *every ω-distributive semisimple left R-module is ω-thick;*

(d) *every ω-distributive semisimple left R-module is ω-good.*

If $\omega \geq 3$ then (a)-(d) are equivalent to:

(e) *every ω-distributive left R-module is ω-hyperdistributive;*

(f) *every ω-distributive semisimple left R-module is ω-hyperdistributive;*

(g) *every ω-distributive semisimple left R-module is ω-quasi-good.*

Proof. (a)$\Rightarrow$(b) follows from Corollary 5.8; (b)$\Rightarrow$(c) is obvious.

(c)$\Rightarrow$(a) Assume $\mathrm{crs}({}_RR) \geq \mathrm{cf}(\omega)$. Then by Lemma 5.5, there exists an ω-distributive semisimple module ${}_RT$, for which $\mathrm{crs}(T) = \mathrm{cf}_T(\omega)$. By Lemma 5.6, the module T is not ω-thick, contradicting condition (c).

(c)$\Leftrightarrow$(d) follows from Lemma 5.1(3), (5).

(b)$\Rightarrow$(e)$\Rightarrow$(f) are obvious: (f)$\Leftrightarrow$(g) follows by Lemma 5.1(4), (5).

For the rest of the proof we assume $\omega \geq 3$.

(f)$\Rightarrow$(a) Assume that, contrary to (a), we have $\mathrm{crs}({}_RR) \geq \mathrm{cf}(\omega)$. Then by Lemma 5.5, there exists an ω-distributive semisimple module ${}_RT$, for which $\mathrm{crs}(T) = \mathrm{qcf}_T(\omega)$. By Lemma 5.7, T is not ω-hyperdistributive, contradicting (f). $\square$

5.11 Corollary. *For a ring R the following are equivalent:*

(a) $\operatorname{crs}({}_RR) = 1$*;*

(b) *for $\omega \geq 2$, all ω-distributive left R-modules are ω-thick;*

(c) *for some $n \geq 2$, all n-distributive semisimple left R modules are n-thick;*

(d) *for $\omega \geq 2$, all ω-distributive left R-modules are ω-hyperdistributive;*

(e) *for some $n \geq 3$, all n-distributive semisimple left R-modules are n-hyperdistributive;*

(f) *for $\omega \geq 2$, all ω-hyperdistributive left R-modules are ω-thick;*

(g) *all distributive semisimple left R-modules are uniserial.*

Proof. The equivalence of (a)-(e) follows from Theorem 5.10 and the fact that $\operatorname{cf}(n) = 2 \leq \operatorname{cf}(\omega)$, for any finite $n \geq 2$ and any $\omega \geq 2$.

The equivalence of (a), (f) and (g) follows from Theorem 5.9. □

5.12 Corollary. *For any ring R the following are equivalent:*

(a) $\operatorname{crs}({}_RR) < \aleph_0$*;*

(b) *for infinite ω, all ω-distributive left R-modules are ω-thick;*

(c) *all $\aleph_0$-distributive semisimple left R-modules are $\aleph_0$-thick;*

(d) *for infinite ω, all ω-distributive left R-modules are ω-hyperdistributive;*

(e) *all $\aleph_0$-distributive semisimple left R-modules are $\aleph_0$-hyperdistributive;*

(f) *for infinite ω, all ω-hyperdistributive left R-modules are ω-thick;*

(g) *all $\aleph_0$-hyperdistributive semisimple left R-modules are $\aleph_0$-thick.*

Proof. The equivalence of (a)-(e) follows from Theorem 5.10 and the fact that $\operatorname{cf}(\aleph_0) = \aleph_0 \leq \operatorname{cf}(\omega)$, for any infinite ω.

The equivalence of (a), (f) and (g) follows from Theorem 5.9. □

5.13 Corollary. *For R commutative and $n \geq 2$, the following are equivalent:*

(a) *R is n-distributive;*

(b) *the localization of R at any maximal ideal is an n-thick ring.*

Proof. Let R_m be the localization of R at the ideal $m \in \max({}_RR)$ and ρ_m the canonical map of the lattice of ideals of R to the lattice of ideals of R_m, defined by

$$\rho_m(I) = \{a/b \mid a \in I,\, b \in R\backslash m\}, \text{ for } I \in \mathcal{L}({}_RR).$$

It is well-known that ρ_m is a lattice homomorphism and the family $\{\rho_m(I)\}_{m\in\max(R)}$ uniquely determines the ideal I. Therefore the n-distributivity of R is equivalent to the n-distributivity of all rings R_m, where $m \in \max(R)$. It remains to notice that, by Corollary 5.11, any local ring is n-distributive if and only if it is n-thick. □

5.14 Remark. For $\omega = 2$, Corollary 5.3 and 5.8 lead to a well-known result about distributivity of modules [16, Lemma 16]. Similarly Corollary 5.11 is a generalization of [10, Lemma 1.11], Corollary 5.12 is a generalization of [3, Corollary 5], and Corollary 5.13 generalizes [13, Lemma 1]. Compare also [25].

6 ω-thick, ω-noetherian and ω-Bézout modules

We will need the obvious

6.1 Lemma. *If the module* ${}_RM$ *satisfies* $\operatorname{crs}(S) = 1$, *for any semisimple subfactor* S *of* M *(e.g., if* $\operatorname{crs}(M) = 1$*), then for any* $\omega \geq 2$ *we have:*

(1) *M is ω-noetherian if and only if M is homogeneously ω-noetherian;*

(2) *M is ω-Bézout if and only if M is homogeneously ω-Bézout.*

6.2 Lemma. *For a semisimple module* ${}_RT$ *and any infinite cardinal* ϱ, *the following are equivalent:*

(a) *T is ϱ-generated;*

(b) $\dim(T) \leq \varrho$.

Proof. (a)$\Rightarrow$(b) Assume $\omega = \dim(T) > \varrho$. Then considering for T a generating set of cardinality ϱ, it is easy to see that $\omega \leq \varrho\aleph_0 = \varrho < \omega$, a contradiction.

(b)$\Rightarrow$(a) is obvious. □

6.3 Lemma. *For a semisimple module* ${}_RT$, $\omega \geq 2$, *and an abstract class* $\mathcal{S}$ *of semisimple left* R*-modules, consider the conditions*

(1) *T is ω-thick relative to $\mathcal{S}$;*

(2) *T is ω-noetherian relative to $\mathcal{S}$;*

(3) *T is ω-Bézout relative to $\mathcal{S}$.*

Then (1)$\Rightarrow$(2)$\Rightarrow$(3). *If ω is infinite and $\mathcal{S}$ is weakly hereditary then* (3)$\Rightarrow$(1).

Proof. (1)$\Rightarrow$(2) For $S \in \mathcal{S} \cap \mathcal{L}(T)$ put $\varrho = \dim S$. Then S is ϱ-generated and hence $\varrho < \omega$ by (1).

(2)$\Rightarrow$(3) is obvious.

(3)$\Rightarrow$(1) It suffices to show that $\dim(S) < \omega$, for any $S \in \mathcal{S} \cap \mathcal{L}(T)$. Assume that $\dim S \geq \omega$, for some $S \in \mathcal{S} \cap \mathcal{L}(T)$. Since $\mathcal{S}$ is weakly hereditary there exists some submodule $V \subseteq S$, $V \in \mathcal{S}$, with $\dim V = \omega$. From (3) and Lemma 6.2 it follows that $\dim V \leq \varrho$, for some cardinal $\varrho < \omega$, a contradiction. □

6.4 Corollary. *For a semisimple module* ${}_RT$ *and* $\omega \geq 2$ *consider the conditions:*

(1) T is ω-thick;	*(4) T is ω-distributive;*
(2) T is ω-noetherian;	*(5) T is homogeneously ω-noetherian;*
(3) T is ω-Bézout;	*(6) T is homogeneously ω-Bézout.*

Then (1)$\Rightarrow$(2)$\Rightarrow$(3) *and* (4)$\Rightarrow$(5)$\Rightarrow$(6).
If ω is infinite, then (3)$\Rightarrow$(1) *and* (6)$\Rightarrow$(4).

6.5 Corollary. *Let R be a ring, ω infinite, and $\mathcal{S}$ a weakly hereditary abstract class of semisimple left R-modules. Then all left R-modules which are ω-Bézout relative to $\mathcal{S}$ are ω-thick relative to $\mathcal{S}$.*

In particular, all ω-Bézout left R-modules are ω-thick and all homogeneously ω-Bézout left R-modules are ω-distributive.

Proof. Since all semisimple subfactors of a module M which are ω-Bézout relative to $\mathcal{S}$, are ω-Bézout relative to $\mathcal{S}$, it suffices to apply Lemmata 6.3 and 4.1. □

6.6 Example. Let $R = K_n$ - the (n, n)-matrix ring over a field K. Then ${}_RR \simeq P^n$, where ${}_RP$ is a simple R-module. The module ${}_RR$ is fully cyclic (and hence n-noetherian, Bézout, n-Bézout) however it is not n-distributive (and hence not n-thick). This shows that the condition $\omega \geq \aleph_0$ in the formulation of Lemma 6.3 and Corollaries 6.4, 6.5 is essential.

6.7 Lemma. *For any ring R and $n \geq 2$, every n-thick left R-module is n-Bézout.*

Proof. If ${}_RM$ is an n-thick module, then by Corollary 3.5, for any n-generated submodule $A = \sum_{i=1}^n Ra_i \subseteq M$, there exists $i \in \{1, 2, \ldots, n\}$, such that

$$Ra_i \subseteq \sum_{\substack{1 \leq j \leq n \\ j \neq i}} Ra_j.$$

Therefore A is $(n-1)$-generated. □

6.8 Remark. Lemma 6.7 generalizes the well-known fact that every uniserial module is Bézout (e.g., [1, Theorem 55.1(2)(ii)]).

6.9 Lemma. *Let ${}_RT$ be a semisimple module and $n \geq 2$. If T is n-distributive and finitely generated, then T is $(n-1)$-generated.*

Proof. For any $\alpha \in \{0, 1\}$ and module ${}_RM$, denote by $M^{[\alpha]}$ the left R-module 0 if $\alpha = 0$, and the module M itself if $\alpha = 1$. Without loss of generality we may assume that $T \neq 0$. Notice that for some pairwise non-isomorphic simple left R-modules $P_1, \ldots, P_r$ and natural numbers $k_1, \ldots, k_r < n$, we have an isomorphism $T \simeq \bigoplus_{i=1}^r P_i^{k_i}$. Putting $k = \max_{1 \leq i \leq r} k_i$, we can write $T \simeq \bigoplus_{j=1}^k Q_j$, where

$$Q_j = \bigoplus_{i=1}^r P_i^{[\alpha_{ij}]}, \quad \alpha_{ij} \in \{0, 1\}, \quad k_i = \sum_{j=1}^k \alpha_{ij}, \quad 1 \leq i \leq r, 1 \leq j \leq k.$$

Since $k < n$ and the modules $Q_1, \ldots, Q_k$ are square free, it remains to show that any finitely generated semisimple square free left R-module is cyclic. By [1, Theorem 9.12] it suffices to verify that $V_i + \bigcap_{i \neq j} V_j = R$ $(1 \leq i \leq q)$, for any maximal left ideals $V_1, \ldots, V_q$ of R, for which the left R-modules $R/V_1, \ldots, R/V_q$ are pairwise non-isomorphic. Assuming the contrary, there exists $i \in \{1, 2, \ldots, q\}$ such that $\bigcap_{j \neq i} V_j \subseteq V_i$. Considering the natural epimorphism $R/\bigcap_{j \neq i} V_j \to R/V_i$ and the embedding $R/\bigcap_{j \neq i} V_j \to \bigoplus_{j \neq i}(R/V_j)$, we conclude that R/V_i embeds into $\bigoplus_{j \neq i}(R/V_j)$, which is not possible. □

6.10 Corollary. *Let ${}_RT$ be a semisimple module and $\omega \geq 2$. If T is ω-distributive and* $\mathrm{crs}(T) < \aleph_0$*, then T is ϱ-generated, for some cardinal $\varrho < \omega$.*

Proof. If $\omega < \aleph_0$ apply Lemma 5.1(5) and 6.9.
If $\omega \geq \aleph_0$ apply Corollary 5.8 and Lemma 5.1(3). □

6.11 Theorem. *For a ring R the following are equivalent:*

(a) *R is generalized left quasi-invariant;*

(b) *for any maximal left ideal V of R and $a, b \in R$,*

$$(1 + V : a) \cap (V : b) = \emptyset \quad \textit{or} \quad (V : a) \cap (1 + V : b) = \emptyset;$$

(c) *all semisimple cyclic left R-modules are square free;*

(d) *for any semisimple module ${}_RT$ and $n \geq 2$, the module T is $(n-1)$-generated if and only if T is n-distributive and finitely generated;*

(e) *for some $n \geq 2$, all $(n-1)$-generated semisimple left R-modules are n-distributive;*

(f) *for $\omega \geq 2$, all homogeneously ω-Bézout left R-modules are ω-distributive;*

(g) *all homogeneously Bézout left R-modules are distributive;*

(h) *all left Bézout-modules are distributive;*

(i) *for some $n \geq 2$, all n-Bézout semisimple left R-modules are n-distributive.*

Proof. (a)⇔(c) This follows from the observation that for semisimple modules, quasi-invariance is equivalent to being square free.

(b)⇔(c) Condition (c) is obviously equivalent to: for any maximal left ideal V of R, the left R-module $(R/V)^2$ is not cyclic. In other words, there is no epimorphism of left R-modules $f : R \to (R/V)^2$. Given any homomorphism $f : R \to (R/V)^2$, put $(1)f = (a+V, b+V)$. It remains to recall that surjectivity of f is equivalent to the existence of $x, y \in R$, for which $(1 + V, V) = (x)f$ and $(V, 1 + V) = (y)f$, i.e.,

$$x \in (1 + V : a) \cap (V : b) \quad \text{and} \quad y \in (V : a) \cap (1 + V : b).$$

(c)⇒(d) By Lemma 6.9, an n-distributive and finitely generated semisimple module T is $(n-1)$-generated. Conversely, if we know that T is $(n-1)$-generated then $T = \sum_{i=1}^{n-1} Ra_i$, for suitable $a_1, \ldots, a_{n-1} \in T$. Applying (c) and Lemma 5.1(5) we conclude that T is n-distributive.

(d)⇒(e) Obvious.

(e)⇒(c) Assuming the contrary there exists a simple module ${}_RP$ such that P^2 is cyclic. Then the module P^n is $(n-1)$-generated and hence n-distributive. This is impossible by Lemma 5.1(5).

(d)⇒(f) If $\omega \geq \aleph_0$ we apply Corollary 6.5. Consider the case $\omega = n < \aleph_0$. Since all semisimple subfactors of a homogeneously n-Bézout module are homogeneously n-Bézout, by Lemma 4.1, it suffices to prove that any homogeneously n-Bézout semisimple module ${}_RT$ is n-distributive. Assuming the contrary, using Lemma 5.1(5), we conclude that the module T contains a submodule S isomorphic to P^n, where P is some simple R-module. Therefore the module S is $(n-1)$-generated and n-distributive which is impossible by Lemma 5.1(5).

(f)⇒(g)⇒(h)⇒(i) are obvious, and $(i) \Rightarrow (e)$ follows from the fact that any $(n-1)$-generated semisimple module ${}_RT$ is n-Bézout. □

6.12 Remark. Theorem 6.11 implies the known fact: *Over a left quasi-invariant ring all left Bézout modules are distributive* (see [15]).

6.13 Corollary. *For a semisimple module ${}_RT$ and $n \geq 2$, consider the conditions*

(1) *T is n-distributive;*

(2) *T is n-Bézout;*

(3) *T is homogeneously n-Bézout.*

Then (1) ⇒(2) ⇒(3). *If the ring R is generalized left quasi-invariant then* (3) ⇒(1).

Proof. (1) ⇒(2) Any n-generated submodule S of T is $(n-1)$-generated by Lemma 6.9.
(2) ⇒(3) is obvious, and (3) ⇒(1) follows from Theorem 6.11. □

6.14 Lemma. *Let R be a ring, $\omega \geq 2$ and $n \geq 2$. Then:*

(1) *If R is semilocal, then all n-distributive left R-modules are n-Bézout and all noetherian n-distributive left R-modules are n-noetherian.*

(2) *If R is left perfect, then all ω-distributive left R-modules are ω-noetherian.*

Proof. (1) Let ${}_RM$ be an n-distributive module. By Lemma 4.1 and Corollary 6.10, for any finitely generated submodule A of M, the corresponding left $R/J(R)$-module $A/J(R)A$ is n-distributive, semisimple and $(n-1)$-generated. Therefore, by the Nakayama Lemma, A is $(n-1)$-generated.

(2) Consider any submodule A in an ω-distributive module ${}_RM$. Applying Lemma 4.1 and Corollary 6.10, we observe that the left $R/J(R)$-module $A/J(R)A$ is ω-distributive, semisimple and ϱ-generated, for some cardinal $\varrho < \omega$. Therefore A is ϱ-generated by the generalized Nakayama Lemma [1, Theorem 43.5]. □

6.15 Remark. Lemma 6.14 generalizes the following results from [15]:
- *over any semilocal ring all distributive left modules are Bézout;*
- *over any left perfect ring all distributive left modules are fully cyclic.*

6.16 Example. Let P denote the set of all prime numbers. Then the semisimple $\mathbb{Z}$-module $\bigoplus_P \mathbb{Z}_p$, where $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, is distributive (and hence $\aleph_0$-distributive) by Lemma 5.1(5), however it is not $\aleph_0$-Bézout (and not noetherian). Therefore for infinite cardinality the analogue of Corollary 6.13 is not true in general. Moreover we observe that the left perfectness of the ring R is essential in the statement of Lemma 6.14(2).

6.17 Theorem. *For any module ${}_RM$ and $\omega \geq 2$, consider the conditions*

(1) *M is ω-noetherian;*

(2) *M is homogeneously ω-noetherian;*

(3) *M is ω-Bézout;*

(4) *M is homogeneously ω-Bézout;*

(5) *M is ω-distributive;*

(6) *M is ω-hyperdistributive;*

(7) *M is ω-thick.*

Then the following assertions hold:

(i) *If R is a left quasi-invariant left perfect ring, then* (1)-(5) *are equivalent.*

(ii) *If R is a local perfect ring, then* (1)-(7) *are equivalent.*

(iii) *If R is a left perfect ring and ω is infinite, then* (1)-(7) *are equivalent.*

(iv) *If R is a left quasi-invariant semilocal ring and ω is finite, then* (3)-(5) *are equivalent.*

(v) *If R is a local ring and ω is finite, then* (3)-(7) *are equivalent.*

Proof. First observe that the implications (1) ⇒ (2), (3) ⇒ (4), (1) ⇒ (3), (2) ⇒ (4), (7) ⇒ (6) ⇒ (5) hold true for any ring R.

(i). (4) ⇒ (5) by Theorem 6.11; (5) ⇒ (1) by Lemma 6.14(2).

(ii). The equivalence of (1)-(5) follows from (i); (5) ⇒ (7) by Corollary 5.11.

(iii). (3) ⇒ (7), (4) ⇒ (5) by Corollary 6.5; (5) ⇒ (1) by Lemma 6.14(2).

(iv). (4) ⇒ (5) by Theorem 6.11; (5) ⇒ (3) by Lemma 6.14(1).

(v). The equivalence of (3)-(5) follows from (iv); (5) ⇒ (7) by Corollary 5.11. □

6.18 Corollary. *For a left noetherian local ring R and $n \geq 2$, the following are equivalent:*

(a) *R is left n-noetherian;*

(b) *R is left n-distributive;*

(c) *R is left n-thick.*

Proof. (a)⇒(b) by Theorem 6.11; (b)⇒(a) by Lemma 6.14(1); (b)⇔(c) by Theorem 6.17(v). □

6.19 Remarks. From Theorem 6.11 it is easy to see that a semilocal ring is generalized left quasi-invariant if and only if it is left quasi-invariant. Moreover it is clear that left quasi-invariant semilocal are in particular those rings R, for which the factor ring $R/J(R)$ is isomorphic to a finite product of skew fields. The question of the existence of generalized left quasi-invariant rings which are not left quasi-invariant remains open.

Theorem 6.17 generalizes known results about distributive modules ([15, Proposition 2]; [16, Lemma 18]) and about $\aleph_0$-distributive modules (see [3, Lemma 5]).

In [6], Corollary 6.18 is shown for left noetherian local rings R under the additional assumptions that R is commutative or $\bigcap_{i=1}^{\infty} J(R)^i = 0$.

7 Hom-functor and ω-distributive modules

Let $_RM$ and $_RU$ be fixed modules and $E = \mathrm{End}_R(U)$. Consider the functors

$$\begin{aligned} h^U &= \mathrm{Hom}_R(U, -) : R\text{-}Mod \to E\text{-}Mod, \text{ and} \\ h_U &= \mathrm{Hom}_R(-, U) : R\text{-}Mod \to Mod\text{-}E. \end{aligned}$$

For the investigation of the relationships between ω-distributivity of M and properties of the modules $h^U(M)$ and $h_U(M)$ it is convenient to refer to the canonical maps

$$\mathcal{L}(h^U(M)) \underset{\mathbf{Q}}{\overset{\mathrm{Im}}{\rightleftharpoons}} \mathcal{L}(M) \underset{\mathrm{Ke}}{\overset{\mathbf{N}}{\rightleftharpoons}} \mathcal{L}(h_U(M)),$$

defined by the conditions

$$\begin{aligned} \mathbf{Q}(A) &= \{\varphi \in h^U(M) \mid \mathrm{Im}\,\varphi \subseteq A\}; \qquad \mathrm{Im}\, I = \textstyle\sum_{\varphi \in I} \mathrm{Im}\,\varphi; \\ \mathbf{N}(A) &= \{\psi \in h_U(M) \mid \mathrm{Ke}\,\psi \supseteq A\}; \qquad \mathrm{Ke}\, J = \textstyle\bigcap_{\psi \in J} \mathrm{Ke}\,\psi. \end{aligned}$$

Recall some definitions from [24]. The module U is called *M-finitely generated* if $\mathrm{Im}\,\varphi$ is finitely generated for any $\varphi \in h^U(M)$. Dually, U is called *M-finitely cogenerated* if $\mathrm{Im}\,\psi$ is finitely cogenerated for any $\psi \in h_U(M)$. The module U is called *M-intrinsically projective* if every diagram

$$\begin{array}{ccccc} & & U & & \\ & & \downarrow & & \\ U^m & \to & A & \to & 0, \end{array}$$

where m is a natural number, $A \in \mathcal{L}(M)$, and the row is exact, can be extended commutatively by some homomorphism $U \to U^m$. Dually, U is called *M-intrinsically injective* if every diagram

$$\begin{array}{ccccc} 0 & \to & M/A & \to & U^m \\ & & \downarrow & & \\ & & U & & \end{array}$$

where m is a natural number, $A \in \mathcal{L}(M)$, and the row is exact, can be extended commutatively by some homomorphism $U^m \to U$. The module U is called *intrinsically projective* (*intrinsically injetive*) if U is U-intrinsically projective (respectively U-intrinsically injective). The module U is called an *M-generator* if U generates any submodule of M. Dually, U is called *M-cogenerator* if U cogenerates any factor module of M.

We recall properties of the canonical mappings $\mathbf{Q}$, $\mathbf{N}$, Im, Ke which will be indispensable for our investigation.

7.1 Lemma. *For any modules ${}_RM$ and ${}_RU$ we have:*

(1) $\mathbf{Q}$ *and* Im *establish a Galois correspondence between* $\mathcal{L}(M)^*$ *and* $\mathcal{L}(h^U(M))$.

(2) $\mathbf{N}$ *and* Ke *establish a Galois correspondence between* $\mathcal{L}(M)$ *and* $\mathcal{L}(h_U(M))$.

(3) $\mathbf{Q}(\bigcap_\Lambda A_\lambda) = \bigcap_\Lambda \mathbf{Q}(A_\lambda)$, $\quad \mathbf{Q}(\sum_\Lambda A_\lambda) \supseteq \sum_\Lambda \mathbf{Q}(A_\lambda)$,
$\mathbf{N}(\sum_\Lambda A_\lambda) = \bigcap_\Lambda \mathbf{N}(A_\lambda)$, $\quad \mathbf{N}(\bigcap_\Lambda A_\lambda) \supseteq \sum_\Lambda \mathbf{N}(A_\lambda)$,
for any family $\{A_\lambda\}_\Lambda$ *of submodules of* M.

(4) *If U is M-projective, then* $\mathbf{Q}$ *is a lattice homomorphism.*

(5) *If U is M-injective, then* $\mathbf{N}$ *is a lattice anti-homomorphism.*

(6) *If U is M-intrinsically projective then* $\mathbf{Q}(\mathrm{Im}\, I) = I$*, for all finitely generated submodules* $I \subseteq h^U(M)$.

(7) *If U is M-intrinsically injective, then* $\mathbf{N}(\mathrm{Ke}\, J) = J$*, for all finitely generated submodules* $J \subseteq h_U(M)$.

(8) *If U is finitely M-generated and M-intrinsically projective, then* $\mathbf{Q}(\mathrm{Im}\, I) = I$*, for all submodules* $I \subseteq h^U(M)$.

(9) *If U is finitely M-cogenerated and M-intrinsically injective and M is an $AB5^*$ module, then* $\mathbf{N}(\mathrm{Ke}\, J) = J$*, for all submodules* $J \subseteq h_U(M)$.

Proof. The assertions (1)-(3) are well-known and obvious. (4), (5) are shown in [26, Proposition 5.4]. (6)-(9) are proved in [24, Theorem 2.10, 2.10*, 2.17, 2.18]. □

For the dualization of some assertions about ω-distributive modules we need the following definitions. We call a lattice $\mathcal{L}$ *ω-codistributive* if the dual lattice $\mathcal{L}^*$ is ω-distributive. A module M is called *ω-codistributive* if the lattice $\mathcal{L}(M)$ is ω-codistributive. Since any modular lattice $\mathcal{L}$ is n-codistributive if and only if $\mathcal{L}$ is n-distributive (by [5, Proposition 3.1]) we have:

7.2 Lemma. *For a module ${}_RM$ and $n \geq 2$, the following are equivalent:*

(a) *M is n-distributive;*

(b) *M is n-codistributive.*

Before explaining what ω-codistributivity of a module means for $\omega \geq \aleph_0$ we recall that the following was proved in [3]:

7.3 Lemma. *Each $AB5^*$ module is ω-distributive, for any infinite ω.*

Since any module satisfies the $AB5$ condition we have dually to Lemma 7.3:

7.4 Lemma. *Every module is ω-codistributive, for any infinite ω.*

The following lemma generalizes the arguments used in [14, Lemma 2.7].

7.5 Lemma. *For any module ${}_RM$ and $\omega \geq 2$, the following assertions hold true:*

(1) *M is ω-codistributive if and only if*

$$A \cap \sum_{\lambda\in\Lambda} B_\lambda = \sum_{\lambda\in\Lambda}(A \cap \sum_{\mu\in\Lambda\backslash\{\lambda\}} B_\mu),$$

for any cyclic submodules A, $B_\lambda \subseteq M$, $\lambda \in \Lambda$, $|\Lambda| = \omega$.

(2) *M is ω-distributive if and only if*

$$A + \bigcap_{\lambda\in\Lambda} B_\lambda = \bigcap_{\lambda\in\Lambda}(A + \bigcap_{\mu\in\Lambda\backslash\{\lambda\}} B_\mu),$$

for any ω-generated submodules A, $B_\lambda \subseteq M$, $\lambda \in \Lambda$, $|\Lambda| = \omega$.

Proof. (1) The necessity is obvious. Let us prove the sufficiency. For this we verify that

$$C \cap \sum_{\lambda\in\Lambda} D_\lambda = \sum_{\lambda\in\Lambda}(C \cap \sum_{\mu\in\Lambda\backslash\{\lambda\}} D_\mu),$$

for any submodules C, $D_\lambda \subseteq M$ $(\lambda \in \Lambda)$.

Indeed, if $a \in C \cap \sum_{\lambda\in\Lambda} D_\lambda$, then $a = b_{\lambda_1} + \ldots + b_{\lambda_m} \in C$ for some $b_{\lambda_i} \in D_{\lambda_i}$. Putting $b_\lambda = 0$ for $\lambda \in \Lambda\backslash\{\lambda_1, \ldots, \lambda_m\}$, $A = Ra \subseteq C$, and $B_\lambda = Rb_\lambda \subseteq D_\lambda$ $(\lambda \in \Lambda)$, we easily obtain that $a \in \sum_{\lambda\in\Lambda}(C \cap \sum_{\mu\in\Lambda\backslash\{\lambda\}} D_\mu)$. So we have

$$C \cap \sum_{\lambda\in\Lambda} D_\lambda \subseteq \sum_{\lambda\in\Lambda}(C \cap \sum_{\mu\in\Lambda\backslash\{\lambda\}} D_\mu).$$

The converse inclusion is obvious.

(2) The necessity is clear. We prove the sufficiency. Since the converse inclusion is obvious it only remains to verify that

$$C + \bigcap_{\lambda\in\Lambda} D_\lambda \supseteq \bigcap_{\lambda\in\Lambda}(C + \bigcap_{\mu\in\Lambda\backslash\{\lambda\}} D_\mu),$$

for any submodules C, $D_\lambda \subseteq M$ $(\lambda \in \Lambda)$.

Indeed, if $x \in \bigcap_{\lambda\in\Lambda}(C + \bigcap_{\mu\in\Lambda\backslash\{\lambda\}} D_\mu)$, then for every $\lambda \in \Lambda$, there exist $a_\lambda \in C$ and $b_\lambda \in \bigcap_{\mu\in\Lambda\backslash\{\lambda\}} D_\mu$, such that $x = a_\lambda + b_\lambda$. Putting $A = \sum_{\lambda\in\Lambda} Ra_\lambda \subseteq C$ and $B_\lambda = \sum_{\mu\in\Lambda\backslash\{\lambda\}} Rb_\mu \subseteq D_\lambda$ $(\lambda \in \Lambda)$, we easily conclude that $x \in C + \sum_{\lambda\in\Lambda} D_\lambda$. □

7.6 Lemma. *Let $\omega \geq 2$, and let ${}_RM$, ${}_RU$ be modules such that U is M-intrinsically projective and M-projective. Then:*

(1) *If M is ω-codistributive, then the left $\mathrm{End}_R(U)$-module $\mathrm{Hom}_R(U, M)$ is ω-codistributive.*

(2) *If M is ω-distributive and U is M-finitely generated, then the left $\mathrm{End}_R(U)$-module $\mathrm{Hom}_R(U, M)$ is ω-distributive.*

7.7 Lemma. *Let $\omega \geq 2$ and ${}_RM$, ${}_RU$ modules such that U is M-intrinsically injective and M-injective. Then:*

(1) *If M is ω-distributive, then the right $\mathrm{End}_R(U)$-module $\mathrm{Hom}_R(M, U)$ is ω-co-distributive.*

(2) *If M is ω-codistributive and $AB5^*$ and U is M-finitely cogenerated, then the right $\mathrm{End}_R(U)$-module $\mathrm{Hom}_R(M, U)$ is ω-distributive.*

In view of the duality of Lemma 7.6 and 7.7 it suffices to give the

Proof of Lemma 7.7. (1) As shown in Lemma 7.4, we may assume, without loss of generality, that ω is finite. By Lemma 7.5(1), it suffices to verify that

$$I \cap \sum_{\lambda\in\Lambda} J_\lambda \subseteq \sum_{\lambda\in\Lambda}(I \cap \sum_{\mu\in\Lambda\backslash\{\lambda\}} J_\mu),$$

for any cyclic submodules I, $J_\lambda \subseteq h_U(M)$ ($\lambda \in \Lambda$, $|\Lambda| = \omega$). By Lemma 7.1(7), for suitable A, $B_\lambda \in \mathcal{L}(M)$ ($\lambda \in \Lambda$), we have $I = \mathbf{N}(A)$ and $J_\lambda = \mathbf{N}(B_\lambda)$ ($\lambda \in \Lambda$). Therefore, applying Lemma 7.1(3), we conclude that

$$I \cap \sum_{\lambda\in\Lambda} J_\lambda \subseteq \mathbf{N}(A + \bigcap_{\lambda\in\Lambda} B_\lambda).$$

By distributivity of M and Lemma 7.1(3),(5) we obtain the desired inclusion.

(2) We prove that

$$I + \bigcap_{\lambda\in\Lambda} J_\lambda = \bigcap_{\lambda\in\Lambda} (I + \bigcap_{\mu\in\Lambda\setminus\{\lambda\}} J_\mu),$$

for any submodule I, $J_\lambda \subseteq h_U(M)$ ($\lambda \in \Lambda$, $|\Lambda| = \omega$). Since by Lemma 7.1(9), $I = \mathbf{N}(A)$ and $J_\lambda = \mathbf{N}(B_\lambda)$, for some A, $B_\lambda \in \mathcal{L}(M)$, we conclude by Lemma 7.1(3),(5) that

$$I + \bigcap_{\lambda\in\Lambda} J_\lambda = \mathbf{N}(A \cap \sum_{\lambda\in\Lambda} B_\lambda).$$

To complete the proof it remains to make use of the ω-codistributivity of M and Lemma 7.1(3),(5). □

7.8 Remark. In view of Lemma 7.4 the assertions (1) in Lemma 7.6 and 7.7 only hold for finite ω. The assertions (2) in Lemma 7.6 and 7.7 can only be used for infinite ω, since by Lemma 7.2, for finite cardinality they are weaker than the corresponding assertions (1).

For infinite cardinality we obtain immediately from Lemma 7.7(2) and 7.4:

7.9 Corollary. *If ω is infinte, ${}_RM$ is an $AB5^*$ module, and ${}_RU$ is M-finitely cogenerated, M-intrinsically injective and M-injective, then the right $\mathrm{End}_R(U)$-module $\mathrm{Hom}_R(M,U)$ is ω-distributive.*

7.10 Proposition. *If ${}_RM$ is an $AB5^*$ module and ${}_RU$ is M-finitely cogenerated, M-intrinsically injective and M-injective, then the right $\mathrm{End}_R(U)$-module $\mathrm{Hom}_R(M,U)$ is an $AB5^*$ module.*

Proof. Let us prove that $h_U(M)$ satisfies the condition

$$I + \bigcap_{\Lambda} J_\lambda = \bigcap_{\Lambda} (I + J_\lambda),$$

for all submodules I and inverse systems $\{J_\lambda\}_\Lambda$ of submodules.

Putting $A = \mathrm{Ke}\, I$ and $B_\lambda = \mathrm{Ke}\, J_\lambda$ ($\lambda \in \Lambda$), we obtain a submodule $A \subseteq M$ and a direct system $\{B_\lambda\}_\Lambda$ of submodules of M. Since $I = \mathbf{N}(A)$ and $J_\lambda = \mathbf{N}(B_\lambda)$ ($\lambda \in \Lambda$) by Lemma 7.1(9), we easily deduce by Lemma 7.1(3),(5) that

$$I + \bigcap_{\Lambda} J_\lambda = \mathbf{N}(A \cap \sum_{\Lambda} B_\lambda).$$

To complete the proof apply the $AB5$ condition and Lemma 7.1(3), (5). □

7.11 Lemma. *For modules ${}_RM$ and ${}_RU$, the following assertions hold:*

(1) *If U is M-projective then $h^U(A/B) \simeq \mathbf{Q}(A)/\mathbf{Q}(B)$, for any subfactor A/B of M, where $B \subseteq A \subseteq M$.*

(2) *If U is M-injective then $h_U(A/B) \simeq \mathbf{N}(A)/\mathbf{N}(B)$, for any subfactor A/B of M, where $B \subseteq A \subseteq M$.*

Proof. In view of the duality of (1) and (2) it is enough to prove (2). For this apply the functor $\mathrm{Hom}_R(-,U)$ to the exact sequence

$$0 \to A/B \to M/B \to M/A \to 0,$$

and recall the canonical identification $\mathbf{N}(A) = \mathrm{Hom}_R(M/A, U)$. □

7.12 Lemma. *For modules ${}_RM$, ${}_RU$, and an (M,Λ)-family $\mathcal{A} = (V/B, \{A_\lambda/B_\lambda\}_\Lambda)$, where $B \subseteq A_\lambda \subseteq V \subseteq M$ for all $\lambda \in \Lambda$, the following hold:*

(1) *If the (M,Λ)-family $\mathcal{A}$ is independent, then the $(h^U(M),\Lambda)$-family $(\mathbf{Q}(V)/\mathbf{Q}(B), \{\mathbf{Q}(A_\lambda)/\mathbf{Q}(B)\}_\Lambda)$ is independent.*

(2) *If the (M,Λ)-family $\mathcal{A}$ is weakly coindependent, then the $(h_U(M),\Lambda)$-family $(\mathbf{N}(B)/\mathbf{N}(V), \{\mathbf{N}(A_\lambda)/\mathbf{N}(V)\}_\Lambda)$ is independent.*

Proof. In view of the duality of (1) and (2) it suffices to prove (2). Indeed, weak coindependence of the (M,Λ)-family $\mathcal{A}$ means that $A_\lambda + \bigcap_{\Lambda\setminus\{\lambda\}} A_\mu = V$ for all $\lambda \in \Lambda$. Applying to both sides of this equality the mapping $\mathbf{N}$ and recalling Lemma 7.1(3), we obtain

$$\mathbf{N}(A_\lambda) \cap \sum_{\Lambda\setminus\{\lambda\}} \mathbf{N}(A_\mu) \subseteq \mathbf{N}(V).$$

The converse inclusion is obvious by Lemma 7.1(2). □

7.13 Lemma. *Let $\omega \geq 2$ and ${}_RM$, ${}_RU$ modules such that U is M-projective and U generates the simple subfactor P of M. If the left $\mathrm{End}_R(U)$-module $\mathrm{Hom}_R(U,M)$ is ω-distributive, then M is ω-thick relative to $\mathcal{S}_1(P)$.*

7.14 Lemma. *Let $\omega \geq 2$ and ${}_RM$, ${}_RU$ modules such that U is M-injective and U cogenerates the simple subfactor P of M. If the right $\mathrm{End}_R(U)$-module $\mathrm{Hom}_R(M,U)$ is ω-distributive, then M is ω-thick relative to $\mathcal{S}_1(P)$.*

In view of the duality of Lemma 7.13 and 7.14 it is enough to give the

Proof of Lemma 7.14. Assuming the contrary and recalling Corollary 3.4, suppose that there exists a weakly coindependent (M,Λ)-family $(V/B, \{A_\lambda/B\}_\Lambda)$, where $|\Lambda| = \omega$ and $V/A_\lambda \simeq P$, for all $\lambda \in \Lambda$. By Lemma 7.12(2), the $(h_U(M),\Lambda)$-family $(\mathbf{N}(B)/\mathbf{N}(V), \{\mathbf{N}(A_\lambda)/\mathbf{N}(V)\}_\Lambda)$ is independent. Applying Lemma 7.11(2) we conclude that the submodules $\mathbf{N}(A_\lambda)/\mathbf{N}(V) \subseteq \mathbf{N}(B)/\mathbf{N}(V)$ are non-zero and pairwise isomorphic. By Corollary 3.7, this contradicts the ω-distributivity of $h_U(M)$. □

7.15 Lemma. *Let ${}_RM, {}_RU$ be modules and $K \subset U$ a fully invariant submodule.*

(1) *If U is quasi-projective, then the lattice of submodules of the left $\mathrm{End}_R(U/K)$-module $h^{U/K}(M)$ and the left $\mathrm{End}_R(U)$-module $\mathbf{N}(U/K)$ are isomorphic.*

(2) *If U is quasi-injective, then the lattice of submodules of the right $\mathrm{End}_R(K)$-module $h_K(M)$ and the right $\mathrm{End}_R(U)$-module $\mathbf{Q}(K)$ are isomorphic.*

Proof. In view of the duality of (1) and (2) it is enough to prove (2). Using the full invariance of the submodule $K \subset U$ it is easy to verify that $\mathbf{Q}(K)$ is a submodule of the right $\mathrm{End}_R(U)$-module $h_U(M)$. Considering $\mathbf{Q}(K)$ as a right $\mathrm{End}_R(U)$-module we define mappings

$$\xi : \mathcal{L}(h_K(M)) \rightleftharpoons \mathcal{L}(\mathbf{Q}(K)) : \eta,$$

putting

$$\xi(I) = \{fi \mid f \in I\} \quad \text{and} \quad \eta(J) = \{f \in h_K(M) \mid fi \in J\},$$

where $I \in \mathcal{L}(h_K(M))$, $J \in \mathcal{L}(\mathbf{Q}(K))$ and $i : K \to U$ is the natural inclusion.

Notice that by the full invariance of $K \subseteq U$, any $\alpha \in \mathrm{End}_R(U)$ induces some $\beta \in \mathrm{End}_R(K)$. Because of the quasi-injectivity of U the converse is also true: any $\beta \in \mathrm{End}_R(K)$ induces some $\alpha \in \mathrm{End}_R(U)$. In both situations we have a commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{\beta} & K \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ U & \xrightarrow{\alpha} & U \end{array}.$$

Now it is easy to see that the mappings ξ and η are well-defined, isotone and inverse to each other. □

7.16 Theorem. *For any module $_RM$ and $\omega \geq 2$, the following are equivalent:*

(a) *M is ω-distributive;*

(b) *$\mathrm{Hom}_R(U, M)$ is an ω-distributive left $\mathrm{End}_R(U)$-module, for any M-finitely generated, M-intrinsically projective, M-projective $_RU$;*

(c) *$\mathrm{Hom}_R(U, M)$ is an ω-distributive left $\mathrm{End}_R(U)$-module, for any finitely generated projective module $_RU$;*

(d) *$\mathrm{Hom}_R(U, M)$ is an ω-distributive left $\mathrm{End}_R(U)$-module, for any finitely generated quasi-projective module $_RU$ which has a projective cover;*

(e) *$\mathrm{Hom}_R(U, M)$ is an ω-distributive left $\mathrm{End}_R(U)$-module, for any module $_RU$ from some class $\mathcal{U}$ of left R-modules with the properties*

  (i) *$\mathcal{U}$ generates all simple subfactors of M;*

  (ii) *every $U \in \mathcal{U}$ is M-finitely generated, M-intrinsically projective and M-projective.*

If ω is finite these conditions are equivalent to:

(f) *$\mathrm{Hom}_R(U, M)$ is an ω-distributive left $\mathrm{End}_R(U)$-module, for any M-intrinsically projective, M-projective module $_RU$;*

(g) *$\mathrm{Hom}_R(U, M)$ is an ω-distributive left $\mathrm{End}_R(U)$-module, for any projective module $_RU$;*

(h) *$\mathrm{Hom}_R(U, M)$ is an ω-distributive left $\mathrm{End}_R(U)$-module, for any quasi-projective module $_RU$ which has a projectve cover;*

(i) *$\mathrm{Hom}_R(U, M)$ is an ω-distributive left $\mathrm{End}_R(U)$-module, for any module $_RU$ from some class $\mathcal{U}$ of left R-modules with the properties*

  (i) *$\mathcal{U}$ generates all simple subfactors of M;*

  (ii) *every $U \in \mathcal{U}$ is M-intrinsically projective and M-projective.*

If the ring R is semiperfect then (a)-(e) are equivalent to:

(j) *$\mathrm{Hom}_R(U, M)$ is an ω-distributive left $\mathrm{End}_R(U)$-module, for any finitely generated quasi-projective module $_RU$;*

(k) *$\mathrm{Hom}_R(U, M)$ is an ω-thick left $\mathrm{End}_R(U)$-module, for the projective cover $_RU$ of any simple left R-module;*

(l) *$\mathrm{Hom}_R(U, M)$ is an ω-thick left $\mathrm{End}_R(U)$-module, for the projective cover $_RU$ of any simple subfactor of M;*

(m) *for any primitive idempotent $e \in R$, the left eRe-module eM is ω-thick;*

If the ring R is semiperfect and ω is finite, then (a)-(m) are equivalent to:

(n) *$\mathrm{Hom}_R(U, M)$ is an ω-distributive left $\mathrm{End}_R(U)$-module, for any quasi-projective module $_RU$.*

Proof. (a)⇒(b) follows from Lemma 7.6(2).

(c)⇒(d) Any finitely generated quasi-projective module with a projective cover is isomorphic to the factor module of some finitely generated projective module by a fully invariant submodule [1, 19.10 (i)]. It remains to apply Lemma 7.15(1).

(e)⇒(a) From (e) we obtain immediately that for every simple subfactor P of M there exists a module $U \in \mathcal{U}$ which generates P. It remains to apply Lemma 7.13 and 1.4.

(a)⇒(f) follows from Lemma 7.6(1) and 7.2.

(g)⇒(h) As in the proof of the implication (c)⇒(d), it suffices to apply [1, 19.10(7)(i)] and Lemma 7.15(1).

(i)⇒(a) is shown similarly to the implication (e)⇒(a).

(j)⇒(k) follows from well-known properties of the projective cover of simple modules [22, Proposition 17.19] and Corollary 5.11.

(k)⇔(m) is clear by the obvious semilinear isomorphism between the left $\mathrm{End}_R(Re)$-module $\mathrm{Hom}_R(Re, M)$ and the left eRe-module eM, where e is an idempotent of the ring R.

It remains to notice that the implications (b)⇒(c)⇒(e), (d)⇒(c), (f)⇒(g)⇒(i), (h)⇒(g), (k)⇒(l)⇒(e) and the equivalences (d)⇔(j), (h)⇔(n) are obvious. □

A similar proof yields:

7.17 Theorem. *For any module ${}_RM$ and $n \geq 2$, the following are equivalent:*

(a) *M is n-distributive;*

(b) $\mathrm{Hom}_R(M, U)$ *is an n-distributive right* $\mathrm{End}_R(U)$*-module, for any M-intrinsically injective M-injective module ${}_RU$;*

(c) $\mathrm{Hom}_R(M, U)$ *is an n-distributive right* $\mathrm{End}_R(U)$*-module, for any injective module ${}_RU$;*

(d) $\mathrm{Hom}_R(M, U)$ *is an n-distributive right* $\mathrm{End}_R(U)$*-module, for any quasi-injective module ${}_RU$;*

(e) $\mathrm{Hom}_R(M, U)$ *is an n-thick and n-Bézout right* $\mathrm{End}_R(U)$*-module, for the injective envelope ${}_RU$ of any simple subfactor P of M;*

(f) $\mathrm{Hom}_R(M, U)$ *is an n-distributive right* $\mathrm{End}_R(U)$*-module, for any module ${}_RU$ from some class $\mathcal{U}$ of left R-modules with the properties*

  (i) *$\mathcal{U}$ cogenerates all simple subfactors of M;*

  (ii) *every $U \in \mathcal{U}$ is M-intrinsically injective and M-injective.*

7.18 Remark. Theorem 7.17 is obtained by dualizing the part of Theorem 7.16 which is related to the case of finite ω. This does not apply to the part for infinite cardinality in Theorem 7.16 for the following reasons. Firstly, for $\omega \geq \aleph_0$, by Lemma 7.7(2), we have to assume that M is an $AB5^*$ module. Secondly, for $\omega \geq \aleph_0$, according to Lemma 7.4, the ω-codistributivity of the module is trivial and so the dualization of Theorem 7.16 leads to an assertion which contains nothing new in comparison with Lemma 7.7(2).

Observing that any quasi-projective (quasi-injective) module is intrinsically projective (intrinsically injective) [24, Lemma 2.1] we obtain from the Lemmata 7.6, 7.7, 7.13, 7.14, Proposition 7.10 and the Theorems 7.16, 7.17:

7.19 Corollary. *For a module ${}_RU$ with endomorphism ring $E = \mathrm{End}_R(U)$, $\omega \geq 2$, and $n \geq 2$ we have:*

(1) *If ${}_RU$ is n-distributive and quasi-projective, then E is left n-distributive.*

(2) *If ${}_RU$ is ω-distributive, finitely generated, and quasi-projective, then E is left ω-distributive.*

(3) *If ${}_RU$ is quasi-projective and generates all its simple subfactors, then*

  (i) *if E is left ω-distributive, then ${}_RU$ is ω-distributive;*

  (ii) *${}_RU$ is n-distributive if and only if E is left n-distributive.*

(4) *If ${}_RU$ is finitely generated, quasi-projective, and generates all its simple subfactors, then ${}_RU$ is ω-distributive if and only if E is left ω-distributive.*

(5) *If ${}_RU$ is n-distributive and quasi-injective, then E is right n-distributive.*

(6) *If ${}_RU$ is an ω-codistributive, finitely cogenerated, quasi-injective $AB5^*$ module, then E is a right ω-distributive right $AB5^*$ ring.*

(7) *If ${}_RU$ is quasi-injective and cogenerates all its simple subfactors, then*

   (i) *if E is right ω-distributive, then ${}_RU$ is ω-distributive;*

   (ii) *${}_RU$ is n-distributive if and only if E is right n-distributive.*

(8) *If R is left n-distributive and ${}_RU$ is injective, then U_E is n-distributive.*

(9) *If R is a left ω-codistributive and left $AB5^*$ ring, and ${}_RU$ is finitely cogenerated and injective, then U_E is an ω-distributive $AB5^*$ module.*

(10) *If ${}_RU$ is an injective cogenerator in R-Mod, then*

   (i) *if U_E is ω-distributive, then R is left ω-distributive;*

   (ii) *R is left n-distributive if and only if U_E is n-distributive.*

7.20 Remark. The Lemmata 7.6, 7.7, 7.13, 7.14, the Theorems 7.16, 7.17 and Corollary 7.19 generalize many known results about distributivity of modules ([11, Lemma 4], [12, Lemma 3], [16, Theorem 2], a.o.).

7.21 Remark. For any module ${}_RM$ and $n \geq 2$, the class of all modules ${}_RU$ for which $\mathrm{Hom}_R(M, U)$ is a right n-distributive right $\mathrm{End}_R(U)$-module is closed under direct products and direct summands. In view of Lemma 1.1 this can be shown in the same way as [16, Lemma 11], where this is proved for $n = 2$. It is also straightforward to show the dual assertions about the class of all modules ${}_RU$, for which $\mathrm{Hom}_R(U, M)$ is an n-distributive left $\mathrm{End}_R(U)$-module. These assertions give the possibility to extend the list of equivalent conditions in the Theorems 7.16, 7.17 and Corollary 7.19.

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