OBERSEMINAR ALGEBRAIC GEOMETRY WINTER SEMESTER 2022/23: ELLIPTIC SURFACES

STEFAN SCHRÖER

Overview: Roughly speaking, an *elliptic surface* is an algebraic surface X, endowed with a morphisms $f : X \to C$ to some algebraic curve C, such that the geometric generic fiber is an elliptic curve E. Except for surfaces of general type, this appears in all Kodaira dimensions. Actually, for $\mathrm{Kod}(X) = 1$ there is a canonical such structure, defined by the dualizing sheaf. For rational surfaces, abelian surfaces, K3 surfaces, Enriques surface, bielliptic surfaces the understanding of elliptic structures is often crucial. For example, every Enriques surface admits at least one elliptic fibration, and often there are infinitely many such structures.

For an elliptic fibration $f : X \to C$, there is an amazing interplay between the arithmetic of the generic fiber X_η, which is a genus-one curve over the function field $F = k(C)$, and the geometry of the surface X. One major insight is the classification of all possible degenerate fibers $X_b = f^{-1}(b)$, their determination with the Tate Algorithm [10], and the Tate–Shioda Formula that relates the Picard group $\mathrm{Pic}(X)$ with the Mordell–Weil group $\mathrm{MW}(X/C)$.

The goal of this Oberseminar is to gain familiarity with these notions and the basic structure of elliptic surfaces, by working through Miranda's monograph [4], a beautiful text with a lively, hands-on approach. There are numerous other recommendable sources, for example the exposition [8] or the textbook [1]. Among other things, we will learn about the classification of rational elliptic surfaces that are extremal, that is, where the Mordell–Weil group $\mathrm{MW}(X/C)$ is finite. At the end we have a glimpse at Kondo's classification of the finite groups that occur as automorphism groups of Enriques surfaces [3].

Time and Place: Monday, 12:30-13:30, seminar room 25.22.03.73.

Schedule: (all dates are tentative, as shifts may likely to occur, for example due to guest talks)

Talk 1: (10. Oktober), Daniel Harrer:
Definitions and examples.
Introduce the notion of elliptic surfaces, and explain the structure of closed fibers, after [4], Lecture I.

Talk 2: (17. Oktober), Jakob Bergqvist:
Weierstraß equations.
Discuss the Weierstraß equations (observe correct spelling) and the resulting invertible sheaf $\underline{\mathrm{Hom}}(R^1 f_*(\mathscr{O}_X), \mathscr{O}_C)$ (always use sheaf language), as in [4], Lecture II.

Talk 3: (24. October), Jan Hennig:
Global aspects.
Discuss the dualizing sheaf, the formulas for the Chern invariants and the cohomological invariants $h^i(\mathscr{O}_X)$, together with the Kodaira dimension, following [4], Lecture III.

Talk 4: (31. Oktober), NN:
Local aspects.
Describe the relation between the singularities of certain curves with the Kodaira symbols of the degenerate fibers ([4], Lecture IV).

Talk 5: (14. November), Thuong Dang:
The J-map.
Discuss to what extend the J-map $C \dashrightarrow \mathbb{A}^1 = \operatorname{Spec} k[j]$ determines the elliptic surface $f : S \to C$, and touch the topic of twists, following [4], Lecture V.

Talk 6: (21. November), Cesar Hilario:
Monodromy.
Discuss the effect of walking around a degenerate fiber, after [4], Lecture VI.

Talk 7: (28. November), Thor Wittich:
Néron–Severi group and Mordell–Weil lattice.
Introduces these two groups, together with their bilinear forms, and explain the Tate–Shioda Formula that relates them, as in [4], Lecture VII.

Talk 8: (5. December), Stefan Schröer:
Extremal rational elliptic surfaces.
Introduce this notation, and explain their classification (after [4], Lecture VIII, compare also [2] and [5]).

Talk 9: (12. December), Fabian Korthauer:
Lattice theory.
Provide the background and results from lattice theory that was relevant in the theory of extremal surfaces.

Talk 10: (9. Januar), Ivo Kroon:
Enriques surfaces with finite automorphism groups.
For an Enriques surface Y with finite automorphism group, and elliptic structure must be extremal. Summarize how Kondo [3] used this to classify the possible automorphism groups $\mathrm{Aut}(Y)$.

Talk 11: (16. Januar), NN:
TBA

Talk 12: (23. Januar), NN:
TBA

References

[1] L. Bădescu: Algebraic surfaces. Springer, New York, 2001.
[2] A. Beauville: Les familles stables de courbes elliptiques sur P^1 admettant quatre fibres singulières. C. R. Acad. Sci. Paris Sér. I Math. 294 (1982), 657–660.
[3] S. Kondo: Enriques surfaces with finite automorphism groups. Japan. J. Math. 12 (1986), 191–282.
[4] R. Miranda: The basic theory of elliptic surfaces. ETS Editrice, Pisa, 1989.
[5] R. Miranda, U. Persson: On extremal rational elliptic surfaces. Math. Z. 193 (1986), 537–558.
[6] R. Miranda: Persson's list of singular fibers for a rational elliptic surface. Math. Z. 205 (1990), 191–211.
[7] U. Persson: Configurations of Kodaira fibers on rational elliptic surfaces. Math. Z. 205 (1990), 1–47.
[8] M. Schütt, T. Shioda: Elliptic surfaces. In: J. Keum, S. Kondo, K. Konno, K. Oguiso (eds.), Algebraic geometry in East Asia—Seoul 2008, pp. 51–160. Math. Soc. Japan, Tokyo, 2010.
[9] M. Schütt, T. Shioda: Mordell–Weil lattices. Springer, Singapore, 2019.

[10] J. Tate: Algorithm for determining the type of a singular fiber in an elliptic pencil. In: B. Birch, W. Kuyk (eds.), Modular functions of one variable IV, pp. 33–52. Springer, Berlin, 1975.