Lindström's Theorem

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Slides available at: http://homepages-nw.uni-regensburg.de/∼arp13290/

I. The statement

Theorem (Lindström)

There is no logic that is more expressive than classical first order logic and that satisfies both the Compactness and the Löwenheim-Skolem properties.

From: Per Lindström, On extensions of elementary logic, Theoria 35, p.1-11, 1969

I. The statement

Theorem (Lindström, 1969)

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Plan:

- **I. The statement** (abstract logics, expressivity, compactness and Löwenheim-Skolem properties)
- **II. The proof** (back-and-forth method, theorem of Fraïssé, Lindström's proof)
- **III. Other variants** (different characterizations, topological reformulation, results for fragments and extensions of first order logic/modal logics)

Signatures, S-structures

Definition:

- A *signature* S is a set of relation symbols, function symbols (each with arities) and constant symbols. $S = \{R, \dots, f, \dots, c, \dots\}$

- An *S-structure* is a set M together with interpretations of the relation/function/constant symbols as actual relations/functions/constants

Notation for the interpretations of symbols in an S-structure $\mathfrak{M}$:

$$R^{\mathfrak{M}},\ f^{\mathfrak{M}},\ c^{\mathfrak{M}} \dots$$

Example: Let $S = \{<, s, 0\}$ be a signature with a binary relation, a unary function symbol and a constant symbol. A well-known S-structure is $\mathfrak{Nat} := (\mathbb{N}, <, succ(-), 0)$.

Reducts and isomorphims of S-structures

Definition: Let $S_0 \subseteq S_1$, and $\mathfrak{M}$ an S_1-structure. Then $\mathfrak{M}|_{S_0}$ denotes the *reduct* of $\mathfrak{M}$ to S_0, i.e. the S_0-structure obtained by forgetting the interpretations of symbols from $S_1 \setminus S_0$.

Definition: An *isomorphism* of S-structures $\mathfrak{M} = (M, R^{\mathfrak{M}}, f^{\mathfrak{M}}, c^{\mathfrak{M}}, ...)$, $\mathfrak{N} = (N, R^{\mathfrak{N}}, f^{\mathfrak{N}}, c^{\mathfrak{N}}, ...)$ is a bijection $h \colon M \cong N$ such that

(1) $R^{\mathfrak{N}}(h(m_1), \ldots, h(m_k))$ iff $R^{\mathfrak{M}}(m_1, \ldots, m_k)$ for each relation symbol R

(2) $h(f^{\mathfrak{M}}(m_1, \ldots, m_k)) = f^{\mathfrak{N}}(h(m_1), \ldots, h(m_k))$ for each function symbol f

(3) $h(c^{\mathfrak{M}}) = c^{\mathfrak{N}}$ for each constant symbol c

1st order language

Given a signature S, we can build S-*terms* from variables, constant symbols and function symbols.

Atomic first order S-*formulas*: $t_1 = t_2$ or $R(t_1, \ldots, t_n)$ for terms $t_1, \ldots, t_n$.
General first order S-*formulas*: Atomic or $\neg\varphi, \varphi \wedge \psi, \exists x \varphi$ for previously built formulas φ, ψ

A *sentence* is a formula with no free variables.

$\rightsquigarrow$ $L(S) := \{S\text{-sentences}\}$ — the set of all first order S-sentences.

1st order satisfaction relation

For $\mathfrak{M}$ an S-structure and $\varphi \in L(S)$ one defines the *satisfaction relation*:

- Atomic sentences: $\mathfrak{M} \vDash R(t_1, \dots, t_n) \;:\Leftrightarrow R^{\mathfrak{M}}(t_1^{\mathfrak{M}}, \dots, t_n^{\mathfrak{M}})$ and
  $\mathfrak{M} \vDash t_1 = t_2 \;:\Leftrightarrow t_1^{\mathfrak{M}} = t_2^{\mathfrak{M}}$
- $\mathfrak{M} \vDash \neg\varphi :\Leftrightarrow$ not $\mathfrak{M} \vDash \varphi$
- $\mathfrak{M} \vDash \varphi \wedge \psi :\Leftrightarrow \mathfrak{M} \vDash \varphi$ and $\mathfrak{M} \vDash \psi$
- $\mathfrak{M} \vDash \exists x \varphi(x) :\Leftrightarrow$ there exists $m \in M$ with $\mathfrak{M} \vDash \varphi(m)$

For $\Phi \subseteq L(S)$ write $\mathfrak{M} \vDash \Phi$ iff $\mathfrak{M} \vDash \varphi$ for all $\varphi \in \Phi$. One then says that $\mathfrak{M}$ is a model of Φ. If Φ has a model, it is called *satisfiable*.

Two S-structures $\mathfrak{M}$, $\mathfrak{N}$ are called *elementary equivalent* if
$\forall \varphi \in L(S) :\; \mathfrak{M} \vDash \varphi \Leftrightarrow \mathfrak{N} \vDash \varphi$

Properties of the 1st order satisfaction relation

Theorem (Downward Löwenheim-Skolem, Löwenheim 1915/Skolem 1920)

If $\varphi \in L(S)$ has a model, then it has a countable model.

Reason: One can take a syntactic model. Applications: Smaller models are better to handle...

See the course by Nate Ackerman (LOW), next in this room

(Stronger version: Let S be a signature, $\Phi \subseteq L(S)$ and $\kappa > |S|$ an infinite cardinal. If Φ has an infinite model $\mathfrak{M}$, then $\mathfrak{M}$ has a submodel of cardinality κ)

Properties of the 1st order satisfaction relation

Theorem (Compactness theorem, Gödel 1930/Maltsev 1936)

$\Phi \subseteq L(S)$ *is satisfiable if and only if every finite subset of* Φ *is satisfiable.*

Application 1: Let $S := \{+, \cdot, -, 0, 1\}$ and $\varphi \in L(S)$. If φ is satisfied in every field of characteristic zero, then there exists a $p > 0$ such that φ is satisfied in every field of characteristic $> p$.

Proof: $\{field\ axioms\} \cup \{\neg(1+1=0), \neg(1+1+1=0), \neg(1+1+1+1=0), \ldots\} \cup \{\neg\varphi\}$ is not satisfiable. Hence a finite subset, which w.l.o.g contains $\{field\ axioms\} \cup \{\neg\varphi\}$, is not satisfiable. Hence this finite subset with $\neg\varphi$ removed (which *is* satisfiable) implies φ. □

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Application 2: Upward Löwenheim-Skolem: If Φ has an infinite model, then it has models of arbitrary cardinality. Proof: Add constant symbols and the axioms $\neg(c = c')$...$\square$

See the course by David Pierce (PAC), 18h, Room I

Abstract Logics

Definition: An *abstract logic* $\mathcal{L}$ consists of a function L: signatures $\rightarrow$ sets (elements of $L(S)$ are called the *S-sentences of $\mathcal{L}$*) ...

Abstract Logics

Definition: An *abstract logic* $\mathcal{L}$ consists of a function L: signatures $\to$ sets and a binary relation $\vDash_{\mathcal{L}}$ betwen S-structures and elements of $L(S)$ (written $\mathcal{M} \vDash_{\mathcal{L}} \varphi$), such that

(a) If $S_0 \subseteq S_1$ then $L(S_0) \subseteq L(S_1)$
(b) If $\mathfrak{M} \vDash_{\mathcal{L}} \varphi$ and $\mathfrak{M} \cong \mathfrak{N}$ then $\mathfrak{N} \vDash_{\mathcal{L}} \varphi$
(c) If $S_0 \subseteq S_1$, $\varphi \in L(S_0)$ and $\mathfrak{M}$ is an S_1-structure, then $\mathcal{M} \vDash_{\mathcal{L}} \varphi$ iff $\mathcal{M}|_{S_0} \vDash_{\mathcal{L}} \varphi$

For $\varphi \in L(S)$ we write $\mathrm{Mod}_{\mathcal{L}}(\varphi) := \{\mathfrak{M} \in S - \mathrm{structures} \mid \mathfrak{M} \vDash \varphi\}$

Abstract Logics: Examples

(1) First order logic with $L(S)$ and $\models$ as defined before.

Abstract Logics: Examples

(2) The second order logic $\mathcal{L}^{2nd}$:

For $L^{2nd}(S)$-formulas we adopt the generation rules of first order S-formulas. Additionally we have *relation variables* of all arities and declare:

(a) If X is an n-ary relation variable and $t_1, \dots, t_n$ are terms, then $X(t_1, \dots, t_n)$ is an S-formula

(b) If φ is an S-formula, and X is a relation variable, then $\exists X \varphi$ is an S-formula.

(c) An $L^{2nd}(S)$-sentence is a $L^{2nd}(S)$-formula without free variables.

Satisfaction relation: For first order formation rules as usual. Additionally declare for an n-ary relation variable:

$\mathfrak{M} \models_{\mathcal{L}^{2nd}} \exists X \varphi \;:\Leftrightarrow$ there is an $R \subseteq M^n$ such that $\mathfrak{M} \models_{\mathcal{L}^{2nd}} \varphi(R/X)$

Abstract Logics: Examples

(3) The logics $\mathcal{L}_{\kappa\lambda}$:

For cardinals $\kappa \geq \lambda$ define the $L_{\kappa\lambda}(S)$-formulas as for first order logic, plus:
– for a set $\{\varphi_i \mid i \in I\}$, $|I| \leq \kappa$, one has a formula $\bigwedge \varphi_i$
– for a set of variables $\{x_i \mid i \in I\}$, $|I| \leq \lambda$ and a formula φ one has a formula $\exists(x_i \mid i \in I)\varphi$.

Satisfaction relation: For first order formation rules as usual. Additionally
– $\mathfrak{M} \models_{\mathcal{L}_{\kappa\lambda}} \bigwedge \varphi_i \;:\Leftrightarrow \mathfrak{M} \models_{\mathcal{L}_{\kappa\lambda}} \varphi_i$ for all $i \in I$
– $\mathfrak{M} \models_{\mathcal{L}_{\kappa\lambda}} \exists(x_i \mid i \in I)\varphi \;:\Leftrightarrow$ there is $\{m_i \mid i \in I\} \subseteq M$ such that $\mathfrak{M} \models_{\mathcal{L}_{\kappa\lambda}} \varphi(m_i/x_i)$

1. Note that $\mathcal{L}_{\omega\omega}$ is classical first order logic.
2. One also allows the case κ or $\lambda = \infty$ where one imposes no cardinality restriction.

Abstract Logics: Examples

(4) $\mathcal{L}_{\omega\omega}(Q_1)$:= usual 1st order logic enhanced with the quantifier Q_1, interpreted as "there exist uncountably many"

(5) $\mathcal{L}_{\omega\omega}(Q^R)$:= usual 1st order logic enhanced with the *binary* quantifier Q^R, interpreted as
$\mathfrak{M} \vdash_{\mathcal{L}_{\omega\omega}(Q^R)} Q^R xy\, [\varphi(x), \psi(y)] :\Leftrightarrow card\{m \in M \mid \mathfrak{M} \vdash \mathcal{L}_{\omega\omega}(Q^R)\varphi(m)\} < card\{m \in M \mid \mathfrak{M} \vdash \mathcal{L}_{\omega\omega}(Q^R)\psi(m)\}$

(6) Weak second order logic $\mathcal{L}^{w2nd}$: Same syntax as $\mathcal{L}^{2nd}$ but relation variables are only interpreted as ranging over *finite* subsets of M^n.

Abstract Logics: Non-example

NOT an example: start from a *2nd order signature* **S** containing relation/function/constant symbols as before, and additionally second order relation symbols interpreted as relations between *subsets* of the domain of interpretation.

There are obvious notions of **S**-structure, and of isomorphism of **S**-structures.

One can set up a language $L(\mathbf{S})$ from such a 2nd order signature **S** (best done using sorts) and define the obvious satisfaction relation between **S**-structures and $L(\mathbf{S})$-sentences (example: one can define the theory of topological spaces).

Our logics always are based on first order signatures!

Expressivity of abstract logics

Definition: Let $\mathcal{L}$, $\mathcal{L}'$ be abstract logics. We say that $\mathcal{L}'$ *has at least the same expressive power* as $\mathcal{L}$, written $\mathcal{L}' \geq \mathcal{L}$, if for every S and every $\varphi \in L(S)$ there is a $\psi \in L'(S)$ with $\mathrm{Mod}_{\mathcal{L}}(\varphi) = \mathrm{Mod}_{\mathcal{L}'}(\psi)$.

We write $\mathcal{L}' \sim \mathcal{L}$ (equal expressive power), if $\mathcal{L}' \geq \mathcal{L}$ and $\mathcal{L}' \leq \mathcal{L}$. We write $\mathcal{L}' > \mathcal{L}$ if $\mathcal{L}' \geq \mathcal{L}$ and not $\mathcal{L}' \sim \mathcal{L}$.

Example: In $\mathcal{L}^{2nd}$ we can characterize $\mathbb{R}$ up to isomorphism by adding to the theory of ordered fields the sentence
$\forall X((\exists x X(x) \wedge \exists y \forall z(X(z) \rightarrow z < y)) \rightarrow \exists y(\forall z(X(z) \rightarrow (z < y \vee z = y)) \wedge \forall x(x < y \rightarrow \exists z(x < z \wedge X(z)))))$
("every nonempty subset which is bounded above has a supremum")

By Löwenheim-Skolem we can not characterize $\mathbb{R}$ up to isomorphism in first order language. Hence $\mathcal{L}^{2nd} > \mathcal{L}_{\omega\omega}$.

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Another Example: In $\mathcal{L}_{\omega_1\omega}$ we can characterize the class of fields of characteristic 0 by adding to the theory of fields the sentence $\bigvee\{1+1=0, 1+1+1=0, 1+1+1+1+1=0, \ldots\}$

By Application 1 of the compactness theorem, there is no first order sentence characterizing fields of characteristic 0. Hence $\mathcal{L}_{\omega_1\omega} > \mathcal{L}_{\omega\omega}$.

Properties of abstract logics

Definition: For an abstract logic $\mathcal{L}$ we abbreviate:

- $\mathrm{LöSko}(\mathcal{L})$ ("$\mathcal{L}$ has the Löwenheim-Skolem property") $:\Leftrightarrow$ If $\varphi \in L(S)$ has a model, then it has a model which is at most countable.
- $\mathrm{Comp}(\mathcal{L})$ ("$\mathcal{L}$ has the compactness property") $:\Leftrightarrow$ If $\Phi \subseteq L(S)$ and every finite subset of Φ is satisfiable, then Φ is satisfiable.

Definition: For an abstract logic $\mathcal{L}$ we abbreviate:

- $\mathrm{Bool}(\mathcal{L})$ ("$\mathcal{L}$ contains Boolean connectives") $:\Leftrightarrow$
  (1) For every $\varphi \in L(S)$ there is a $\chi \in L(S)$ such that for all S-structures $\mathfrak{M}$: $\mathfrak{M} \vDash \varphi \Leftrightarrow$ not $\mathfrak{M} \vDash \chi$
  (2) For every $\varphi, \psi \in L(S)$ there is a $\chi \in L(S)$ such that for all S-structures $\mathfrak{M}$: $\mathfrak{M} \vDash \chi \Leftrightarrow \mathfrak{M} \vDash \varphi$ and $\mathfrak{M} \vDash \psi$

Properties of abstract logics

Definition: For an abstract logic $\mathcal{L}$ we abbreviate:

- $\mathrm{Repl}(\mathcal{L})$ ("$\mathcal{L}$ admits replacement of function symbols and constants by relation symbols"):
  From a signature S we get a new signature S^r by replacing n-ary function (resp. constant) symbols with $(n+1)$-ary (resp. unary) relation symbols.
  From an S-structure $\mathfrak{M}$ we get an S^r-structure $\mathfrak{M}^r$ by interpreting the new relation symbols as the graphs of the functions $f^{\mathfrak{M}}$.
  Then: $\mathrm{Repl}(\mathcal{L}) :\Leftrightarrow$ For every $\varphi \in L(S)$ there is a $\chi \in L(S^r)$ such that for all S-structures $\mathfrak{M}$ we have $\mathfrak{M} \models \varphi \Leftrightarrow \mathfrak{M}^r \models \chi$.

Properties of abstract logics

Definition: For an abstract logic $\mathcal{L}$ we abbreviate:

- $\mathrm{Rel}(\mathcal{L})$ ("$\mathcal{L}$ admits relativization"):
  For an S-structure $\mathfrak{M}$ and an S-closed subset $A \subseteq M$ we get a sub-S-structure $\mathfrak{M}|_A$ with underlying set A.
  We also get an $S \cup \{U\}$-structure $\mathfrak{M}^{U \rightsquigarrow A}$ (U a new unary relation symbol), with underlying set M, where U is interpreted as the subset A.
  Then: $\mathrm{Rel}(\mathcal{L}) :\Leftrightarrow$ For every $\varphi \in L(S)$ there is a $\chi \in L(S \cup \{U\})$ such that $\mathfrak{M}|_A \models \varphi \Leftrightarrow \mathfrak{M}^{U \rightsquigarrow A} \models \chi$

Definition: An abstract logic satisfying Bool, Repl and Rel is called regular.

Lindström's Theorem

Theorem (Lindström's Theorem)

For a regular abstract logic $\mathcal{L}$ with $\mathcal{L}_{\omega\omega} \leq \mathcal{L}$ one has: If $LöSko(\mathcal{L})$ and $Comp(\mathcal{L})$ then $\mathcal{L} \sim \mathcal{L}_{\omega\omega}$.

Equivalently: $\mathcal{L}_{\omega\omega}$ is the most expressive regular abstract logic having the Löwenheim-Skolem and Compactness properties.

(The first form can be read as a no-go theorem, the second as a characterization of $\mathcal{L}_{\omega\omega}$)

Idea of the proof: Assume that $\mathcal{L}_{\omega\omega} < \mathcal{L}$. Then there exist S and a $\psi \in L(S)$ which is not equivalent to any first order sentence, i.e. $\nexists \varphi \in L_{\omega\omega}(S)$ s.t. $\mathrm{Mod}_{\mathcal{L}_{\omega\omega}}(\varphi) = \mathrm{Mod}_{\mathcal{L}}(\psi)$.
We get S-structures $\mathfrak{M}$, $\mathfrak{N}$ with $\mathfrak{M} \vDash_{\mathcal{L}} \psi$, $\mathfrak{N} \vDash_{\mathcal{L}} \neg\psi$. By $\mathrm{LöSko}(\mathcal{L})$ both $\mathfrak{M}$ and $\mathfrak{N}$ are $\mathcal{L}$-elementary equivalent to countable structures.
From the fact that $\mathfrak{M}$ and $\mathfrak{N}$ are indistinguishable by first order formulas we get $\mathfrak{M} \cong_m \mathfrak{N}$ (:=there is a set of partial isomorphisms which are extendable m times with any choice of argument/value) – here we use $\mathcal{L}_{\omega\omega} \leq \mathcal{L}$ and that $\mathcal{L}$ is regular to handle first order formulas inside $\mathcal{L}$.
From compactness we get $\mathfrak{M} \cong_m \mathfrak{N} \Rightarrow \mathfrak{M} \cong_p \mathfrak{N}$ (:= there is a set of partial isomorphisms extendable countably many times). For countable structures $\mathfrak{M} \cong_p \mathfrak{N}$ implies $\mathfrak{M} \cong \mathfrak{N}$.
But isomorphic structures behave identically for any abstract logic - contradiction to $\mathfrak{M} \vDash_{\mathcal{L}} \psi$, $\mathfrak{N} \vDash_{\mathcal{L}} \neg\psi$.