ON THE MULTIPLICITY OF ILLUMINATION OF CONVEX BODIES BY POINT SOURCES

O. V. Bogopol'skii and V. A. Vasil'ev

In this paper we investigate a combinatorial geometry problem on the multiplicity of illumination of sets by point sources. Being of interest in itself, the problem considered is orientated towards problems of nonsymmetric analogs of Shapley value in cooperative games with accessory payments [1].

In Theorem 1 we give the exact lower bound of a number of sources that are strictly separated from a convex body of $\mathbb{R}^n$ and illuminate it with a multiplicity not less than two. We obtain an analogous result for nonconvex bodies with a finite set of nonsmooth points (see Proposition 1).

We have established the hypothesis on the minimal number of sources for the general case, that is, for a multiplicity of illumination more than two.

For a nonempty subset $X \subseteq \mathbb{R}^n$ we denote the convex hull of X by $\operatorname{co} X$; $B_\varepsilon(x)$ is a closed ball (in l_2-norm) of radius $\varepsilon > 0$ and with center at a point $x \in \mathbb{R}^n$; $L[x, y)$ is a ray on a vertex x passing through a point y; $\Pi(x, y)$ is a straight line passing through the points x and y; $[x, y]$ is the segment connecting points x and y.

By a body $V \in \mathbb{R}^n$ we mean a compact set V such that for any point x of its boundary ∂V and for any $\varepsilon > 0$ we have $B_\varepsilon(x) \cap \operatorname{int} V \neq \varnothing$. Using the usual terminology of combinatorial geometry [2, 3], we introduce the following definition.

Definition 1. Let V be a body of $\mathbb{R}^n$. We say that a point $y \in \partial V$ is illuminated by a point (a source) $x \in \mathbb{R}^n \setminus V$, if there exists $\varepsilon > 0$ such that for any $z \in B_\varepsilon(y) \cap \partial V$ we have $[x, z] \cap V = \{z\}$.

For convex bodies Definition 1 is equivalent to the following definition.

Definition 1′. Let V be a convex body of $\mathbb{R}^n$. We say that a point $y \in \partial V$ is illuminated by a point (a source) $x \in \mathbb{R}^n \setminus V$, if $[x, y] \cap V = \{y\}$ and $L[x, y) \cap \operatorname{int} V \neq \varnothing$.

Let V be a body of $\mathbb{R}^n$ and X be a subset of $\mathbb{R}^n$ such that $V \cap X = \varnothing$. A *multiplicity of illumination of a point* $y \in \partial V$ (with respect to X) is a cardinality $m_X(y)$ of the set of points of X that illuminate y.

Definition 2. The quantity $m_X(V) = \sup\{m_X(y) \mid y \in \partial V\}$ is called the *multiplicity of illumination of the body* V by the set X.

By V_x we denote the set of points $y \in \partial V$ that are illuminated by a point x.

Remark 1. The condition $m_X(V) = 1$ is equivalent to the condition $V_x \cap V_{x'} = \varnothing$ for $x, x' \in X$, $x \neq x'$.

Theorem 1. *Let $V \subseteq \mathbb{R}^n$ be a convex body, $X \subseteq \mathbb{R}^n$ a set of points such that $|X| \geq n + 1$ and $\operatorname{co} X \cap V = \varnothing$. Then the multiplicity of illumination of the body V by the set X is not less than two.*

We first prove the following lemma.

Lemma. *Let V be a convex body of $\mathbb{R}^n$, $x \in \mathbb{R}^n \setminus V$, and $x' \in \operatorname{co}(\{x\} \cup V) \setminus V$. Then $V_{x'} \subseteq V_x$.*

Proof. It is sufficient to prove that $z \in \partial V$ and $[x', z] \cap V = \{z\}$ imply $[x, z] \cap V = \{z\}$. If $z \in L[x, x')$, then the latter is obvious. Therefore, we consider that $z \notin L[x, x')$. We denote by t a point lying on the ray $L[x, x')$ and such that $[x, t] \cap V = \{t\}$.

Suppose there exists a point $v \in [x, z) \cap V$. The point t lies on the continuation of a side of triangle $xx'z$, and the point v lies inside $[x, z]$. Then by Pasch's axiom of the system of Hilbert's axioms of Euclidean geometry the segment $[v, t]$ intersects the side $[x', z]$ at some interior point u. But since the body V is convex and $v, t \in V$, it follows that $u \in [v, t] \subseteq V$, contradicting $[x', z] \cap V = \{z\}$.

Proof. It is sufficient to prove the theorem for the case $|X| = n + 1$. We apply the induction over n. For $n = 1$ the statement of the theorem is obviously valid. Let us go over by induction from $n - 1$ to n. Let the body $V \subseteq \mathbb{R}^n$ and the set $X = \{x_1, \ldots, x_{n+1}\}$ satisfy the hypotheses of the theorem, and $m_X(V) = 1$. Since $\operatorname{co} X \cap V = \varnothing$, we see that there exists a hyperplane H strictly separating X and V. Take some point $v \in \operatorname{int} V$ and put $x_i' = L[x_i, v) \cap H$, $i = 1, \ldots, n + 1$. Then by the lemma, the set $X' = \{x_1', \ldots, x_{n+1}'\}$, as well as the set X, illuminate V with multiplicity one and, furthermore, $X' \subseteq H$, $H \cap V = \varnothing$.

Therefore, without loss of generality we can consider in the sequel that $X \subseteq H$. Let us show that without loss of generality we can also assume that $x_1 \in \operatorname{int}_H \operatorname{co}(X \setminus \{x_1\}$.

Let H^0 be a hyperplane of support to V that is parallel to the plane H. For $\tau \in \mathbb{R}$ by H^τ we denote a hyperplane parallel to H^0 and lying at a distance $|\tau|$ from H^0 in the same half-space (with respect to H^0) as the body V for $\tau \leq 0$, or in the other half-plane for $\tau > 0$.

Since $\operatorname{int} V \neq \varnothing$, there exists $\tau_0 < 0$ such that

1) for $\tau_0 < \tau < 0$ the set $V^\tau = H^\tau \cap V$ is a (convex) body in the hyperplane H^{τ_0}, and
2) the point $v \in \operatorname{int} V$ and the hyperplane H^0 are strictly separated by the hyperplane H^{τ_0}.

For $\tau > \tau_0$ we define the points $x_2^\tau = L[x_2, v) \cap H^\tau$, $\ldots$, $x_{n+1}^\tau = L[x_{n+1}, v) \cap H^\tau$. Further, we put $X^\tau = \{x_2^\tau, \ldots, x_{n+1}^\tau\}$. Then either $X^\tau \cap V^\tau \neq \varnothing$, or $X^\tau \subseteq H^\tau \setminus V^\tau$. Let us show that $\operatorname{co} X^\tau \cap V^\tau \neq \varnothing$ in both cases. Toward this end we prove the following statement:

for any $\tau_0 < \tau < 0$ and $x \in H^\tau \setminus V^\tau$

$$(V^\tau)_x \subseteq V_x. \tag{1}$$

Let $y \in (V^\tau)_x$. Then by Definition 1′, $L[x, y) \cap \operatorname{int}_{H^\tau} V^\tau \neq \varnothing$. To prove the inclusion $y \in V_x$, it is sufficient to show that $\operatorname{int}_{H^\tau} V^\tau \subseteq \operatorname{int} V$. However, this is obvious since $\operatorname{co}(\{v, w\} \cup V^\tau) \subseteq V$, where $w \in V^0$.

Therefore, in the case $X^\tau \subseteq H^\tau \setminus V^\tau$ $(\tau_0 < \tau < 0)$ it follows from (1) and $m_X(V) = 1$ that $m_{X^\tau}(V^\tau) = 1$. But then by the inductive hypothesis we have $\operatorname{co} X^\tau \cap V^\tau \neq \varnothing$.

Thus, under our assumptions for all $\tau \in (\tau_0, 0)$ we have $\operatorname{co} X^\tau \cap V^\tau \neq \varnothing$.

Now for each τ, $\tau_0 < \tau < 0$, we choose a point $y^\tau \in \operatorname{co} X^\tau \cap V^\tau$ and coefficients $\alpha_2^\tau, \ldots, \alpha_{n+1}^\tau$ such that $y^\tau = \sum_{i=2}^{n+1} \alpha_i^\tau x_i^\tau$, $\sum_{i=2}^{n+1} \alpha_i^\tau = 1$, $\alpha_2^\tau \geq 0, \ldots, \alpha_{n+1}^\tau \geq 0$. Since the simplex $\{x \in \mathbb{R}_+^n \,|\, \sum_{i=1}^n x_i = 1\}$ is compact, we see that there exist coefficients $\alpha_2^0, \ldots, \alpha_{n+1}^0$ such that $\sum_{i=2}^{n+1} \alpha_i^0 = 1$, $\alpha_2^0 \geq 0, \ldots, \alpha_{n+1}^0 \geq 0$ and the vector $(\alpha_2^0, \ldots, \alpha_{n+1}^0)$ is a limit vector for the sequence of vectors $(\alpha_2^\tau, \ldots, \alpha_{n+1}^\tau)$, $\tau = \frac{\tau_0}{2}, \frac{\tau_0}{3}, \ldots$. Put $y^0 = \sum_{i=2}^{n+1} \alpha_i^0 x_i^0$. Then y^0 is the limit point of the sequence y^τ, $\tau = \frac{\tau_0}{2}, \frac{\tau_0}{3}, \ldots$, and, consequently, $y^0 \in V^0$, since V is closed.

Further, we fix $\varepsilon > 0$. Then $y^0 \in \operatorname{int} \operatorname{co} \bigcup_{i=2}^{n+1} B_\varepsilon(x_i^0) \cap V^0$ and for the point $x_1' = H^\mu \cap L[x_1, y^0)$, with sufficiently small $\mu > 0$, we have $x_1' \in \operatorname{int} \operatorname{co} \bigcup_{i=2}^{n+1} B_\varepsilon(x_i^0) \cap H^\mu$.

Now for each $i = 2, \ldots, n + 1$ we choose a point x_i'' of the sets

$$(B_\varepsilon(x_i^\mu) \cap H^\mu) \cap \operatorname{co}(\{x_i\} \cup V)$$

so that

$$x_1' \in \operatorname{int} \operatorname{co}\{x_2', \ldots, x_{n+1}'\}. \tag{2}$$

By the choice of x_i' and the lemma we have $V_{x_i'} \subseteq V_{x_i}$ for $i = 1, 2, \ldots, n + 1$. It follows from this and from $m_X(V) = 1$, that $m_{X'}(V) = 1$, where $X' = \{x_1', \ldots, x_{n+1}'\}$. Moreover, $X' \subseteq H^\mu$ and the inclusion (2) is true.

So, without loss of generality we can assume that the set X lies in a hyperplane H not intersecting V, $m_X(V) = 1$ and

$$x_1 \in \operatorname{int} \operatorname{co}\{x_2, \ldots, x_{n+1}\}. \tag{3}$$

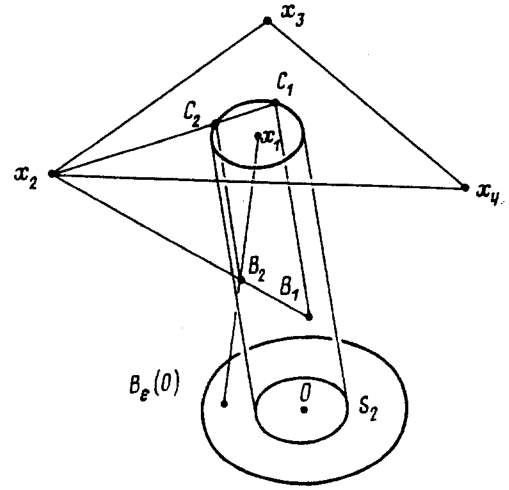

FIG. 1

Let $A \in V$ be a nearest point to the point x_1 with respect to Euclidean metric l_2, and let $h = l(x_1, A)$ be the distance between x_1 and A. Then there exists a ball $B_\varepsilon(O) \subseteq \operatorname{int} V$ such that $\varepsilon < \frac{h}{9}$ and $l(x_1, O) < \frac{4}{3}h$. There also exists ε_1 such that $0 < \varepsilon_1 < \frac{\varepsilon}{2}$ and $l(x_j, B_{\varepsilon_1}(x_1)) > 6\varepsilon_1$ for $j = 2, \ldots, n+1$.

Let us put $S_1 = \partial(B_{\varepsilon_1}(x_1) \cap H)$ and $S_2 = S_1 + \overrightarrow{x_1 O}$. We build a cylinder R on the bases S_1 and S_2 and say that the sets $\left\{S_2 + t \cdot \overrightarrow{Ox_1} \mid 0 \le t \le \frac{1}{2}\right\}$ and $\left\{S_2 + t \cdot \overrightarrow{Ox_1} \mid 0 \le t \le \frac{1}{3}\right\}$ are the lower half part and the lower third part of R, respectively. Then

1) the lower half part of R lies in $K = \operatorname{co}(\{x_1\} \cup B_\varepsilon(O))$, since $\varepsilon_1 < \frac{\varepsilon}{2}$, and
2) $R \cap V$ lies in the lower third of R, otherwise $l(x_1, V) < \varepsilon_1 + \frac{2}{3} l(x_1, O) < \frac{\varepsilon}{2} + \frac{8}{9}h < h$.

Let us define a function $F\colon B_{\varepsilon_1}(x_1) \cap H \to \mathbb{R}$ in such a way that $F(x)$ is the distance from the point x to V along the direction $\overrightarrow{x_1 O}$, that is,

$$F(x) = l(x, V \cap \Pi_x), \qquad x \in B_{\varepsilon_1}(x_1) \cap H,$$

where Π_x is a straight line passing through the point x and parallel to the straight line $\Pi(x_1, O)$. Since the body V is convex, we see that the finite-valued function F is convex and, therefore, continuous [4]. By virtue of the compactness of S_1 the lower bound $d = \inf\limits_{x \in S_1} F(x)$ is realized. Let $C_1 \in S_1$ and $B_1 \in R \cap V$ be points such that $\overrightarrow{C_1 B_1} \parallel \overrightarrow{x_1 O}$ and $l(C_1, B_1) = d$. It is clear that B_1 lies in the lower third part of the cylinder R.

Let us choose a point $x_i \in \{x_2, \ldots, x_{n+1}\}$ so that a point $C_2 \in B_{\varepsilon_1}(x_1)$ lies inside the segment $[x_i, C_1]$ (see Fig. 1). This is possible, otherwise the sets $\{x_2, \ldots, x_{n+1}\}$ and $B_{\varepsilon_1}(x_1)$ are separated by a hyperplane passing through C_1, contradicting the condition (3).

Now let us consider the triangle $x_i C_1 B_1$. Let B_2 be an interior point of the segment $[x_i, B_1]$ such that $\Pi(C_2, B_2) \parallel \Pi(C_1, B_1)$. Then $[C_2, B_2] \subseteq R$. Since $l(C_2, B_2) < l(C_1, B_1)$, we have $B_2 \notin V$. On the other hand,

$$l(C_2, B_2) > \frac{3}{4} l(C_1, B_1), \tag{4}$$

since

$$\frac{l(C_1, B_1)}{l(C_2, B_2)} = \frac{l(x_i, C_1)}{l(x_i, C_2)} = \frac{l(x_i, C_2) + l(C_2, C_1)}{l(x_i, C_2)} = 1 + \frac{l(C_2, C_1)}{l(x_i, C_2)} < 1 + \frac{2\varepsilon_1}{6\varepsilon_1} = \frac{4}{3}.$$

It follows from (4) and from B_1 lying in the lower third part of the cylinder R that the point B_2 lies in the lower half part of R and, hence, in K. By the lemma we have $V_{B_2} \subseteq V_{x_1}$ and $V_{B_2} \subseteq V_{x_i}$. Besides, $V_{B_2} \neq 0$, since $B_2 \notin V$. From here it follows that $V_{x_1} \cap V_{x_i} \neq 0$ and $m_X(V) \geq 2$, contradicting the hypothesis. The theorem is proved.

Definition 3. We say that a point y of the boundary of the body V is smooth, if at this point there exists a unique hyperplane that is tangent to V.

Proposition 1. *Let V be a body of $\mathbb{R}^n$ (possibly, nonconvex) with a finite set of nonsmooth points of ∂V; let X be a set of points of $\mathbb{R}^n$ such that*

1) $|X| \geq 3$, *and*
2) $\operatorname{co} X \cap V = \varnothing$.

Then the multiplicity of illumination of the body V by the set X is not less than two.

If all points of ∂V are smooth or $n = 1$, then condition 1) *may be replaced by*

1') $|X| \geq 2$.

Proof. For $n = 1$ the proposition is obvious. Let $n \geq 2$, $|X| \geq 3$, and let $X_1 = \{x_1, x_2, x_3\}$ be a three-element subset of X. Also let $z \in \partial V$ be the nearest point to $\operatorname{co} X_1$, and d the distance from z to $\operatorname{co} X_1$. Since d, the neighborhood of the set $\operatorname{co} X_1$, is also a convex set, we see that there are no points of V inside segments connecting z with any point of $\operatorname{co} X_1$. Therefore, if z is a smooth point, then $z \in V_{x_1} \cap V_{x_2}$ and $m_X(V) \geq 2$.

Suppose z is a nonsmooth point and $B_\varepsilon(z)$ is a ball such that $\operatorname{co}(B_\varepsilon(z) \cup X_1)$ does not contain any nonsmooth points except for z.

Let us prove that one can choose a point $t \in T = \operatorname{co}(B_\varepsilon(z) \cup X_1) \cap \partial V$ such that for some distinct $x_i, x_j \in X_1$, the set $\operatorname{co}\{x_i, x_j, t\} \cap \partial V$ does not contain the point z. We first choose a point $y \in T$ that is different from z. The following two cases are possible:

a) $z \notin \operatorname{co}\{x_i, x_j, y\}$ for some distinct $x_i, x_j \in X_1$, or
b) $z \in \operatorname{co}\{x_i, x_j, y\}$ for some distinct $x_i, x_j \in X_1$.

In case a) we put $t = y$.

In case b) the points x_1, x_2, x_3, y and z lie in the same affine plane P of dimension 2. In addition, in this case $z \in [x_i, y]$ for some $x_i \in \{x_1, x_2, x_3\}$.

If there exists a point of T not lying in P, then as t we choose any such point.

But if $T \subseteq P$, then as t we take a point of T such that t and some point $x_j \in X_1 \setminus \{x_i\}$ lie in the same half-plane defined by the straight line $\Pi(x_i, y)$, and $t \notin (x_i, y)$. The triple x_i, x_j, t is required.

Thus, the set $\operatorname{co}\{x_i, x_j, t\} \cap \partial V$ is nonempty, compact, and does not contain nonsmooth points of ∂V. Let v be a point of $\operatorname{co}\{x_i, x_j, t\} \cap \partial V$ such that the sum of the distances from v to the points x_i and x_j is minimal. Then v is a smooth point and there exist no points of V on the segments $[x_i, v]$ and $[x_j, v]$ except for v. Hence, $v \in V_{x_i} \cap V_{x_j}$, that is, $m_X(V) \geq 2$ also in this case.

If all points of ∂V are smooth, then the proof of the assumption is analogous to that of the first section of the present proof.

Definition 4. For $n, k \in \mathbb{N}$, $n \geq 1$, $k \geq 2$, by $f(n,k)$ we denote a minimal natural number such that for a convex body of $\mathbb{R}^n$ and for any set $X \subseteq \mathbb{R}^n$ of points such that

1) $|X| \geq f(n,k)$, and
2) $\operatorname{co} X \cap V = \varnothing$,

the set X illuminates the body V with a multiplicity not less than k.

Conjecture. $f(n,k) = n(k-1)+1$.

Remark 2. The conjecture is true for $k = 2$ (Theorem 1) and for $n = 2$.

Remark 3. The following example shows that

$$f(n,k) \geq n(k-1)+1.$$

As V we take a simplex $\{x \in \mathbb{R}_+^n \mid \sum_{i=1}^n x_i \leq 1\}$, and as X a set of points with the properties

1) X lies in a plane $P = \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = -1\}$,
2) $X = X_1 \cup \cdots \cup X_n$, where $X_i \cap X_j = \varnothing$ for $i \neq j$; $|X_i| = k-1$ for $i = 1, \dots, n$; and points of X_i (and only these points of X) illuminate the interior of the face of the body V,

$$\Gamma_i = \left\{ x \in \mathbb{R}_+^n \mid \sum_{i=1}^n x_s \leq 1, x_i = 0 \right\} \qquad i = 1, \dots, n.$$

It is easy to understand that there exists such a set X of the cardinality $n(k-1)$ that this set illuminates the body V with multiplicity $k-1$, and, if we add a new point of P to X, then the new set illuminates V with multiplicity k.

References

1. V.A.Vasil'ev, *Characterization of kernels and generalized solutions for some classes of cooperative games*, Trudy Inst. Matematiki Akad. Nauk SSSR, Sib. Otd. **10** (1988).
2. V.G.Boltianskii, *The problem on the illumination of convex body boundary*, Izv. Mold. Ril. Akad. Nauk SSSR **76** (1960), no. 10, 77–84.
3. V.G.Boltianskii and P.S.Soltan, *Combinatorial geometry of different classes of convex sets*, "Shtiintsa", Kishinev, 1978.
4. R.T.Rockafellar, *Convex analysis*, "Mir", Moscow, 1973.