The number of non-solutions to an equation in a group and non-topologizable torsion-free groups

(Klyachko and Trofimov, 2004)

① Definitions and Markov's theorem

<u>Def. 1</u> A group G is called <u>topologizable</u> if it admits a non-discrete Hausdorff group topology.

<u>Def. 2</u> A subset $S \subseteq G$ is called <u>elementary algebraic</u> if there exist elements $a_1, \dots, a_m \in G$ and numbers, $\varepsilon_1, \dots, \varepsilon_n \in \mathbb{Z}$, s.t. $S = Sol(a_1 x^{\varepsilon_1} a_2 x^{\varepsilon_2} \dots a_m x^{\varepsilon_m} = 1)$.

<u>Thm (Markov, 1946)</u> A countable group G is non-topologizable

$\Leftrightarrow$ $G \setminus \{1\}$ is a finite union of elementary algebraic sets

$\Leftrightarrow$ $\exists$ a finite subset $\mathcal{F} \subseteq G$:

$G \setminus \mathcal{F}$ is a finite union of elementary algebraic sets

<u>Lem</u> Let X be a Hausdorff top. space and let $\mathcal{F} = \{x_1, \dots, x_n\} \subseteq X$ be a finite subset.

Then $\mathcal{F}$ is open $\Leftrightarrow$ each $\{x_i\}$, $i = 1, \dots, n$ is open.

<u>Proof</u> $\exists\, O(x_1)$: $x_2, \dots, x_n \notin O(x_1)$. Then

$\{x_1\} = \mathcal{F} \cap O(x_1)$.

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Main theorem [KT]. $\exists$ f.g. $\boxed{\text{torsion-free}}$ group H and $\boxed{\text{one equation}}$ $w(x)=1$ with one variable and koefficients from H, s.t. $H \setminus \{1\} = Sol(w(x)=1)$.

Remark 1) By Löwenheim-Skolem, f.g. can be replaced by any infinite cardinality.

2) We will use later that $exp_x(w)=0$.

Earlier was known

Olšh (1989): $G \setminus \{g_1, \dots, g_n\} = \bigcup_{i=1}^{n} Sol(x^n = g_i)$,

G — f.g. periodic

$n \geq 665$, odd.

Trofimov (2004): $G \setminus \{g_1, \dots, g_{2n}\} = Sol([x,a]^n = 1)$

G — f.g., $\pi(G) = \mathbb{N} \cup \{\infty\}$.

a — some element of G.

Thm 1 For any two cardinals s and n with $s+n=\infty$, there exists a group G of cardinality $s+n$ and an equation $u(x)=1$ over G, s.t.

$$|Sol(u(x)=1)| = s \quad \& \quad |Sol(u(x) \neq 1)| = n.$$

Exercise $(2,1)$ is not realizable: in $\mathbb{Z}_3$ any equation $u(x)=1$ has exactly $0, 1$, or 3 solutions

Proof: Consider an equation $x^n = a$, where $\langle a \rangle = \mathbb{Z}_3$.
If $x_1^n = a = x_2^n$, $x_1 \neq x_2$, then $(x_1 x_2^{-1})^n = e \Rightarrow n \vdots 3 \Rightarrow a=1$
$\Rightarrow 3$ solutions

Thm 1 For any two cardinals s and n with $s+n=\infty$, there exists a group G of cardinality $s+n$ and an equation $u(x)=1$ over G, s.t.

$$|Sol(u(x)=1)|=s \quad \& \quad |Sol(u(x)\neq 1)|=n.$$

Proof

1) $n=0$. Equation $1=1$
2) $s=0$. $g=1$, where $g\in G\setminus\{1\}$.
3) $s=1$ $x=1$.
4) $1<s\leq n$ ($\Rightarrow n$ is infinite)

$$G=\underbrace{A}_{\substack{\text{abelian}\\ |A|=s}} * \underbrace{B}_{\substack{\text{any}\\ |B|=n}}$$

Take $a\in A\setminus\{1\}$. Then

$$Sol_G(xa=ax)=A$$
$$Sol_G(xa\neq ax)=G\setminus A.$$

5) $s>n>0$ ($\Rightarrow s$ is infinite)

$$G=\underbrace{H}_{\substack{\text{from main Thm}\\ \text{but cardinal}\\ s}}\times\underbrace{K}_{\substack{\text{any}\\ |K|=n}}$$

Solutions of $u(x)=1$: $(H\setminus\{1\})\times K$

Nonsolutions: $\{1\}\times K$.

Proof: $[a,hk]=[a,h]$

$\Rightarrow k$ does not influence.

From the proof of the main thm:

$H=\langle a,b,c_1,\dots,c_e \mid \mathcal{R}\rangle$

Equation:

$$\underbrace{[c_1 v([a,x])c_1^{-1}, v([b,x])]}_{u(x)}=1,$$

where c_1,a,b are as above, v is a word, depending on generators of H and one variable y.

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② Ol'shanskii's approach to constructing groups with prescribed properties

A graded presentation $G(\infty) = \langle \mathcal{A} \mid \mathcal{R} \rangle$ with a filtration on the set of relators : $\mathcal{R} = \bigcup_{i=0}^{\infty} \mathcal{R}_i$, $\emptyset = \mathcal{R}_0 \subseteq \mathcal{R}_1 \subseteq \cdots$

Relators from $\mathcal{R}_i \setminus \mathcal{R}_{i-1}$ are called relators of rank i,

$G(i) = \langle \mathcal{A} \mid \mathcal{R}_i \rangle$ is also considered as a graded presentation with $\mathcal{R}_i = \mathcal{R}_{i+1} = \cdots$

Def. A graded presentation $G(\infty)$ satisfies $\boxed{\text{condition } R}$ with parameters α, h, d, n if for each $i = 0, 1, 2, \cdots$, there exists a set $X_i \subseteq F(\mathcal{A})$ of words (called periods of rank i), s.t.

1) $W \in X_i \Rightarrow |W| = i$, $W \underset{G(i-1)}{\not\sim} (W')^m$ ← any $\in \mathbb{Z}$

↖ any in $F(\mathcal{A})$ with $|W'| \leq i-1$.

2) $U \neq V$ from $X_i \Rightarrow U \underset{G(i-1)}{\not\sim} V^{\pm 1}$

3) Each relator $R \in \mathcal{R}_i \setminus \mathcal{R}_{i-1}$ has the form

$$R = \prod_{k=1}^{h} (T_k A^{n_k}),$$ where $A \in X_i$ and n_k, T_k, A satisfy :

R1 $|n_k| \geq n$.

R2 $|n_s| / |n_t| \leq 1 + \frac{1}{2} h^{-1}$. (think that $|n_s| = |n_t|$)

R3 The words T_k are not equal in $G(i-1)$ to words of smaller length, and $|T_k| \leq di$

if $|T_k| \ll n$, then T_k is small and A^{n_k} large

R4 T_k is not a power of A_k in $G(i-1)$.

R5–R6

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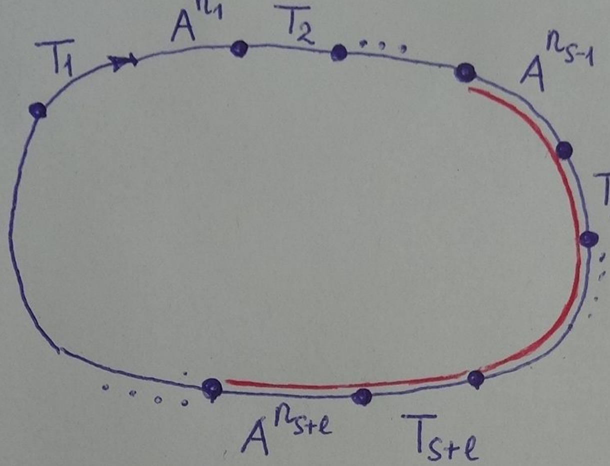
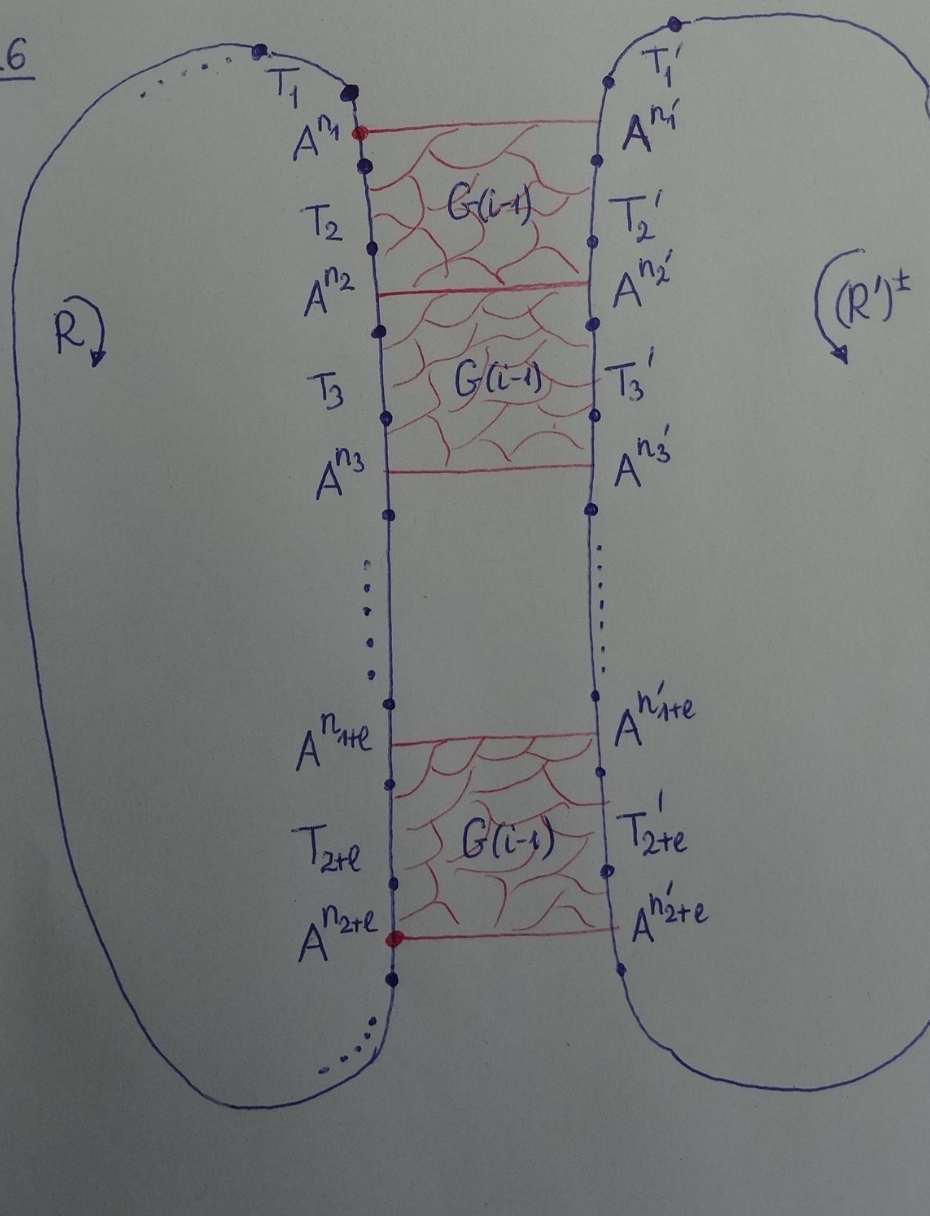
R5. The word R is not a proper power in the free group and any <u>long</u> subword of ~~R~~ the cyclic word R occurs in R <u>only once</u>:



$\ell > \alpha^{-1} - 4$
pairs of syllables.

R6



Suppose:

1) $\ell \geq \alpha^{-1} - 2$,
2) $\mathrm{Sign}(n_i) = \mathrm{Sign}(n_i')$, $i = 1, \dots, 2+\ell$,
3) A contiguity condition (see red)

Then $R \equiv R'$,
$V \equiv V'$
↑
the subword of R from one red point to the other

and V is not a subword of the cyclic word R^{-1}.

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<u>Lem 1</u> (from Ol'sh.) If a graded presentation $G(\infty)$ satisfies cond. R with $1 \ll \alpha^{-1} \ll h \ll d \ll n$, then

Thm 26.5 | 1) all abelian subgroups of $G(\infty)$ are cyclic

Lem 25.2 | 2) the group $G(\infty)$ is torsion-free (in particular $|G(\infty)| = \infty$)

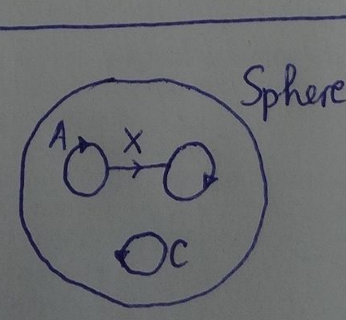
Lem 25.4 | 3) if $X \underset{G(\infty)}{\sim} Y$, then $\exists Z \in G(\infty)$: $X = ZYZ^{-1}$,
$$|Z| \leq \left(\tfrac{1}{2} + \alpha\right)(|X| + |Y|)$$

Lem 22.2 | 4) if A, B, C are nontrivial elements from $G(\infty)$, $X \in G(\infty)$ such that $X^{-1}AXB \sim C$, then the double coset $\langle A \rangle X \langle B \rangle$ contains an element X' of length
$$|X'| < \left(\tfrac{1}{2} + \alpha\right)(|A| + |B| + |C|) + \left\lfloor \tfrac{1}{2}|A| \right\rfloor + \left\lfloor \tfrac{1}{2}|B| \right\rfloor$$

Lem 23.16 | 5) if a word $X \underset{G(\infty)}{=} 1$, then $X = 1$ in the group
$$\langle \mathcal{A} \mid \{R \in \mathcal{R} \mid |R| < (1-\alpha)^{-1} \cdot |X|\} \rangle$$

Lem 25.14 | 6) if $[X, ZXZ^{-1}] \underset{G(\infty)}{=} 1$, then $[X, Z] \underset{G(\infty)}{=} 1$

Geometry for 4):



Useful to read from the book of Ol'shanski

1) Pages 218–226 (B-maps) from Chapter 7.

2) §25 from Chapter 8

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1)

③ Construction of H

We fix $n \gg d \gg h \gg 1$
$\qquad\qquad\qquad\qquad\uparrow$ even number.

Generators $\mathcal{A} = \{a, b, c_1, \dots, c_h\}$.

We set $\mathcal{R}_0 = \mathcal{R}_1 = \mathcal{R}_2 = \emptyset$ and define $G(i) = \langle \mathcal{A} \mid \mathcal{R}_i \rangle$, $i \geq 2$ assuming the presentation $G(i-1) = \langle \mathcal{A} \mid \mathcal{R}_{i-1} \rangle$ is defined.

- Definition of periods $X_i \subseteq F(\mathcal{A})$.

  This is a maximal subset satisfying:

  1) $W \in X_i \Rightarrow |W| = i$

  2) $W \underset{G(i-1)}{\nsim} W'$ if $W \in X_i$ and $|W'| < |W|$

  3) $W_1 \neq W_2 \in X_i \Rightarrow W_1 \underset{G(i-1)}{\nsim} W_2^{\pm 1}$

  4) $W \in X_i \Rightarrow W = ab \cdot (c_1 c_2 \cdots c_h)^{\ell}$ in $G(i-1)/[G(i-1), G(i-1)]$ for some $\ell \in \mathbb{Z}$.

- Definition of $\mathcal{R}_i = \mathcal{R}_{i-1} \cup \mathcal{T}_i$:

  $\mathcal{T}_i$ is a maximal set of pairwise nonconjugate in $G(i-1)$ words from $F(\mathcal{A})$ of the form

  $$R = \prod_{j=1}^{h} \left(T_j A^{(-1)^j n} \right), \quad \text{where } A \in X_i \text{ and}$$

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$T_1, \dots, T_h$ are words from $F(\mathcal{A})$ s.t. $\forall j = 1, \dots, h$:

1) $|T_j| < d_i$ (small)

2) $T_j \underset{G(i-1)}{\neq} \mathcal{U}$ if $|\mathcal{U}| < |T_j|$ (minimal)

3) $\exists\, Z \in F(\mathcal{A})$: $(T_1, \dots, T_h) \underset{G(i-1)}{\sim} Z^{-1}(c_1, \dots, c_h) Z$.

Lem. 2 $H = G(\infty) = \langle \mathcal{A} \mid \mathcal{R} \rangle$, where $\mathcal{R} = \bigcup_{i=0}^{\infty} \mathcal{R}_i$ satisfies R.

Proof Mod $[F(\mathcal{A}), F(\mathcal{A})]$ we have

- $R = c_1 c_2 \cdots c_h$
- $T_j = c_j$, $j = 1, \dots, h$
- $\underset{X_i}{\overset{\Uparrow}{A}} = ab(c_1 c_2 \cdots c_h)^{\ell}$ for some ℓ
- $G(i)/[G(i), G(i)] = \langle a, b, c_1, \dots, c_h \mid c_1 c_2 \cdots c_h = 1 \rangle$, $i > 2$

  $= G(\infty)/[G(\infty), G(\infty)]$

$\Rightarrow$ $\underset{X_i}{\overset{\Uparrow}{A}}$ is not a proper power in $G(i-1)$ and even in $G(i-1)/[G(i-1), G(i-1)]$

$\Rightarrow$ Properties 1), 2), R1 – R4 from the definition of condition R are satisfied.

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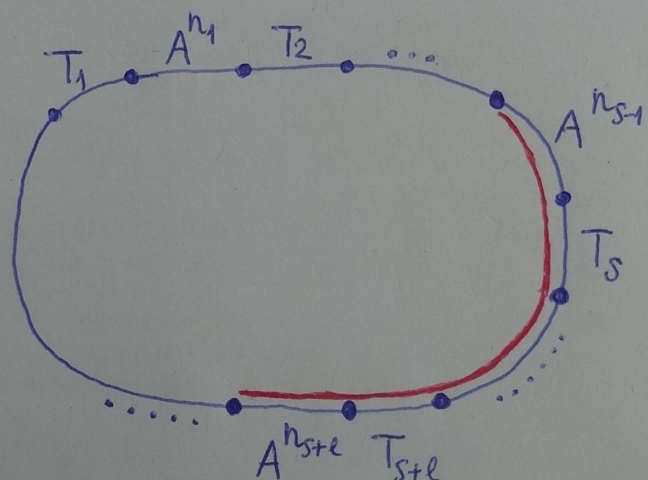
Verification of R5.

<u>R5</u> The word R is not a proper power in the free group and any <u>long</u> subword of the cyclic word R occurs in R <u>only once</u>:



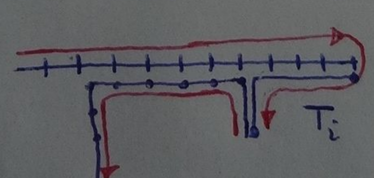
$\ell > \alpha^{-1} - 4$
pairs of syllables.

Attention: There may be cancellations in

$$R = T_1 A^{n_1} T_2 A^{n_2} \cdots T_h A^{n_h}$$

But these cancellations are small:

In each triple $A^{n_{i-1}} T_i A^{n_i}$ if T_i cancells completely, then only a couple (or few) A's cancel further.



<u>Use</u>: long overlapping A-words (powers of A) must ~~match~~ be consistent.

$\Downarrow$

$T_i \in \langle A \rangle \leftarrow$ forbidden by R4.

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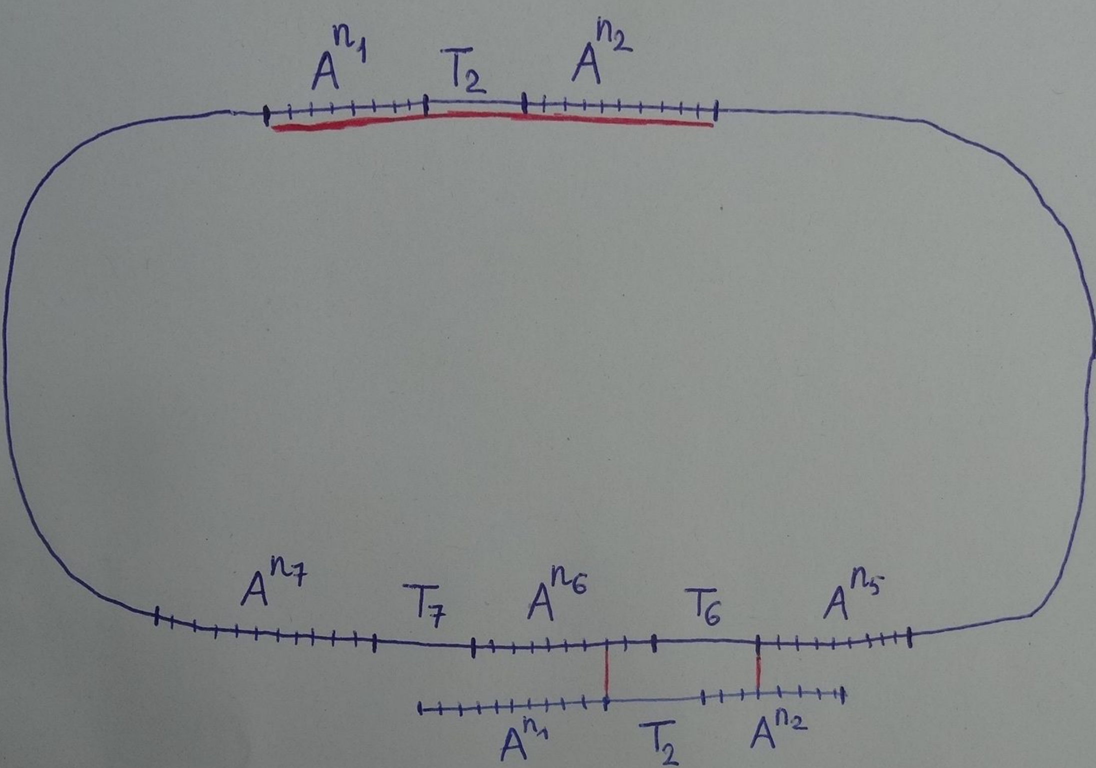
Continue to verify R5



Because of matching, we have $A^s T_2 = T_6 A^t$ for some s, t.

[redacted]

Consider this equality mod $[F(\mathcal{A}), F(\mathcal{A})]$:

$$(ab(c_1c_2\cdots c_k)^{\ell})^{s} \cdot c_2 = c_6 \cdot (ab(c_1c_2\cdots c_k)^{\ell})^{t} \quad \leftarrow \text{impossible.}$$

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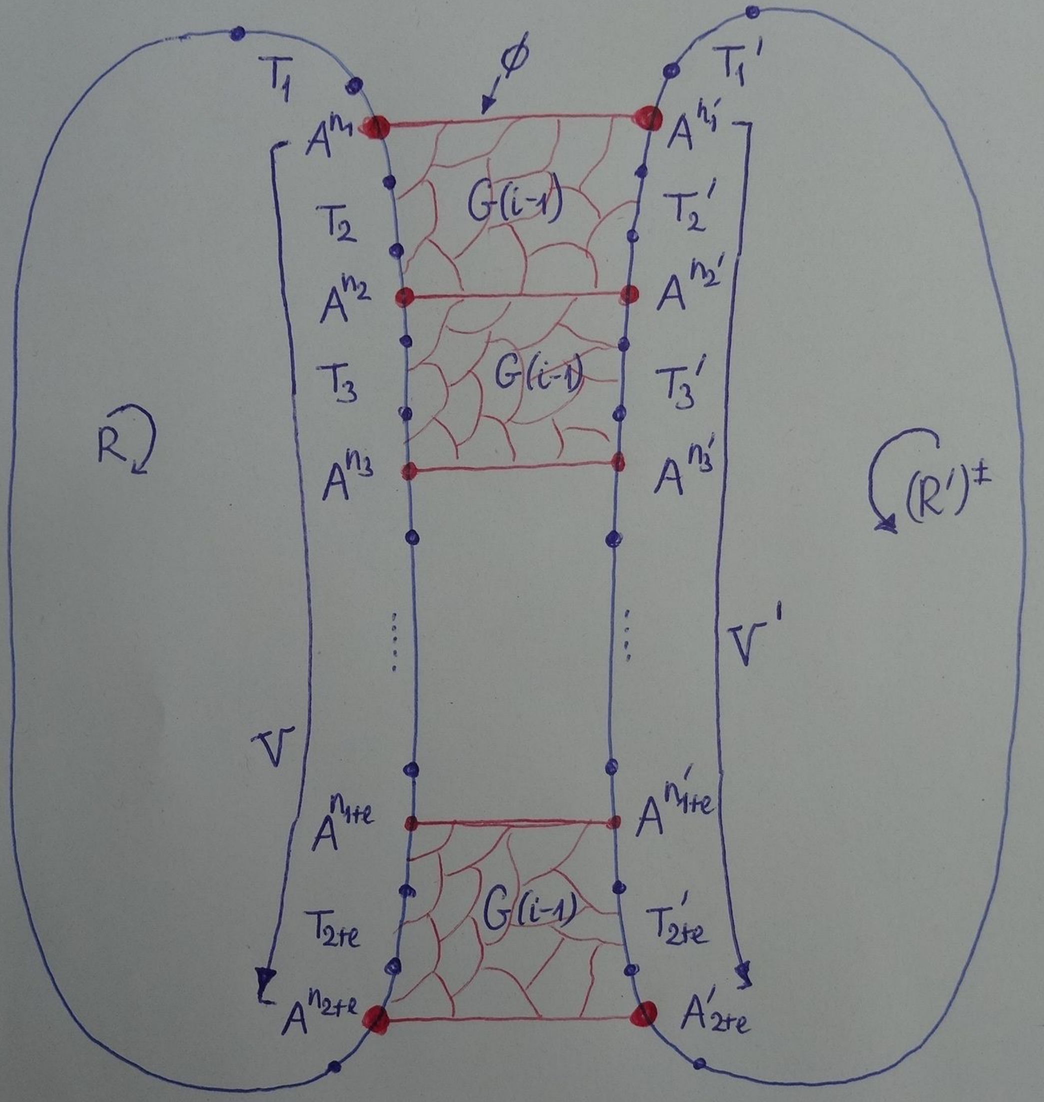
Verification of R6



Formulation of R6: Suppose $\ell \geq \alpha^{-1} - 2$, $\quad \mathrm{sign}(n_i) = \mathrm{sign}(n_i')$, $i = 1, \cdots, 2+\ell$

and there exist a subdivision ~~by red~~ of V and V' by red points such that the vertical sides of each of $\ell+1$ recktangles are equal in $G(i-1)$

Then $R \equiv R'$, $V \equiv V'$ and V is not a subword of the cyclic word R^{-1}.

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6)

We have equations of kind $A^{\alpha} T_j A^{\beta} \underset{G(i-1)}{=} A^{\alpha'} T_{j'}^{\pm 1} A^{\beta'}$ (0)

Rewrite: $T_j A^p \underset{G(i-1)}{=} A^s T_{j'}^{\pm 1}$ (1)

Consider mod $[G(i-1), G(i-1)]$ $\Rightarrow$ $j = j'$, $p = s$, $\pm 1 = 1$

Recal that by def. of T_j, T_j', there exist $Z, Z' \in F(\mathcal{A})$:

$$T_j \underset{G(i-1)}{=} Z c_j Z^{-1}, \quad T_j' \underset{G(i-1)}{=} Z' c_j (Z')^{-1}. \qquad (*)$$

Substitute in (1):

$$Z c_j Z^{-1} A^p \underset{G(i-1)}{=} A^p Z' c_j (Z')^{-1}$$

Rewrite: $c_j \underset{G(i-1)}{\rightleftarrows} Z^{-1} A^p Z'$ (2)

By induction, $G(i-1)$ satisfies condition R, in particular commuting elements of $G(i-1)$ must lie in the same cyclic subgroup (Lem. 1). But c_j is not a proper power in $G(i-1)$ (again mod $[G(i-1), G(i-1)]$).

$\Rightarrow$ $\quad Z^{-1} A^p Z' \underset{G(i-1)}{=} c_j^{k}$ for some k (3)

Consider $j+1, j+2, \ldots$ instead of j and analogously (by ass. of R_6 we have many equalities of kind (0)).

$$Z^{-1} A^{p_1} Z' \underset{G(i-1)}{=} c_{j_1}^{k_1}. \qquad (4)$$

(3)-(4) $Z^{-1} A^{p-p_1} Z \underset{G(i-1)}{=} c_j^{k} c_{j_1}^{-k_1}$. Again mod $[G(i-1), G(i-1)]$ (5)

gives $p = p_1$, $k = k_1 = 0$. Then (3) implies $Z' \underset{G(i-1)}{=} A^{-p} Z$. Then, by $(*)$

[redacted] $T_j' \underset{G(i-1)}{=} A^{-p} T_j A^p \Rightarrow R \underset{G(i-1)}{\sim} R'. \Rightarrow R = R'$ (see def $\mathcal{T}_i$)

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