Actions of groups on hyperbolic spaces

We use [1, 2].

1 Boundaries of metric spaces

Let (X, d) be a metric space. We fix a point $x \in X$.

Definition 1.1 The *Gromov product* of two points $y, z \in X$ with respect to x is

$$(y, z)_x = \frac{1}{2}(d(x, y) + d(x, z) - d(y, z)).$$

In the following we write (x_n) instead of $(x_n)_{n \in N}$.

Definition 1.2

(a) We say that a sequence (x_n) of points in X *converges to infinity* if

$$\liminf_{i,j \to \infty} (x_i, x_j)_x = \infty.$$

(b) We write $(x_n) \sim (y_n)$ if

$$\liminf_{i,j \to \infty} (x_i, y_j)_x = \infty.$$

(c) The (sequential) *boundary* of X is defined as follows:

$$\partial X := \{[(x_n)] \,|\, (x_n) \text{ is a sequence converging to } \infty \text{ in } X\}.$$

(c) Topology on ∂X: For $p \in \partial X$ and $r \in \mathbb{R}_+$ we define

$$V(p, r) := \{q \in \partial X \quad | \text{ for some sequences } (x_n), (y_n) \text{ with } [(x_n)] = p, [(y_n)] = q \text{ we have } \liminf_{i,j \to \infty} (x_i, y_j)_x \geqslant r\}.$$

We take all such sets as a basis of open sets for the topology on ∂X.

d) Topology on $X \cup \partial X$: For $p \in X \cup \partial X$ and $r \in \mathbb{R}_+$ we define a set $U(p, r)$ as follows:
If $p \in X$, then $U(p, r)$ is the ball of radius r in X.
If $p \in \partial X$, then $U(p, r)$ is the union of $V(p, r)$ and the set

$$\{y \in \partial X \,|\, \text{for some sequence } (x_n) \text{ with } [(x_n)] = p, \text{ we have } \liminf_{i,j \to \infty} (x_i, y)_x \geqslant r\}.$$

A metric space is called *proper* if every closed, bounded subspace is compact.

Proposition 1.3 Let (X, d) be a proper δ-hyperbolic metric space. Then the topological spaces ∂X and $X \cup \partial X$ are compact.

2 Limit set of a group

Let G be a group acting by isometries on a δ-hyperbolic metric space (X, d); we write $G \curvearrowright X$. Then $G \curvearrowright \partial X$. Let x be an arbitrary point of X. The *limit set* $\Lambda G \subseteq \partial X$ of G is

$$\Lambda G := \{p \in \partial X \,|\, p = \lim_{n\to\infty} g_n x \text{ for some sequence } g_n \in G\}.$$

The limit set does not depend on the choice of x.

Proposition 2.1 G be a group acting by isometries on a δ-hyperbolic metric space (X, d). Then

1) ΛG is a minimal closed G-invariant subset of ∂X.
2) Suppose that G acts on X properly discontinuously[1]. Then G acts properly discontinuously on $\partial X \setminus \Lambda G$ and on $(X \cup \partial X) \setminus \Lambda G$.

3 Classification of elements and limit sets

Let G be a group acting by isometries on a δ-hyperbolic metric space (X, d); An element $g \in G$ is called *elliptic* if some (equivalently any) orbit of $\langle g \rangle$ is bounded. An element $g \in G$ is called *loxodromic* if for some (for any) $x \in X$ the map $\mathbb{Z} \to X$, $n \mapsto g^n x$ is a quasi-isometry, i.e. there exist constants $A, B > 0$ such that for any $n, m \in \mathbb{N}$ we have

$$|n - m|A - B \leqslant d(g^n x, g^m x) \leqslant |n - m|A + B.$$

Proposition 3.1 Let G be a group acting by isometries on a δ-hyperbolic metric space (X, d). Every loxodromic element of G has exactly 2 limit points $g^{\pm\infty}$ on ∂X.

Definition 3.2 Loxodromic elements g, h are called *independent* if the sets $\{g^{\pm\infty}\}$ and $\{h^{\pm\infty}\}$ are disjoint.

Satz 3.3 Let G be a group acting by isometries on a δ-hyperbolic metric space (X, d). Then one of the following holds:

1) $|\Lambda G| = 0$. Equivalently, G has bounded orbits. In this case G is called *elliptic*.

2) $|\Lambda G| = 1$. Equivalently, G has bounded orbits and contains no loxodromic elements. In this case G is called *parabolic*.

3) $|\Lambda G| = 2$. Equivalently, G contains a loxodromic element and any two loxodromic elements have the same limit points on ∂X.

4) $|\Lambda G| = \infty$. Then G contains loxodromic elements. In turn, this case breaks into two subcases.

   (a) Any two loxodromic elements of G have a common limit point on the boundary. In this case G is called quasi-parabolic.

   (b) G contains infinitely many independent loxodromic elements.

[1]For every $x \in X$ and $r > 0$ the set $\{g \in G \,|\, (gx, x) \leqslant r\}$ is finite.

Literatur

[1] N. Benakli, I. Kapovich, *Boundaries of hyperbolic groups*, Combinatorial and geometric group theory, Contemporary Mathematics, 296 (2002), pp. 3993.

[2] D. Osin, *Acylindrically hyperbolic groups*, 2015. Available in ArXiv.