A proof of the Hanna-Neumann-Conjecture after Igor Mineyev and Warren Dicks

Hanna-Neumann-Conjecture (HNC) (1957) Let G be a free group, $A, B \leq G$ finitely generated subgroups. Then

$$\bar{r}(A \cap B) \leq \bar{r}(A) \cdot \bar{r}(B)$$

where $\bar{r}(G) = \max(\operatorname{rank}(G) - 1, 0)$ for any free group G.

[Hanna Neumann prooved $\bar{r}(A \cap B) \leq 2 \cdot \bar{r}(A) \cdot \bar{r}(B)$ in 1957]

Strengthened-Hanna-Neumann-Conjecture (SHNC) (1990, Walter Neumann) Let G be a free group, $A, B \leq G$ fin. gen. subgroups. Let $S \subseteq G$ be a subset s.t. $\exists$ bijection $S \longrightarrow A \backslash G / B$. Then

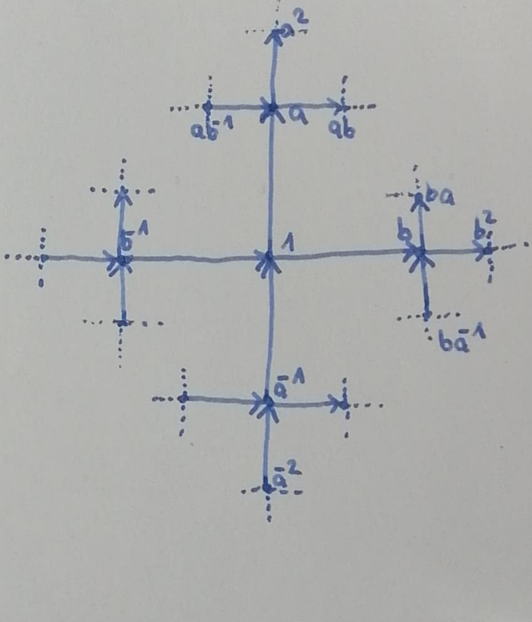
$$\sum_{s \in S} \bar{r}(\underbrace{A^s}_{= s^{-1}As} \cap B) \leq \bar{r}(A) \cdot \bar{r}(B)$$

Some Notation:

- Let $F = \langle a, b \mid \rangle$ denote the free group of rank 2.

  We assume that F is ordered by $<$ (total order and for $g, g_1, g_2 \in F$ with $g_1 < g_2$ $\Rightarrow gg_1 < gg_2$)

- Let Γ denote the Cayley-Graph of F:

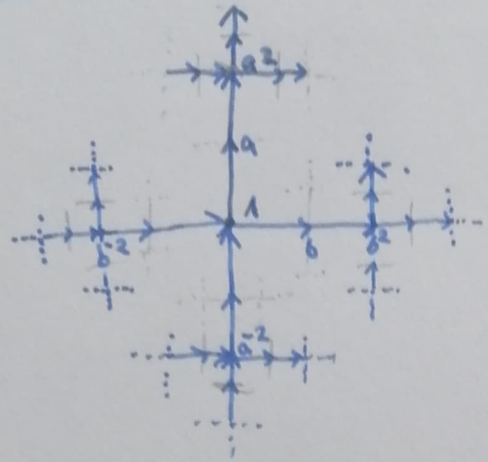


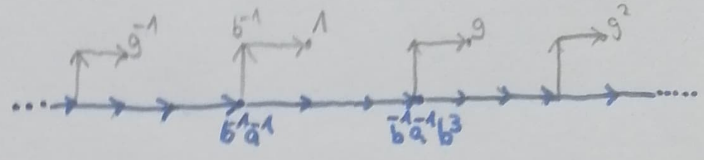
Important:

- F acts on this tree by Leftmultiplication (without inversion of edges)
- $V\Gamma$ (set of vertices of Γ) is ordered by $<$ [$V\Gamma = F$]
- $E\Gamma$ (set of edges of Γ) is ordered by $<$ [$E\Gamma \subseteq F \times F$, use $<$ lexicographic]

• For $G \leq_f F$ denote by $\Gamma(G) = \cap\{T \subseteq \Gamma \mid G \cdot T = T\}$ the minimal G-invariant subtree of Γ

examples: $G_1 = \langle a^2, b^2 \rangle$ $\qquad$ $G_2 = \langle \underbrace{b^{-1}a^{-1}b^3ab}_{=:g} \rangle$





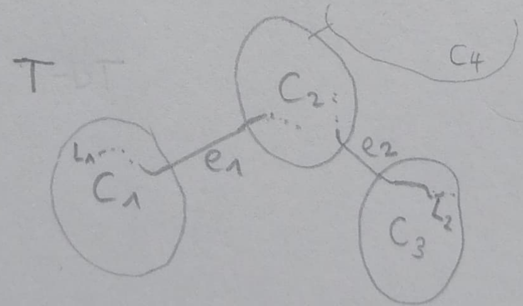
remark: if G is cyclic then $\Gamma(G)$ is a line
(a subtree where every vertex has exactly 2 incident edges)

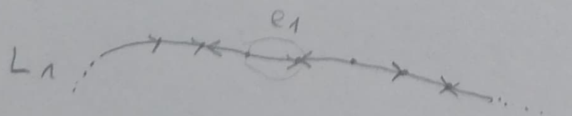
Important fact: If G is finitely generated ($\mathrm{rank}(G) < \infty$) then $|G \backslash E\Gamma(G)| < \infty$
(number of G-Orbits on edge-set is also finite)

New definitions (Let $T \subseteq \Gamma$ be a subtree)

• A bridge of T is an edge $e \in ET$ s.t. $\exists$ a line $L \subseteq T$ with $\max(EL) = e$ ($\forall d \in EL - \{e\} : d < e$). Set of bridges: BT

• An island of T is a connected component of $T - BT$. Set of islands: IT





$\forall d \in EL_1, d \neq e \quad e_1 \succ d$

The island-theorem

Let $G \leq F$ be a subgroup, $G \neq \{1\}$ and $C \in I\Gamma(G)$ an island of $\Gamma(G)$.

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) holds for

(a) $\operatorname{rank}(G) < \infty$

(b) $St_G(C) = \{g \in G \mid g \cdot C = C\} \neq \{1\}$

(the stabilizer of C in G is <u>non-trivial</u>)

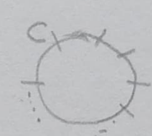
(c) $\operatorname{rank}(St_G(C)) = 1$

(the stabilizer of C in G is <u>cyclic</u>)

<u>proof</u> <u>(a) $\Rightarrow$ (b)</u>:

$\operatorname{rank}(G) < \infty \Rightarrow |G \backslash E\Gamma(G)| < \infty$. (*)

define $\delta C := \{e \in B\Gamma(G) \mid e \text{ is } \underline{\text{incident}} \text{ to } C\}$

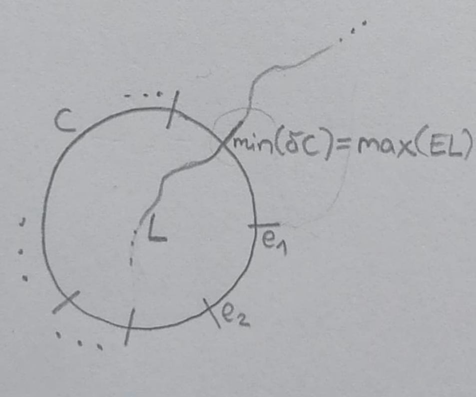


<u>Case 1</u>: $|\delta C| = 0. \Rightarrow C = \Gamma(G)$ and $St_G(C) = G \neq \{1\}$.

<u>Case 2:</u> $0 < |\delta C| < \infty$. Then $\exists \min(\delta C)$ $(\forall e \in \delta C - \{\min(\delta C)\} : e > \min(\delta C))$

$\exists$ Line $L \subseteq \Gamma(G)$: $\max(EL) = \min(\delta C)$.

We have $|EL \cap \delta C| = 1$



$\Rightarrow |EC| = \infty$

because one (infinite) component of $L - \{\min(\delta C)\}$ lies in C

with (*) $|G \backslash EC| < \infty$.

$\exists d, e \in EC$ with $d \neq e$ but $G \cdot d = G \cdot e$.

island!

$\Rightarrow \exists g \in G : g \cdot d = e \quad \Rightarrow e \in EC \cap E(g \cdot C) \quad \Rightarrow C = g \cdot C$

(because $<$ is preserved under Leftmult. bridges stay bridges, islands stay islands under Leftmult.)

$g \neq 1$ (because $e \neq d$) and $g \in St_G(C) \neq \{1\}$.

Case 3: $|\delta C| = \infty$. For $e \in \delta C$ either $\alpha(e) \in VC$ or $\varepsilon(e) \in VC$

beginning of e — end of e

denote this vertex by v_e for $e \in \delta C$. $\Rightarrow |VC| = \infty$

because $|G \setminus E\Gamma(G)| < \infty$, $|G \setminus \{v_e \mid e \in \delta C\}|$

$\exists d, e \in \delta C$ with $v_e \neq v_d$ but $G \cdot v_e = G \cdot v_d$

$\exists g \in G : g \cdot v_d = v_e \Rightarrow v_e \in VC \cap V(g \cdot C) \Rightarrow g \cdot C = C$, $g \in St_G(C) \neq \{1\}$

(b) $\Rightarrow$ (c): We assume $St_G(C) \neq \{1\}$. (remember (c) $\operatorname{rank}(St_G(C)) = 1$)

Choose $v \in VC$ and $\overset{*_1}{h} \in St_G(C)$ s.t. $[v, h \cdot v]$ minimal: (and $h \neq 1$)

$\nexists u \in VC, \underset{*_1}{\tilde{h}} \in St_G(C) : [u, \tilde{h}u] \subsetneq [v, h \cdot v]$

not minimal!

define the line $L := \bigcup_{i \in \mathbb{Z}} [h^i v, h^{i+1} v] \subseteq C$ (it is a line because of the minimality of $[v, hv]$)

[We have $\Gamma(\langle h \rangle) = L$ by definition of $\Gamma(\langle h \rangle)$]

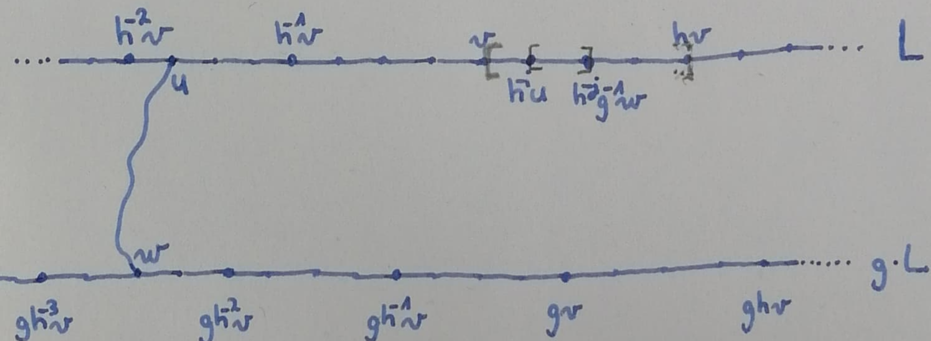
We want to show that $St_G(C) = \langle h \rangle$.

Let $g \in St_G(C)$. Then $g \cdot L \subseteq C$ aswell.

Choose $u \in VL$, $w \in V(g \cdot L)$ s.t. $[u, w]$ is $\subseteq$-minimal.

Then $\exists\, i, j \in \mathbb{Z}$: $h^i u, h^j (g^{-1} w) \in V[v, hv] - \{hv\}$

example



$i = -2$

$j = -3$

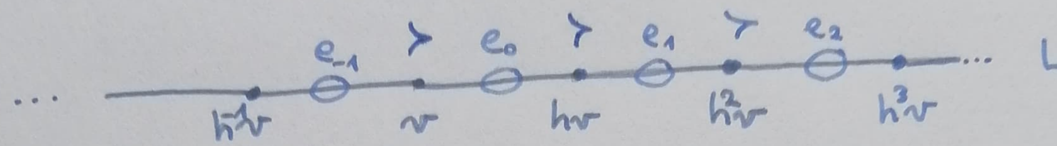
Case 1: $u = w$

then $[\underbrace{h^{-i}w}_{\in VC}, \underbrace{(h^{-j}g^{-1}h^{i})}_{\in St_G(C)}\underbrace{h^{-i}w}] = [h^{-i}u, h^{-j}g^{-1}w] \subsetneq [v, hv]$

$\Rightarrow h^{-j}g^{-1}h^{i} = 1 \Leftrightarrow g = h^{i-j} \in \langle h \rangle$

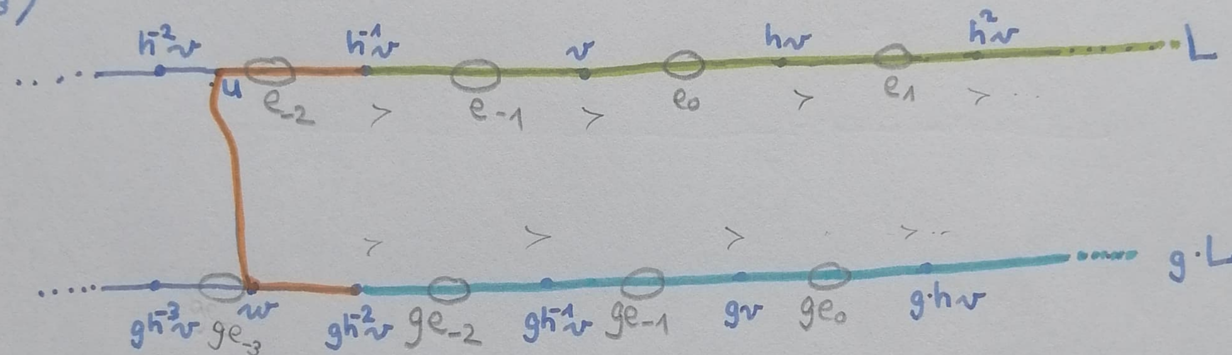
Case 2: $u \neq w$ ($\rightsquigarrow$ L and $g \cdot L$ are disjunct). define: $e_n := \max(E[h^n v, h^{n+1} v])$ (for $n \in \mathbb{Z}$)

without loss of generality: $e_0 < e_{-1}$



[we also have $g \cdot e_{-1} > g \cdot e_0 > g \cdot e_1 > \cdots$]

$\binom{i=-2}{j=-3}$



Define
$$\begin{cases} M_1 := \bigcup_{n=i+1}^{\infty} [h^n v, h^{n+1} v] \\ M_2 := [h^{i+1} v, g h^{j+1} v] \\ M_3 := \bigcup_{n=j+1}^{\infty} [g h^n v, g h^{n+1} v] \end{cases}$$
and $M := M_1 \cup M_2 \cup M_3$

Then M is a line and $M \subseteq C$; with a maximal element ↯ ($B\Gamma(G) \cap I\Gamma(G) = \emptyset$), a contradiction.

$\square$