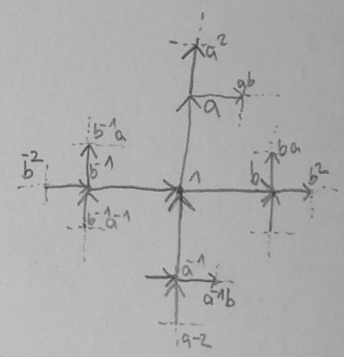
Last time:

- $\mathcal{F} = \langle a, b \rangle$ free group of rank 2, ordered by $\prec$
  [for $g, g_1, g_2 \in \mathcal{F}$ with $g_1 \prec g_2 \Rightarrow gg_1 \prec gg_2$]

- Γ Cayley-graph of $\mathcal{F}$

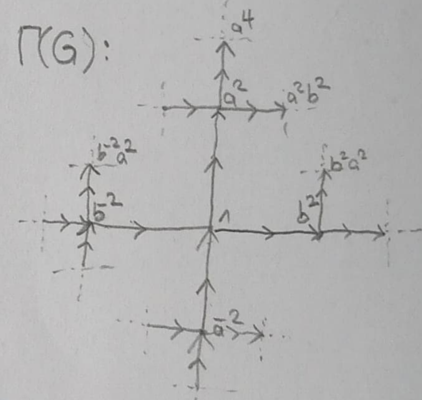


  $V\Gamma$ vertex set, $E\Gamma$ edge set: both ordered by $\prec$ aswell

  $\mathcal{F}$ acts on Γ without inversion of edges by Leftmultiplication

- for $G \leq \mathcal{F}$ denote $\Gamma(G) := \bigcap \{ T \subseteq \Gamma \mid G \cdot T = T \}$
  the smallest G-invariant subtree

$G = \langle a^2, b^2 \rangle$, $\Gamma(G)$:



- <u>fact</u>: if $\mathrm{rank}(G) < \infty$, then $|G \backslash E\Gamma(G)| < \infty$

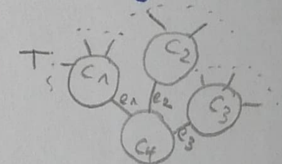
Let $T \subseteq \Gamma$ be a subtree.

<u>def</u> An edge $e \in ET$ is called <u>bridge</u> (of T) if there exists a <u>line</u> $L \subseteq T$ s.t. $\max(EL) = e$ $(\forall d \in EL - \{e\} : d \prec e)$

↳ subtree where every vertex has exactly 2 incident edges

<u>def</u> An <u>island</u> of T is a connected component of $T - BT$ (set of bridges of T)

(set of islands: IT)



<u>Island-theorem</u>

Let $G \leq \mathcal{F}$ be a subgroup, $G \neq \{1\}$ and $C \in I\Gamma(G)$ an island.
Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) for:

(a) $\mathrm{rank}(G) < \infty$

(b) $\mathrm{St}_G(C) \neq \{1\}$ (stabilizer of C in G is non-trivial)
$\mathrm{St}_G(C) = \{ g \in G \mid g \cdot C = C \}$

(c) $\mathrm{rank}(\mathrm{St}_G(C)) = 1$ (stabilizer is cyclic)

The bridge-theorem

Let $G \leq F$ be a subgroup. Then $|G \backslash B\Gamma(G)| = \bar{r}(G) = \max\{\mathrm{rank}(G)-1, 0\}$

proof Case 1: $G = \{1\}$. Then $\Gamma(G) = \emptyset$ and $|G\backslash B\Gamma(G)| = 0 = \bar{r}(G)$

Case 2: $G \neq \{1\}$ and $|G\backslash B\Gamma(G)| = \infty$.

Then $\mathrm{rank}(G) = \bar{r}(G) = \infty$.

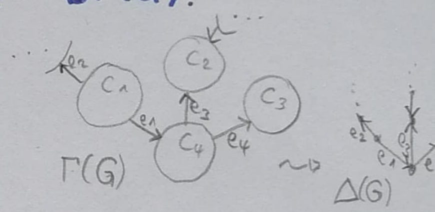
[because $\mathrm{rank}(G) < \infty \Rightarrow |G\backslash B\Gamma(G)| \leq |G\backslash E\Gamma(G)| < \infty$]

Case 3: $G \neq \{1\}$ and $|G\backslash B\Gamma(G)| < \infty$.

Define the tree $\Delta(G)$ as the graph with vertex set: $I\Gamma(G)$

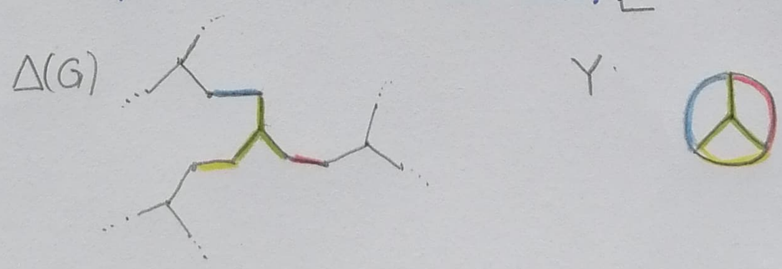
and edge set: $B\Gamma(G)$.

$\to$ G operates on $\Delta(G)$ without invertion of edges;



Then by Bass-Serre-theory G is isomorphic to the fundamental group of the following graph of groups $(\mathcal{G}, Y)$:

- $Y := G\backslash\Delta(G)$ (factorgraph) [$\Delta(G) \to Y$, $x \mapsto G\cdot x$ $(x \in V\Delta(G) \cup E\Delta(G))$] graph-morphism
- T maximal subtree of Y
- $(\tilde{Y}, \tilde{T})$ a lift of Y and T into $\Delta(G)$ [s.t. $\tilde{T}$ is a tree, for $e \in Y - T$ is $\alpha(\tilde{e}) \in \tilde{T}$ or $\varepsilon(\tilde{e}) \in \tilde{T}$]



(for every $x \in VY \cup EY$ denote $\tilde{x}$ the corresponding Lift in $(\tilde{Y}, \tilde{T})$)

- for every vertex or edge $x \in VY \cup EY$ define the corresponding group $G_x := St_G(\tilde{x})$ (stabilizer of $\tilde{x}$ in G)

- for $e \in EY$ $\quad G_e = \{1\}$ because $\tilde{e} \in B\Gamma(G) \subseteq E\Gamma(G) \subseteq E\Gamma$

  and if $g \in G$ with $g \cdot \tilde{e} = \tilde{e} \iff g = 1$

- for $v \in VY$ $\quad G_v = St_G(C)$ for some $C \in \Gamma(G)$

  ↳ by island theorem $\operatorname{rank}(G_v) \leq 1$

  [$\operatorname{rank}(G) < \infty \Rightarrow St_G(C) \neq \{1\} \Rightarrow St_G(C)$ cyclic]

- $|EY| = |G \backslash E\Delta(G)| = |G \backslash B\Gamma(G)| < \infty$ by assumption

$\to |VY| \leq |EY| + 1 < \infty$

- for $e \in EY$ denote by $\alpha_e : G_e \hookrightarrow G_{\alpha(e)}$ the corresponding epimorphism of $(\mathbb{G}, Y)$

fundamental group of $(\mathbb{G}, Y)$ with respect to T:

$$\pi_1(\mathbb{G}, Y, T) = \Big\langle \dot{\bigcup_{v \in VY}} G_v \,\dot{\cup}\, \{t_e \mid e \in EY - ET\} \;\Big|\; t_e = (t_{\bar{e}})^{-1},\; t_e^{-1} \alpha_e(g) t_e = (\alpha_{\bar{e}}(g))^{-1} \;\; (e \in EY - ET,\; g \in \underbrace{G_e}_{=\{1\}}) \Big\rangle$$

$$= \Big\langle \dot{\bigcup_{v \in VY}} G_v \,\dot{\cup}\, \{t_e \mid e \in EY - ET\} \;\Big|\; t_e = (t_{\bar{e}})^{-1} \; (e \in EY - ET) \Big\rangle$$

$$= \underbrace{\Big(\mathop{\ast}_{v \in VY} G_v \Big)}_{=: G_0} \ast \underbrace{\langle t_e \mid e \in EY^+ - ET^+ \rangle}_{=: G_1}$$

$\to G_0$ free of rank $\leq |VY|$, G_1 free of rank $|EY| - |ET| = |EY| - |VY| + 1$

[$\operatorname{rank}(G) < \infty \Rightarrow \operatorname{rank}(St_G(C)) = 1$]

and $G \cong G_0 \ast G_1$, $\Rightarrow \operatorname{rank}(G) < \infty$. $\leadsto$ by island-thm (a)$\Rightarrow$(c): $\operatorname{rank}(G_0) = |VY|$.

$F(G) = \operatorname{rank}(G_0) + \operatorname{rank}(G_1) - 1 = |VY| + |EY| - |VY| + 1 - 1 = |EY|$

$= |G \backslash B\Gamma(G)|$

$\square$

The Strengthened-Hanna-Neumann-Conjecture-Theorem

Let G be a free group, $H, K \leq G$ finitely generated subgroups.

Let $S \subseteq G$ be a subset, s.t. $\exists$ bijection $S \longrightarrow H \backslash G / K$. Then

$$\sum_{s \in S} \bar{r}(H^s \cap K) \leq \bar{r}(K) \cdot \bar{r}(H)$$

$(\bar{r}(G) = \max\{\operatorname{rank}(G) - 1, 0\})$

proof without loss of generality we assume G is a subgroup of the ordered group $F = \langle a, b \mid \rangle$ of rank 2.

Let $\operatorname{rank}(H), \operatorname{rank}(K) > 0$. Let $s \in S$. Then:

$$(*) \quad B\Gamma(H^s \cap K) \underset{\substack{\downarrow \\ \boxed{\begin{array}{l}\text{for } A \leq B \leq F \\ \Gamma(A) \subseteq \Gamma(B)\end{array}}}}{\subseteq} B\Gamma(H^s) \cap B\Gamma(K) \underset{\substack{\downarrow \\ \boxed{\begin{array}{l}(s^{-1}Hs)s^{-1}\Gamma(H) = s^{-1}H\Gamma(H) = s^{-1}\Gamma(H) \\ \Rightarrow \Gamma(H^s) \subseteq s^{-1}\Gamma(H)\end{array}}}}{\subseteq} s^{-1}B\Gamma(H) \cap B\Gamma(K)$$

Define the set $M := \bigcup_{s \in S} \left((H^s \cap K) \backslash B\Gamma(H^s \cap K) \right) := \bigcup_{s \in S} \{ (s, (H^s \cap K)e) \mid e \in B\Gamma(H^s \cap K) \}$

Def. the function $M \longrightarrow (H \backslash B\Gamma(H)) \times (K \backslash B\Gamma(K))$, $(s, (H^s \cap K)e) \longmapsto (Hse, Ke)$

well-defined because of $(*)$, and <u>injective</u>:

Let $s_1, s_2 \in S$, $e_1 \in B\Gamma(H^{s_1} \cap K)$, $e_2 \in B\Gamma(H^{s_2} \cap K)$, then are equivalent:

(1) $Hs_1e_1 = Hs_2e_2$ and $Ke_1 = Ke_2$

(2) $\exists h \in H, k \in K$: $hs_1e_1 = s_2e_2$ and $ke_1 = e_2$

(3) $\exists h \in H, k \in K$: $hs_1k^{-1} = s_2$ and $ke_1 = e_2$

[(2)$\Leftrightarrow$(3): $ke_1 = e_2 \Leftrightarrow k^{-1} = e_1e_2^{-1}$, and $hs_1e_1 = s_2e_2 \Leftrightarrow hs_1e_1e_2^{-1} = s_2$, so $hs_1e_1 = s_2e_2 \Leftrightarrow hs_1k^{-1} = s_2$]

(4) $s_1 = s_2$ and $\exists h \in H, k \in K$: $h^{s_1} = k$ and $ke_1 = e_2$

[(3)$\Leftrightarrow$(4): $hs_1k^{-1} = s_2 \Rightarrow s_1 = s_2$ (def. of S!)

$\rightarrow$ same h and k in (3), (4)

and $hs_1k^{-1} = s_2 \Leftrightarrow s_2^{-1}hs_1 = k$ $= h^{s_1} = k$ (and $ke_1 = e_2$)]

(5) $s_1 = s_2$ and $(H^{s_1} \cap K)e_1 = (H^{s_2} \cap K)e_2$

[(4)$\Leftrightarrow$(5): $h^{s_1} = k \Leftrightarrow k \in H^{s_1} \cap K$]

Finally :

$$\sum_{s\in S} F(H^s \cap K) \overset{\text{bridge-thm.}}{=} \sum_{s\in S} |(H^s\cap K)\setminus B\Gamma(H^s\cap K)| \overset{\text{disjunct union}}{=} |M|$$

$$\overset{\text{injective function}}{\leq} |(H\setminus B\Gamma(H)) \times (K\setminus B\Gamma(K))| \overset{\text{bridge-thm.}}{=} F(H)\cdot F(K) \qquad \square$$

[remember: <u>bridge-thm.</u>: $G\leq F \Rightarrow |G\setminus B\Gamma(G)| = F(G)$

<u>def. of M</u>: $M := \bigcup_{s\in S} ((H^s\cap K)\setminus B\Gamma(H^s\cap K))$]