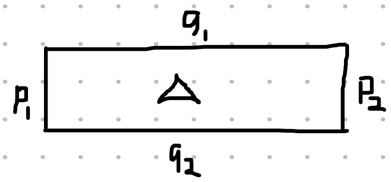
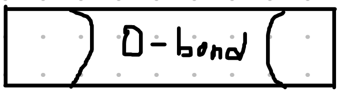
Lemma 17.2



Δ is an A-map, q_1, q_2 are smooth and $|q_1|+|q_2|>0$

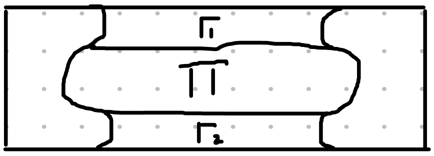
$|p_1|+|p_2|\le \gamma(|q_1|+|q_2|)$

Then either



there exists a 0-bond between q_1 and q_2

or



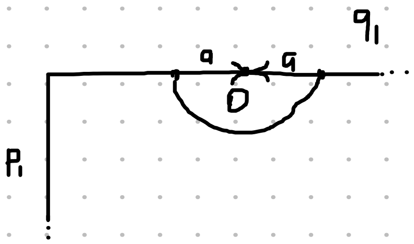
there exists an R-cell Π with

$$(\Pi,\Gamma_1,q_1)+(\Pi,\Gamma_2,q_2)>\bar\beta=1-\beta$$

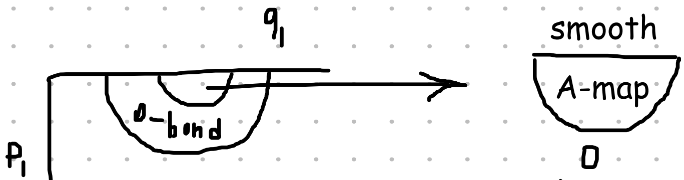
(a β-cell)

Proof

Induction by $|\Delta(2)|$.



impossible since q_1 is reduced

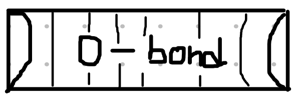


$$\dots\le \frac{1}{\bar\beta}\cdot 0$$

impossible by Theorem 17.1

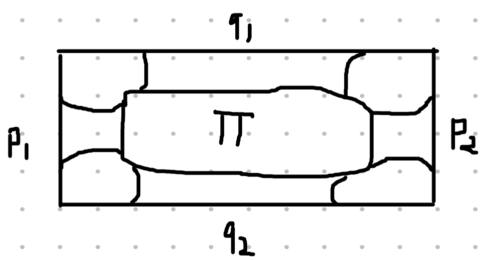
$\Longrightarrow$ no 0-bonds between q_1 and q_1 or q_2 and q_2.

If $|\Delta(2)| = 0$, then all 0-bonds join q_1 and q_2 $\Longrightarrow$



Now $|\Delta(2)| > 0$.

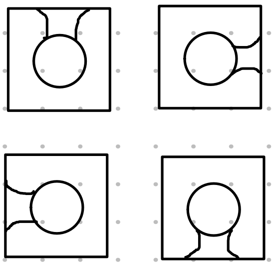
Corollary 16.1 gives us a γ-cell Π:



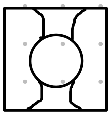
Π - R-cell

sum of contiguity degrees $>\bar\gamma=1-\gamma$

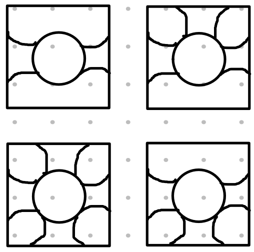
Since some cont. submaps might be missing, we have 2^4-1 cases:



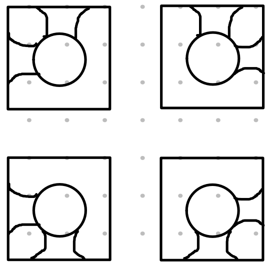
one submap



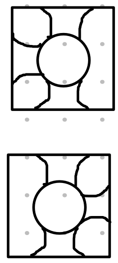
vertical



horizontal



corner

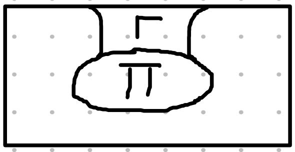


two corners

$$\bar\gamma = 1-\gamma$$
$$\bar\alpha = \tfrac{1}{2}+\alpha$$

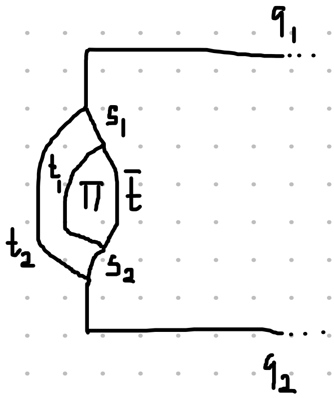
One submap

smooth



$$\bar\gamma < (\Pi,\Gamma,q) \leq \bar\alpha \Longrightarrow \tfrac{1}{2} < \alpha+\gamma$$

$\Longrightarrow$ contradiction

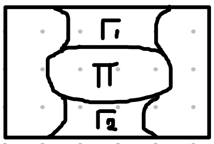


$$(\Pi,\Gamma,p_1) \overset{(\bar\gamma>\bar\alpha)}{\geq} \bar\alpha \overset{L\ 15.6}{\Longrightarrow} |s_1 \bar t s_2| < |t_2|$$

$\Longrightarrow$ cut out Π and Γ, use induction

($|\text{sides}| \leq \gamma(|q_1|+|q_2|)$ holds)

Vertical

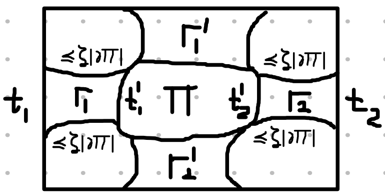


$$(\Pi,\Gamma_1,q_1)+(\Pi,\Gamma_2,q_2) > \bar\gamma \geq \bar\beta \Longrightarrow$$ Π is a β-cell

---

Horizontal

smooth



smooth

$(\Pi,\Gamma_1',q_1)+(\Pi,\Gamma_2',q_2) \leq \bar\beta$, otherwise Π is a β-cell

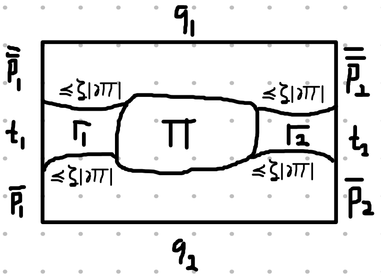
$$\Longrightarrow$$

$$\bar\gamma-\bar\beta < (\Pi,\Gamma_1,p_1)+(\Pi,\Gamma_2,p_2)$$

Theorem 17.1 applied to Γ_1 and Γ_2 gives:

$$|t_1|+|t_2|+4\zeta|\partial\Pi| \geq \bar\beta(|t_1'|+|t_2'|) > \bar\beta(\bar\gamma-\bar\beta)|\partial\Pi| \Longrightarrow$$

$$|t_1|+|t_2| > (\bar\beta(\beta-\bar\gamma)-4\zeta)|\partial\Pi|$$



Theorem 17.1 applied to upper and lower submaps gives:

$$\bar\beta(|q_1|+|q_2|) \le |\bar{\bar p}_1| + |\bar{\bar p}_2| + 2\zeta|\partial\Pi| + |\bar p_1| + |\bar p_2| + 2\zeta|\partial\Pi| + |\partial\Pi|$$

Hypothesis: $|p_1|+|p_2| \le \gamma(|q_1|+|q_2|)$

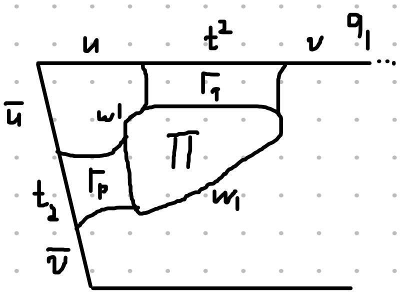
$$(\bar\beta(\beta-\gamma)-4\zeta)|\partial\Pi| + \underbrace{|\bar{\bar p}_1|+|\bar{\bar p}_2|+|\bar p_1|+|\bar p_2|}_{P} < |p_1|+|p_2| \le \gamma(|q_1|+|q_2|)$$

$$\Longrightarrow$$

$$(\bar\beta(\beta-\gamma)-4\zeta)|\partial\Pi| + P < \frac{\gamma}{\bar\beta}(P+(4\zeta+1)|\partial\Pi|)$$ - contradiction

---

Corner



$(\Pi, \Gamma_{q_1}, q_1) < \bar\alpha$ (q_1 is smooth) $\Longrightarrow$

$(\Pi, \Gamma_p, p_1) > \bar\gamma - \bar\alpha$ $\Longrightarrow$

By Lemma 15.4: $|t_2| > (\bar\gamma - \bar\alpha - 2\beta)|\partial\Pi|$

$|w'|+|w_1| \le \gamma|\partial\Pi|$ (from def. of a γ-cell)

$|\bar p_1| \le |\bar v| + (\gamma+2\zeta)|\partial\Pi|$ - estimate on the new left side

$$|p_1|-|\bar p_1| = |\bar v| + |t_2| + |\bar u| - |\bar p_1| > |\bar u| + (\bar\gamma-\bar\alpha-2\beta)|\partial\Pi| - (\gamma+2\zeta)|\partial\Pi| =$$
$$|\bar u| + \left(\frac{1}{2} - \alpha - 2\beta - 2\gamma - 2\zeta\right)|\partial\Pi|$$

New upper side is v. Estimate on the difference:

$$|q_1|-|v| = |ut^2| < \frac{1}{\beta}(|\bar u| + (2\zeta+\gamma+\bar\alpha)|\partial\Pi|) \Longrightarrow$$

$$\gamma(|q_1|-|v|) < \frac{\gamma}{\bar\beta}|\bar u| + \frac{\gamma}{\bar\beta}(2\zeta+\gamma+\bar\alpha)|\partial\Pi| \Longrightarrow$$

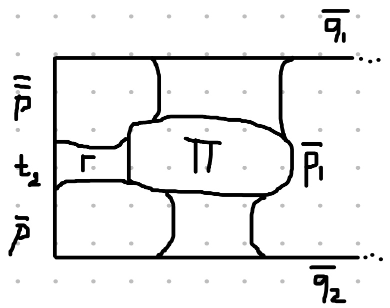
$$\gamma(|q_1|-|v|) < |p_1|-|\bar p_1|$$

Combining with hypothesis $|p_1|+|p_2| \le \gamma(|q_1|+|q_2|)$:

$$|\bar p_1|+|p_2| \le \gamma(|v|+|q_2|)$$

Now remove the corner and apply unduction.

Two corners



$(\Pi, \Gamma, p_1) > \bar\gamma - \bar\beta = \beta - \gamma$, otherwise Π is a β-cell

$|t_2| > \frac{\beta-\gamma}{1+2\beta}|\partial\Pi|$ (by Lemma 15.4, since $\beta-\gamma>\varepsilon$)

$\left(\frac{1}{1+2\beta} > 1-2\beta\right)$

$$|p_1| - |\bar p_1| > |\bar p| + |\bar{\bar p}| + \left((1-2\beta)(\beta-\gamma) - 2\zeta - \gamma\right)|\partial\Pi|$$

$$|q_1| + |q_2| - (|\bar q_1| + |\bar q_2|) \underset{\text{T 17.1}}{\leq} \frac{1}{\beta}\left((1+4\zeta)|\partial\Pi| + |\bar p| + |\bar{\bar p}|\right)$$

$$\Rightarrow$$

$$\gamma\left(|q_1| + |q_2| - (|\bar q_1| + |\bar q_2|)\right) \leq \frac{\gamma}{\beta}\left((1+4\zeta)|\partial\Pi| + |\bar p| + |\bar{\bar p}|\right)$$

$$\Rightarrow$$

$$\gamma\left(|q_1| + |q_2| - (|\bar q_1| + |\bar q_2|)\right) < |p_1| - |\bar p_1|$$

Remove the region on the left and use induction.