Lectures on limit groups

(Oleg Bogopolski)

This is a short introduction to limit groups based on papers [1] and [2]. The mathematics in this area is really nice, but might be difficult for non-experienced students. Therefore, I decided to write a self-contained text that could be helpful for them (the text will be extended).

1 Motivation

Example 1.1 Find all solutions of the following equations in the free group $F(a,b)$:

1) $[a,b][x_1,x_2]=1$,

2) $x_1^2x_2^2x_3^2=1$.

Let $\mathbb{F}$ be a free group and let $a_1,\dots,a_k$ be some elements of $\mathbb{F}$. An *equation* in $\mathbb{F}$ with constants $a_1,\dots,a_k$ and unknowns $x_1,\dots,x_n$ is an expression of kind

$$W(a_1,\dots,a_k,x_1,\dots,x_n)=1, \tag{1}$$

where W is a word in $k+n$ variables. A *solution* of this equation in $\mathbb{F}$ is a tuple of elements $(X_1,\dots,X_n)$ in $\mathbb{F}$ such that $W(a_1,\dots,a_k,X_1,\dots,X_n)=1$.

To describe all solutions, it is convenient to reformulate this problem in terms of homomorphisms. Let $F(x_1,\dots,x_n)$ be the free group with the basis $x_1,\dots,x_n$ and let

$$G:=\langle \mathbb{F}*F(x_1,\dots,x_n)\,|\,W(a_1,\dots,a_k,x_1,\dots,x_n)\rangle.$$

Proposition 1.2 There is a bijection between the set of solutions of the equation (1) in $\mathbb{F}$ and the set of homomorphisms $G\to\mathbb{F}$ sending a_i to a_i for $i=1,\dots,k$.

Proof. If $(X_1,\dots,X_n)$ is a solution of this equation, then there is a homomorphism $\phi:G\to\mathbb{F}$ sending a_i to a_i and x_j to X_j. Conversely, if $\phi:G\to\mathbb{F}$ is a homomorphism sending a_i to a_i, then $(\phi(x_1),\dots,\phi(x_n))$ is a solution of this equation. □

We formulate a weaker version of this proposition.

Proposition 1.3 There is a bijection between the set of solutions of the equation

$$W(x_1,\dots,x_n)=1$$

in $\mathbb{F}$ and the set of homomorphisms $G\to\mathbb{F}$, where

$$G:=\langle \mathbb{F}*F(x_1,\dots,x_n)\,|\,W(x_1,\dots,x_n)\rangle.$$

Therefore the following problem (formulated for an arbitrary f.g. group G) is important.
Problem. Given a finitely generated group G, describe the set $\mathrm{Hom}(G,\mathbb{F})$.

2 A description of $\mathrm{Hom}(G,\mathbb{F})$

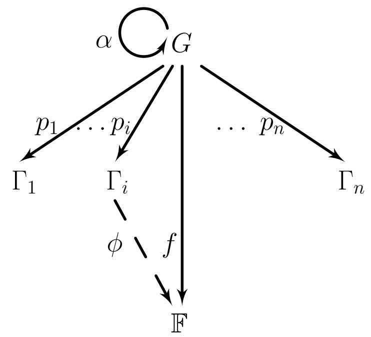
If $G=F(x_1,\dots,x_n)$ is a free group of rank n, then there is a natural bijection

$$\mathrm{Hom}\ (G,\mathbb{F})\to\underbrace{\mathbb{F}\times\dots,\times\mathbb{F}}_{n}.$$

Therefore in the following theorem we consider the case, where G is not free.

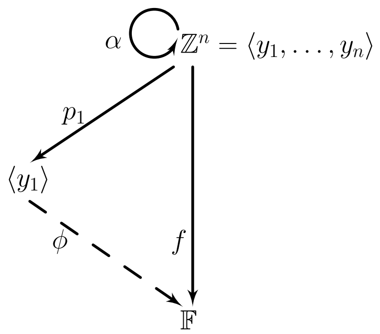
Theorem 2.1 ([Kh,M], [Se]) Let G be a finitely generated non-free group and $\mathbb{F}$ be a free group. There is a finite collection of epimorphisms $\{p_i : G \twoheadrightarrow \Gamma_i\}$ such that

- each Γ_i is a limit group[1] and each $\mathrm{Ker}\,(p_i)$ is nontrivial,
- for any homomorphism $f : G \to \mathbb{F}$, there exists $\alpha \in \mathrm{Aut}(G)$ such that $f \circ \alpha$ factors[2] through some p_i.



$$\forall f\, \exists \alpha\, \exists i\, \exists \phi : \quad f = \phi \circ p_i \circ \alpha$$

Example 2.2 Let $G = \mathbb{Z}^n = \langle y_1, \dots, y_n \rangle$. Let $p_1 : \mathbb{Z}^n \to \mathbb{Z}$ be the projection to the first coordinate. We will show that, for each homomorphism $f : \mathbb{Z}^n \to \mathbb{F}$, there exists $\alpha \in \mathrm{Aut}(\mathbb{Z}^n)$ such that $f \circ \alpha$ factors though p_1.



Since all $f(y_i)$ commute, they lie in the same maximal cyclic subgroup of $\mathbb{F}$.

Thus, there exists $a \in \mathbb{F}$ such that $f(y_i) = a^{\ell_i}$ for some ℓ_i. We write $\ell_i = dk_1$, where $d = g.c.d.(\ell_1, \dots, \ell_n)$ and $1 = g.c.d.(k_1, \dots, k_n)$.

- Define $\alpha : \mathbb{Z}^n \to \mathbb{Z}^n$ by the rule

$$\begin{cases} y_1 \to k_{11}y_1 + \cdots + k_{1n}y_n \\ \dots \\ y_n \to k_{n1}y_1 + \cdots + k_{nn}y_n, \end{cases}$$

  where $k_{11} = k_1, \dots, k_{n1} = k_n$ and the remaining k_{ij} are chosen so that the matrix (k_{st}) is invertible.

- Define $\phi : \mathbb{Z} = \langle y_1 \rangle \to \mathbb{F}$ by the rule $y_1 \mapsto a^d$. It is easy to check that $f = \phi \circ p_1 \circ \alpha$.

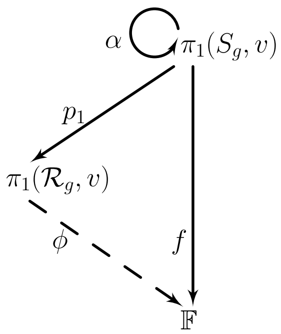
[1]Several definitions of limit groups will be given later.

[2]One says that a homomorphism $\varphi : G \to A$ *factors through* a homomorphism $\psi : G \to B$ if there exists a homomorphism $\theta : B \to A$ such that $\varphi = \theta \circ \phi$.

Example 2.3 Let S_g be a closed orientable surface of genus g. We consider S_g as a sphere with g handles. We fix a retraction $S_g \to \mathcal{R}_g$, where $\mathcal{R}_g$ is a rose with g petals embedded to S_g. Let $p_1 : \pi_1(S_g, v) \to \pi_1(\mathcal{R}_g, v)$ be the induced epimorphism, where v is the unique vertex of the rose. We can choose the objects so that the following will be valid:

$$\begin{aligned} \pi_1(S_g, v) &= \langle x_1, y_1, \dots, x_g, y_g \mid \prod_{i=1}^{g} [x_i, y_i] \rangle; \\ \pi_1(\mathcal{R}_g) &= F(x_1, \dots, x_g); \\ & p_1 : \pi_1(S_g, v) \to \pi_1(\mathcal{R}_g, v) \\ & \qquad x_i \mapsto x_i \\ & \qquad y_i \mapsto 1 \end{aligned}$$

Then for each homomorphism $f : \pi_1(S_g, v) \to \mathbb{F}$, there exists $\alpha \in \mathrm{Aut}(\pi_1(S_g, v))$ such that $f \circ \alpha$ factors though p_1.



Consider the case $g = 2$ in details:

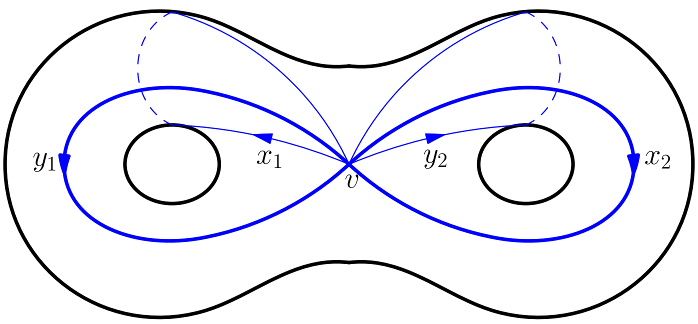


Figure 1. We use notation $[a, b] = a^{-1}b^{-1}ab$.

$$\begin{aligned} \pi_1(S_2, v) &= \langle x_1, y_1, x_2, y_2 \mid [x_1, y_1][x_2, y_2] \rangle; \\ \pi_1(\mathcal{R}_2, v) &= F(y_1, x_2); \\ & p_1 : \pi_1(S_2, v) \to \pi_1(\mathcal{R}_2, v) \\ & \qquad x_1 \mapsto 1 \\ & \qquad y_1 \mapsto y_1 \\ & \qquad x_2 \mapsto x_2 \\ & \qquad y_2 \mapsto 1 \end{aligned}$$

3 Limit groups (first definition)

Definition 3.1 Let G be a finitely generated group. A sequence of homomorphisms

$$(f_i : G \to \mathbb{F})_{i \in \mathbb{N}}$$

is called *stable* if, for all $g \in G$, the sequence of elements $\{f_i(g)\}$ is eventually always 1, or eventually never 1.

	g_1	g_2	g_3	g_4	g_5	...
f_1	$*$	$*$	$*$	$*$	$*$	...
f_2	$*$	$*$	$*$	$*$	$*$	...
f_3	$*$	$*$	$*$	$*$	$*$	...
f_4	$*$	$*$	$*$	$*$	$*$	...
f_5	$*$	$\neq 1$	$*$	1	$*$	...
f_6	1	$\neq 1$	$*$	1	$\neq 1$	...
f_7	1	$\neq 1$	1	1	$\neq 1$	...
f_8	1	$\neq 1$	1	1	$\neq 1$	...
f_9	1	$\neq 1$	1	1	$\neq 1$	...
...	...	...	...	...	...	...

$$\forall g \in G\ \exists n_0 \in \mathbb{N} : \ (\forall n \geqslant n_0\ f_n(g) = 1) \vee (\forall n \geqslant n_0\ f_n(g) \neq 1).$$

Definition 3.2 (stable kernels and limit groups)

1) The *stable kernel* of a stable sequence (f_i) of homomorphisms from $\mathrm{Hom}(G, \mathbb{F})$ is

$$\underset{\longrightarrow}{\mathrm{Ker}}(f_i) := \{g \in G \mid f_i(g) = 1 \text{ for almost all } i\}.$$

2) A finitely generated group Γ is called a *limit group* if there exists a finitely generated group G and a stable sequence (f_i) in $\mathrm{Hom}(G, \mathbb{F})$ such that $\Gamma \cong G/\underset{\longrightarrow}{\mathrm{Ker}}(f_i)$.

4 Residually free and ω-residually free groups

Definition 4.1 1) A finitely generated group G is *residually free* if, for any $1 \neq \gamma \in G$, there exists $f \in \mathrm{Hom}(G, \mathbb{F})$ such that $f(\gamma) \neq 1$.

2) A finitely generated group Γ is *ω-residually free* if for each $n \in \mathbb{N}$ and for each collection of n nontrivial elements $\gamma_1, \ldots, \gamma_n \in G$, there exists $f \in \mathrm{Hom}(G, \mathbb{F})$ such that $f(x_1), \ldots, f(x_n)$ are nontrivial.

We reformulate 2) in a compact way:

2$'$) A finitely generated group G is *ω-residually free* if, for every finite subset $X \subset G$, there exists $f \in \mathrm{Hom}(G, \mathbb{F})$ such that $f|_X$ is injective.

Example 4.2 1) Finitely generated free groups are ω-residually free.

2) Finitely generated free abelian groups are ω-residually free.
Indeed, let $A = \mathbb{Z}^k$ and let $a_1, \ldots, a_n \in A \setminus \{0\}$. Choose $b \in A$ so that all scalar products (a_i, b) are nonzero. Then the homomorphism $A \to \mathbb{Z}$, $a \mapsto (a, b)$ satisfies Definition 4.1 2).

Proposition 4.3 Every subgroup of an ω-residually free group is ω-residually free.

Definition 4.4 A group G is called *commutative transitive* if the following holds:

$$\forall x, y, z \in G \setminus \{1\} \quad ([x,y]=1 \wedge [y,z]=1) \;\Rightarrow\; ([x,z]=1).$$

Proposition 4.5 The following implications are valid:

$$\begin{array}{ccc} & \boxed{G \text{ is } \omega\text{-residually free}} & \\ \swArrow & & \seArrow \\ \boxed{G \text{ is residually free}} & & \boxed{G \text{ is commutative transitive}} \\ \Downarrow & & \Downarrow \\ \boxed{G \text{ is torsion free}} & & \boxed{\text{Every non-trivial abelian subgroup of } G \text{ is contained in a unique maximal abelian subgroup}} \end{array}$$

Example 4.6 The group $F_2 \times \mathbb{Z}$ is residually free, but not ω-residually free.

Theorem 4.7 The only 2-generated noncyclic residually free groups are F_2 and $\mathbb{Z}^2$.

Proof. Let $G = \langle x, y \rangle$ be a noncyclic residually free group. If G is non-abelian, then $[x,y] \neq 1$. Then there exists a homomorphism $f : G \to F_n$ with $f([x,y]) \neq 1$. Then $\langle f(x), f(y) \rangle \cong F_2$. Hence $G \cong F_2$. If G is abelian, then (since G is torsion free) $G \cong \mathbb{Z}^2$.□

Proposition 4.8 Every ω-residually free group G is a limit group.

Proof. Let $G = \langle g_1, \dots, g_k \rangle$. Let $B_n \subset G$ be the ball of radius n about the identity in the corresponding word metric:

$$B_n = \{g \in G \,|\, g = x_1 \dots x_\ell, \text{ where all } x_i \in \{g_1^{\pm}, \dots, g_k^{\pm}\} \text{ and } \ell \leqslant n\}.$$

We have $G = \bigcup\limits_{n \geqslant 1} B_n$. Since G is ω-residually free, there is a homomorphism $f_n : G \to \mathbb{F}$ that is injective on B_n. Then the sequence $(f_n)_{n \in \mathbb{N}}$ is stable and $\underset{\longrightarrow}{\mathrm{Ker}}(f_n) = 1$. Hence $G = G/\underset{\longrightarrow}{\mathrm{Ker}}(f_n)$ is a limit group. □

5 Limit groups are ω-residually free

Let

$$G \overset{\phi}{\twoheadrightarrow} H$$

be an epimorphism of groups, and let F be a third group. Then ϕ determines the map

$$\begin{aligned} \widetilde{\phi}: \quad \mathrm{Hom}(H,F) &\to \mathrm{Hom}(G,F), \\ \psi &\mapsto \psi \circ \phi. \end{aligned}$$

The map $\widetilde{\phi}$ is injective, but not necessarily surjective.
We say that $\alpha \in \mathrm{Hom}(G,F)$ *lifts* to $\psi \in \mathrm{Hom}(H,F)$ if $\alpha = \psi \circ \phi$.
We say briefly that $\alpha \in \mathrm{Hom}(G,F)$ *lifts* if it lifts to some $\psi \in \mathrm{Hom}(H,F)$.

Lemma 5.1 *A homomorphism* $\alpha \in \mathrm{Hom}(G, F)$ *lifts if and only if* $\mathrm{Ker}(\phi) \subseteq \mathrm{Ker}(\alpha)$.

Theorem 5.2 *Let* $\mathbb{F}$ *be a finitely generated free group and let* $G_1 \overset{\phi_1}{\twoheadrightarrow} G_2 \overset{\phi_2}{\twoheadrightarrow} G_3 \overset{\phi_3}{\twoheadrightarrow} \dots$ *be an infinite sequence of epimorphisms between finitely generated groups. Then the sequence*

$$\mathrm{Hom}(G_1, \mathbb{F}) \overset{\widetilde{\phi_1}}{\longleftarrow} \mathrm{Hom}(G_2, \mathbb{F}) \overset{\widetilde{\phi_2}}{\longleftarrow} \mathrm{Hom}(G_3, \mathbb{F}) \overset{\widetilde{\phi_3}}{\longleftarrow} \dots$$

eventually stabilizes, i.e., $\widetilde{\phi_n}$ *is a bijection for all sufficiently large* n.

Proof.

Step 1. We embed $\mathbb{F}$ into $\mathrm{SL}_2(\mathbb{Z})$. Indeed, $\mathbb{F}$ is embeddable into $F(a, b)$, and $F(a, b)$ is embeddable into $\mathrm{SL}_2(\mathbb{Z})$ by $a \mapsto A$, $b \mapsto B$, where

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

It suffices to prove the theorem for $\mathrm{SL}_2(\mathbb{Z})$ instead of $\mathbb{F}$.

Step 2. Let G be a finitely generated group. We describe all homomorphisms $G \to \mathrm{SL}_2(\mathbb{Z})$ as follows. Since G is finitely generated it has a countable presentation:

$$G = \langle g_1, g_2, \dots, g_n \,|\, r_j(g_1, \dots, g_n) = 1, (j \in \mathbb{N}) \rangle.$$

Let

$$A_i = \begin{pmatrix} x_{11}^{(i)} & x_{12}^{(i)} \\ & \\ x_{21}^{(i)} & x_{22}^{(i)} \end{pmatrix}$$

be some matrices from $\mathrm{SL}_2(\mathbb{Z})$, $i = 1, \dots, n$. The map $g_i \mapsto A_i$ can be continued to a homomorphism $G \to \mathrm{SL}_2(\mathbb{Z})$ if and only if

$$r_j(A_1, \dots, A_n) = E$$

for all $j \in \mathbb{N}$, where E is the identity matrix in $\mathrm{SL}_2(\mathbb{Z})$. This gives infinitely many polynomial equations, where the polynomials belong to the ring

$$R := \mathbb{Z}\big[x_{11}^{(1)}, x_{12}^{(1)}, x_{21}^{(1)}, x_{22}^{(1)}, \dots, x_{11}^{(n)}, x_{12}^{(n)}, x_{21}^{(n)}, x_{22}^{(n)}\big].$$

These polynomials generate an ideal $\mathbf{I}$. Zeros of this ideal are in 1-1 correspondence with homomorphisms $G \to \mathrm{SL}_2(\mathbb{Z})$.

Step 3. For $G = G_i$, we denote this ideal by $\mathbf{I}_i$. Since G_{i+1} can be obtained from G_i by putting new relations, we have $I_i \subseteq I_{i+1}$. By Hilbert's Basissatz[3] the chain of ideals

$$\mathbf{I}_1 \subseteq \mathbf{I}_2 \subseteq \mathbf{I}_3 \subseteq \dots$$

stabilizes from some moment n. Then, for all $i \geqslant n$, each homomorphism $G_i \to \mathrm{SL}_2(\mathbb{Z})$ lifts to a homomorphism $G_{i+1} \to \mathrm{SL}_2(\mathbb{Z})$. This proves that $\widetilde{\phi_i}$ is a bijection for $i \geqslant n$. □

[3]**Definition.** A ring K is called *noetherian*, if each non-decreasing chain of ideals in K stabilizes. Equivalently: A ring K is called *noetherian*, if each ideal in K is finitely generated.

Hilbert's Basissatz. If K is a noetherian Ring, then $K[X]$ is too.

Corollary. If K is noetherian, then each (infinite) system of polynomial equations over K is equivalent to some finite subsystem.

Theorem 5.3 *Every limit group is ω-residually free.*

Proof. Let Γ be a limit group. Then there exist a finitely generated group G and a stable sequience of homomorphisms $\mathcal{F} = (f_i : G \to \mathbb{F})_{i\in\mathbb{N}}$ such that $\Gamma = G/\underrightarrow{\mathrm{Ker}}(f_i)$. Let $\phi : G \twoheadrightarrow \Gamma$ be the natural epimorphism with the kernel $\underrightarrow{\mathrm{Ker}}(f_i)$. We enumerate all elements of this kernel:

$$\underrightarrow{\mathrm{Ker}}(f_i) = \{r_1, r_2, \dots\}$$

and set $G_i := G/\langle\langle r_1, r_2, \dots, r_i\rangle\rangle$. Then we have the chain of natural epimorphisms:

$$G \overset{\phi_0}{\twoheadrightarrow} G_1 \overset{\phi_1}{\twoheadrightarrow} G_2 \overset{\phi_2}{\twoheadrightarrow} \dots \twoheadrightarrow \Gamma.$$

Step 1. We show that for each $1 \neq \gamma \in \Gamma$, there exists $\psi \in \mathrm{Hom}(\Gamma, \mathbb{F})$ with $\psi(\gamma) \neq 1$. For that, we choose a preimage $\widetilde{\gamma}$ of γ in G and note the following:

- Since $\gamma \neq 1$, we have $\widetilde{\gamma} \notin \underrightarrow{\mathrm{Ker}}(f_i)$.
- Since $\widetilde{\gamma} \notin \underrightarrow{\mathrm{Ker}}(f_i)$, we have $f_i(\widetilde{\gamma}) \neq 1$ for almost all i.
- Since $r_1 \in \underrightarrow{\mathrm{Ker}}(f_i)$, we have $r_1 \in \mathrm{Ker} f_i$ for almost all i.

By Lemma 5.1, these f_i lift to homomorphisms from G_1 to $\mathbb{F}$.

- More general, for each $n \in \mathbb{N}$, we have $\{r_1, \dots, r_n\} \subset \mathrm{Ker} f_i$ for almost all i.

Lemma 5.1 implies that these f_i lift to homomorphisms from G_n to $\mathbb{F}$.

- By Theorem 5.2, there exists n such that each homomorphism from G_n to $\mathbb{F}$ lifts to a homomorphism from G_{n+1} to $\mathbb{F}$ and further.

All this implies that almost all $f_i : G \to \mathbb{F}$ lift to some homomorphisms $\psi_i : \Gamma \to \mathbb{F}$. Then $\psi_i(\gamma) = \psi_i(\phi(\widetilde{\gamma})) = f_i(\widetilde{\gamma}) \neq 1$ for almost all i. In other words, $\psi_i(\gamma) \neq 1$ for a co-finite subset of indices i in $\mathbb{N}$.

Step 2. To show that Γ is ω-residually free, we must show that for every finite subset $\{\gamma_1, \dots, \gamma_k\} \subset \Gamma \setminus \{1\}$, there exists $\psi \in \mathrm{Hom}(\Gamma, \mathbb{F})$ such that $\{\psi(\gamma_1), \dots, \psi(\gamma_k)\} \subset \mathbb{F} \setminus \{1\}$. This follows from Step 1 and from the fact that the intersection of finitely many co-finite subsets in $\mathbb{N}$ is co-finite. □

Theorem 5.4 A sequence of epimorphisms $G_1 \overset{\phi_1}{\twoheadrightarrow} G_2 \overset{\phi_2}{\twoheadrightarrow} G_3 \overset{\phi_3}{\twoheadrightarrow} \dots$ between residually free groups eventually stabilizes, i.e. ϕ_n are isomorphisms for all sufficiently large n.

Proof. Suppose $\mathrm{Ker}(\phi_i) \neq 1$. Take $x \neq 1$ from $\mathrm{Ker}(\phi_i)$. Then there is a homomorphism $f : G_i \to \mathbb{F}$ with $f(x) \neq 1$. Then f cannot be lifted to a homomorphism from G_{i+1} to $\mathbb{F}$. This contradicts to Theorem 5.2. □

Lemma 5.5 *Every tree that contains infinitely many vertices, each having finite degree, has at least one infinite simple path.*

Using Theorem 5.4 and this lemma, one can deduce the following extended version of Theorem 2.1.

Theorem 5.6 Let G be a finitely generated non-free group and $\mathbb{F}$ be a free group. There is a finite tree of epimorphisms with nontrivial kernels as on Figure 2 such that

- each group in this tree is a limit group with possible exception of G,
- all bottom groups are free groups,
- for any homomorphism $f : G \to \mathbb{F}$, there exists a branch of epimorphisms from the top vertex to some bottom vertex

$$G \overset{\phi_1}{\twoheadrightarrow} \Gamma_1 \overset{\phi_2}{\twoheadrightarrow} \Gamma_2 \overset{\phi_3}{\twoheadrightarrow} \ldots \overset{\phi_k}{\twoheadrightarrow} \Gamma_k$$

and there exist automorphisms $\alpha_0 \in \mathrm{Aut}(G)$, $\alpha_i \in \mathrm{Aut}(\Gamma_i)$, $i = 1, \ldots, k$, such that

$$f = \psi \circ \alpha_k \circ \phi_k \circ \cdots \circ \alpha_2 \circ \phi_2 \circ \alpha_1 \circ \phi_1 \circ \alpha_0$$

for some homomorphism $\psi : \Gamma_k \to \mathbb{F}$.

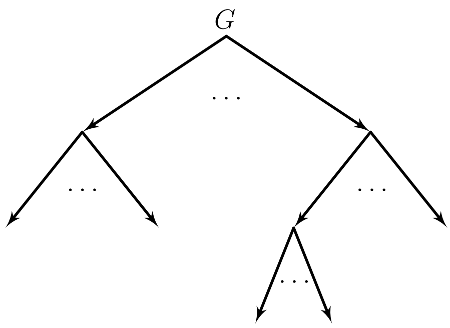


Figure 2

6 Constructible limit groups

Definition 6.1 A generalized abelian decompositon of a group G is a finite graph of groups decomposition of G with abelian edge groups in which some of the vertices are designated *quadratically hanging* (abbreviated **QH**), some others are designated *abelian*, and the remaining are designated *rigid*, and the following holds:

- A **QH**-vertex group is the fundamental group of a compact surface S with boundary and the boundary components correspond to the incident edge groups (they are all infinite cyclic). Further, S is a torus with 1 boundary component (in this case $\chi(S) = -1$) or $\chi(S) \leqslant -2$.

- An abelian vertex group A is abelian Let $P(A)$ be the subgroup of A generated by incident edge groups. The subgroup $\overline{P(A)} := \{a \in A \,|\, \exists n(a) : a^{n(a)} \in A\}$ is called the *peripheral subgroup*. It is easy to understand that there exists A_0 such that $A = A_0 \oplus \overline{P(A)}$.

Definition 6.2 We define a hierarchy of finitely generated groups. If a group belongs to this hierarchy it is called a *constructible limit group* (abbreviated **CLG**).

`Level 0` of the hierarchy consists of finitely generated free groups.
`Level` $\leqslant n+1$ consists of groups G for which one of the following holds:

1) G has a free product decomposition $G = G_1 * G_2$ with G_1 and G_2 of level $\leqslant n$.

2) G has a homomorphism $\rho : G \to G'$ with G' of level $\leqslant n$ and G has a generalized abelian decomposition which satisfies the following properties:

   - For each edge group E at least one of the images of E in a vertex group of the one-edged splitting induced by E is a maximal abelian subgroup.
   - The image of each **QH**-vertex group is a non-abelian subgroup of G'.
   - ρ is injective on each edge group E.
   - ρ is injective on the peripheral subgroup of each abelian vertex group.
   - ρ is injective on the envelope $\widetilde{B}$ of each rigid vertex group B.

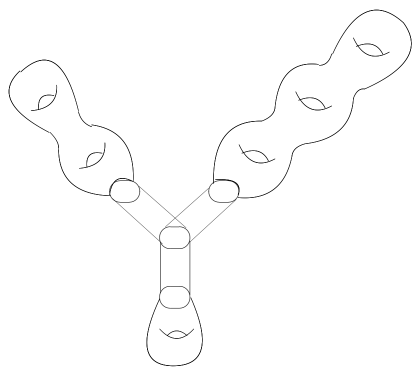
   (The *envelope* $\widetilde{B}$ is defined by first replacing each abelian vertex group with its peripheral subgroup and then letting $\widetilde{B}$ be the subgroup of the resulting group generated by B and by the centralizers of incident edge-groups.)

Example 1. A free abelian group of rank n is a **CLG** of level $n-1$.

Example 2. The fundamental group of a closed surface S with $\chi(S) \leqslant -2$ is a **CLG** of level 1.

Example 3. Let w be an element of a free group F which is not a nontrivial power of another element. Let G be the *double* of F along $\langle w \rangle$, i.e. $G = F *_{\mathbb{Z}} F$, where the generator 1 of $\mathbb{Z}$ is identified with w in both copies of F. Then G is a **CLG** of level 1.

Example 4. Let S be the space obtained from the circle by attaching to it 3 surfaces with one boundary component, with genera 1,2,3. Then $\pi_1(S)$ is a **CLG** of level 2.



Theorem 6.3 The class of constructible limit groups coincides with the class of limit groups.

7 $\mathbb{R}$-trees

Will be extended.

References

[1] M. Bestvina, M. Feighn, *Notes on Selas work: Limit groups and Makanin-Razborov diagrams.*

[2] H. Wilton, *Solutions to Bestvina* & *Feighns Exercises on Limit Groups.*