Group products

$\bigcirc$ is an exact operation, if

- $G = \bigcirc_{i \in I} G_i$,
- $G = \langle G_i \mid i \in I \rangle$,
- $G_i \simeq H_i, \forall i \implies \bigcirc_i G_i \simeq \bigcirc_i H_i$.

We want these properties:

Mal'tsev: $H_i \leq G_i \implies \bigcirc_i H_i \leq \bigcirc_i G_i$,

Associativity: $I = \bigcup I_j \implies \bigcirc_{i \in I} G_i \simeq \bigcirc_j (\bigcirc_{i \in I_j} G_i)$,

Functoriality: $G_i \to H_i \implies \bigcirc_i G_i \to \bigcirc_i H_i$,

Regularity: $G_i \cap \langle G_j \mid j \neq i \rangle^G = 1$.

Mal'tsev's problem: are there any associative Mal'tsev operations besides direct and free products?

Adian constructed such an operation for groups without involutions, but there's a simpler way, suggested by S.V. Ivanov.

Odd $n > 10^{10}$.

Free product: $F = *_i G_i$. Let $C \trianglelefteq F$ such that $F/C \simeq \bigtimes_i G_i$.

$N = \langle X^n \mid X^n \in C \rangle$.

Define $\bigcirc_i^n G_i = F/N$.

Theorem

$\bigcirc^n$ is exact, associative, functorial and regular.

Proof

Easy (no cancellation theory) □

Theorem

$\bigcirc^n$ is Mal'tsev in the class of groups without involutions.

Proof

A graded presentation for $\bigcirc_i^n G_i$:

$G(0) = F = *_i G_i$,

$\mathfrak{X}_i$ are periods of rank i,

$\mathfrak{R}_i = \mathfrak{R}_{i-1} \cup \{A^{\mathrm{lcm}(n,|A|)} \mid A \in \mathfrak{X}_i\}$

$\bigcirc_i^n G_i = G(\infty) = \langle F \mid \bigcup_i \mathfrak{R}_i \rangle$

Use §18 and §19. □

Properties of $\bigcirc^n$

G_i have no involutions, $G = \bigcirc_i^n G_i$

- at least two $G_i \neq 1 \implies |G| = \infty$,
- G_i are periodic $\implies$ G is periodic,
- inside $\mathfrak{B}_n$ it is a free product,
- $G_i \cap G_i^x = 1$, $x \notin G_i$,
- $A \leq G$ and A is abelian or finite $\implies$ A is cyclic or conj. to a subgr. of G_i,
- $x \in G_i$, $x \neq 1 \implies C_G(x) \leq G_i$,
- $x \in G_i$ not conj. to G_j, $j \neq i \implies C_G(x)$ is cyclic,
- ... and other things following from condition R.

Periodic product

n is large, can be even. Periodic product: $G = \prod_i^n G_i$

$G(0) = F = *_i G_i$

$\mathfrak{X}_i$ are periods, not products of two involutions in rank $i - 1$

$\mathfrak{R}_i = \mathfrak{R}_{i-1} \cup \{A^n \mid A \in \mathfrak{X}_i\}$

$\prod_i^n G_i = G(\infty) = \langle F \mid \bigcup_i \mathfrak{R}_i \rangle$

Theorem

$\prod^n$ is an exact associative Mal'tsev operation on all groups.

This gives the solution to Mal'tsev's problem.

General properties

$G = \prod_i^n G_i$, any big n, G_i may have involutions.

- at least two $G_i \neq 1 \implies |G| = \infty$,
- every period has order n,
- $x \in G$ is either conj. to G_i, or conj. to a period (so $x^n = 1$), or $|x| = \infty$ and $x = st$, $s^2 = t^2 = 1$,
- Some props. about finite/abelian subgroups and centralizers?

If n is odd and G_i have no involutions, then $\prod^n$ coincides with Adian's periodic product.

Odd n, no involutions

In addition to the previous slide:

- $x \in G$, either $x^n = 1$ or x is conj. in some G_i.
- if G_i are periodic, then G is periodic,
- $\prod^n$ coincides with $\bigcirc^n$ inside $\mathfrak{B}_n$, and hence is a free product,
- $A \leq G$ abelian $\implies$ A is cyclic of conj. to a subgr. of G_i,
- $x \in G_i$, $x \neq 1 \implies C_G(x) \leq G_i$,
- $x \in G$ not conj. to G_i $\forall i \implies C_G(x)$ is cyclic.

Constructed by Adian in 1976.

Simplicity

Theorem (Adian, 1978)

Suppose that $n \geq 665$ is odd and G_i have no involutions.
Periodic product $\prod_i^n G_i$ is simple $\iff$ $G_i = G_i^n \ \forall i$.

Corollary

Odd $n \geq 665$, odd $k > 1$, $\gcd(n, k) = 1$.
In $\mathfrak{B}_{nk}$ there are infinitely many f.g. nonabelian simple groups.

Proof

Groups $H_r = \prod_{i=1,\ldots,r}^n \mathbb{Z}_k$ lie in $\mathfrak{B}_{nk}$ and are all pairwise nonisomorphic (different number of conj. classes of $\mathbb{Z}_k$). □

Solves Hanna Heumann's problem.

Hopf property

Theorem (Adian–Atabekyan, 2014)

Odd $n \geq 665$, G_i have no involutions, $G = \prod_i^n G_i$.
If $1 < N \trianglelefteq G$, then $G^n \leq N$.

Corollary

If $G_i^n \neq 1$ for some i, then G is Hopfian.

Proof

If $G/N \simeq G$, $N > 1$, then $(G/N)^n = 1 \implies G_i^n = 1$. □

It's not known whether $B(r, n)$ is Hopfian.

$\prod_{i=1,\ldots,r}^n \mathbb{Z}_m$, $m = nk$, $k, r > 1$ is Hopfian, not simple and not residually finite.

Amenability

$G = \langle S \rangle$, $|S| < \infty$.

$\mathrm{Fol}_S(G) = \inf_A \frac{|\partial_S(A)|}{|A|}$, where $\partial_S(A) = \{a \in A \mid ax \notin A, x \in S^{\pm 1}\}$

G is amenable $\iff$ $\mathrm{Fol}_S(G) = 0$
G is uniformly non-amenable $\iff$ $\inf_S \mathrm{Fol}_S(G) > \epsilon$ for sm. $\epsilon > 0$

Theorem (Adian–Atabekyan, 2015)

Odd $n \geq 1003$, G_i have no involutions, $G = \prod_i^n G_i$.
If $H \leq G$, H is f.g. and H not conj. to G_i, then H is uniformly non-amenable.

Generalizes an earlier theorem of Atabekyan, 2009.

$G = \langle S \rangle$, $|S| < \infty$.
$\beta(G, S, k) = |\{x \in G \mid |x| \leq k\}|$, $\omega(G, S) = \lim_{k\to\infty} \sqrt[k]{\beta(G, S, k)}$.

G has uniform exponential growth, if

$$\inf\{\omega(G, S) \mid S \text{ is a fin. set of generators}\} > 1$$

Corollary

Odd $n \geq 1003$, G_i have no involutions, $G = \prod_i^n G_i$.
If $H \leq G$, H is f.g. and H not conj. to G_i, then H has uniform exponential growth.

Answers a question of de la Harpe.

Also possible:
Not simple + Hopfian + not residually finite + uniformly non-amenable + period m,

Simple + uniformly non-amenable + period m.

Even n, no involutions

$G = \prod_i^n G_i$, $n \geq 2^{48}$ and divisible by 2^9, G_i have no involutions.

- at least two factors $\implies |G| = \infty$,
- if $x \in G$, then either $x^n = 1$ or x is conj. to G_i,
- if $A \leq G$ finite, then either $A^n = 1$ or A is conj. to sbgr. of G_i,
- G is simple $\iff G_i = G_i^n \; \forall i$.

Proved in D. Sonkin's thesis, 2005.

Second property: we knew that $x^n = 1$, x is conj, to G_i or $x = st$, $|x| = \infty$, $s^2 = t^2 = 1$, so Sonkin's theorem is stronger.

Last property generalizes Adian's theorem on simplicity.

Recognizibility by spectrum

Spectrum $\omega(G) = \{|x| \mid x \in G\}$
G is recogn. by spectr. if $\omega(G) = \omega(H)$ implies $G \simeq H$

$\mathrm{PSL}_2(2^m)$, $m \geq 2$ is recognizable by spectrum

Theorem (Mazurov–Ol'shanskii–Sozutov, 2015)
$m = 2^{10}k \geq 2^{49}$, $q = m + e$, $e = \pm 1$, and q is a prime power.
Then $\mathrm{PSL}_2(q)$ is not recognizable by spectrum.

Proof
$G = H *^n H$, $n = m/2$, for some finite H. $\square$

$\mathrm{PSL}_2(q)$ is recognizable among finite groups.

Baer–Suzuki theorem

Baer–Suzuki theorem

$|G| < \infty$. If $\forall g \in G$ $\langle x, x^g \rangle$ is a p-group, then x^G is a p-group.

A. Borovik asked if it holds for periodic G and $p = 2$.

Theorem (Mazurov–Ol'shanskii–Sozutov, 2015)

$n = 2^m \geq 2^{48}$, G_i periodic without involutions. Then

- $G = \prod_i^n G_i$ is a simple periodic group,
- $\forall s, t : |s| = |t| = 2 \implies \langle s, t \rangle$ is a 2-group,
- G is not a 2-group.

Sozutov, 2018: boundedly engel elements don't always form a subgroup (similar construction).

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