Survey of some results deduced with the help of Ol'shanskii's technique. Part I.

O. Bogopolski

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1 A countable non-topoligizable group

Any set X with $|X| \geqslant 2$ can be endowed by at least two topologies. In the *maximal* topology all subsets of X, including one-point subsets, are open. Such topology is called *discrete*. In the *minimal* topology the only open subsets are $\emptyset$ and X. The discrete topology is Hausdorff, and the minimal one is not. We will work only with Hausdorff topologies.

Exercise 1.1. Each one-point subset of a Hausdorff space X is closed. Hence any finite subset of a Hausdorff space is closed.

Definition 1.2. Recall that a topology on a group G is called a *group topology* if the group multiplication and inversion are continuous with respect to this topology.

• A group G is called *topologizable* if it admits a non-discrete Hausdorff group topology.

• A group G is called *non-topologizable* if it does not admits a non-discrete Hausdorff group topology.

Examples. All finite groups are non-topologizable. The following infinite groups are topologizable:

1) All infinite residually finite groups (easy).
2) Infinite countable locally finite groups [1].
3) Groups containing infinite normal solvable subgroups [5].
4) Every acylindrically hyperbolic group is topologizable (see [6, Lemma 5.1]).

Problem (Markov, 1944): Does there exist an infinite non-topologizable group?

In 1980, Shelah [12] constructed an uncountable non-topologizable group.

In 1980, Ol'shanskii [10] constructed an infinite 2-generated non-topologizable group.

Theorem 1.3. (Ol'shanskii [10], see [9, Theorem 31.5]) There exists an infinite finitely generated non-topologizable group.

Sketch of the proof. Ol'shanskii constructs a special extension

$$1 \to Z_n \to G \to G/Z_n = B(m,n),$$

where the cyclic group Z_n lies in the center of G and $B(m,n)$ is the free Burnside group with $m \geqslant 2$ generators and of exponent n, where n is a sufficiently large odd number.

Clearly, if $g \in G$ then $g^n \in Z_n$. Ol'shanskii proves (not easy) that the following property is satisfied.

Property:

$$g \in G \setminus Z_n \quad \Longrightarrow \quad g^n \in Z_n \setminus \{1\}.$$

This can be written in terms of solutions of equations:
Equations:

$$G \setminus \{1\} = \bigcup_{a \in Z_n \setminus \{1\}} \mathbf{Sol}(x^n = a) \ \bigcup (Z_n \setminus \{1\}). \tag{1.1}$$

Now, let T be a Hausdorff group topology on G. We shall show that T discrete. It suffices to show that $\{1\}$ is open (then $\{g\}$ will be open for each $g \in G$). Thus, it suffices to show that the set $G \setminus \{1\}$ is closed. This can be easily deduced from equation (1.1) and the following two claims.

Claim 1. The set $Z_n \setminus \{1\}$ is closed.

Claim 2. For any $a \in Z_n \setminus \{1\}$ the set $\mathbf{Sol}(x^n = a)$ is closed.

Proofs. Claim 1 follows from the finiteness of $Z_n \setminus \{1\}$ and Exercise 1.1. To prove Claim 2, we shall prove that the set $S_a = \{g \in G \mid g^n \neq a\}$ is open. Consider an arbitrary $g \in S_a$. By Hausdorff propery, there exists an open neighborhood $U(g^n)$ such that $a \notin U(g^n)$. Obviously, there exists an open neighborhood $V(g)$ such that $(V(g))^n \subseteq U(g^n)$. Then $a \notin (V(g))^n$ and hence $V(g) \subseteq S_a$, i.e. S_a is open. □

Definition 1.4. A subset S of a group G is called *elementary algebraic* if there exist elements $a_1, \dots, a_n \in G$ and numbers $\varepsilon_1, \dots, \varepsilon_n \in \mathbb{Z}$ such that

$$S = \mathbf{Sol}(a_1 x^{\varepsilon_1} a_2 x^{\varepsilon_2} \dots a_n x^{\varepsilon_n} = 1).$$

Exercise 1.5. Prove that if $G \setminus \{1\}$ is a finite union of elementary algebraic sets, then G is non-topologizable.

Theorem 1.6. (Markov, 1946; see [7,8]) A countable group is non-topologizable if and only if $G \setminus \{1\}$ is a finite union of elementary algebraic sets.

A simple proof of Markov's theorem was given in [13].

Remark. We can rewrite equation (1.1) in the style of Markov's theorem:

$$G \setminus \{1\} = \bigcup_{a \in Z_n \setminus \{1\}} \mathbf{Sol}(x^n = a) \bigcup \bigcup_{b \in Z_n \setminus \{1\}} \mathbf{Sol}(x = b).$$

2 Tarski Monsters and (non-)topologizability

A group G has *exponent* n if $g^n = 1$ for any $g \in G$. The following definition is more general than that given in Wikipedia.

Definition 2.1. (see [6, Introduction]) A group G is called *Tarski Monster* if it is infinite, simple and all proper subgroups of G are finite cyclic.

Note that any Tarski Monster is necessarily finitely generated and non-amenable. The existence of Tarski Monsters was first proved by Ol'shanskii in 1980, see [11]. Moreover, Ol'shanskii constructed there continuum non-isomorphic Tarski Monsters of exponent p for each prime $p > 10^{75}$.

Theorem 2.2. (see [6, Theorem 1.2 and comments after it])
There exists a non-topologizable Tarski Monster of a bounded exponent.

Theorem 2.3. (see [6, Theorem 1.4]) For every sufficiently large odd $n \in \mathbb{N}$ there exists a topologizable Tarski Monster of exponent n.

3 Zariski and Markov topologies

Definition 3.1. Let G be a group.

1) *Zariski topology* $\mathcal{Z}_G$ on G is defined so that the set of elementary algebraic subsets in G forms a subbasis of the set of closed subsets for $\mathcal{Z}_G$.

   In other words, all closed sets in this topology are exactly arbitrary intersections of finite unions of elementary algebraic subsets, see Definition 1.4.

2) *Markov topology* $\mathcal{M}_G$ on G is defined as the infimum of all Hausdorff group topologies on G:

$$\mathcal{M}_G = \inf\{T \mid T \text{ is a Hausdorff group topology on } G\}.$$

   This infimum is equal to the intersection of all Hausdorff group topologies on G.

Note that the centralizer $C_G(S)$ of any subset $S \subseteq G$ is closed for $\mathcal{Z}_G$. Obviously,

$$\mathcal{Z}_G \subseteq \mathcal{M}_G.$$

Theorem 3.2. (Markov) For any countable group G we have $\mathcal{Z}_G = \mathcal{M}_G$.

Theorem 3.3. (see [4]) For any abelian group G we have $\mathcal{Z}_G = \mathcal{M}_G$.

Remark 3.4. 1) Markov topology on a group G is not necessarily Hausdorff and not necessarily a group topology. However, Markov topology is T1 (for any two points $x, y \in G$, there exists an open set U such that $x \in U$ and $y \notin U$) and is closed under the left and the right multiplications and the inverse operations.

2) Zariski topology on the multiplicative group $\mathbb{R}^*$ of the real numbers is not Hausdorff and is not a group topology. Zariski topology on $\mathbb{Z}$ is a Hausdorff group topology.

3) If a countable group G is not topologizable, then Markov topology on G is discrete and, by Theorem 3.2, Zariski topology on G is discrete as well.

More information about topologies on groups can be found in the surveys [2,3].

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