Survey of some results deduced with the help of Ol'shanskii's technique. Part II.

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1 Quasi-finite and quasi-countable groups

Definition 1.1. A group G is called *quasi-finite* if it is infinite and any proper subgroup of G is finite.

Exercise 1.2. (easy) Every quasi-finite group is countable.

Examples.

1) For any prime number p the *quasi-cyclic* group C_{p^∞} is defined as follows:

$$C_{p^\infty} = \{z \in \mathbb{C} \mid z^{p^k} = 1,\ k = 1, 2, \dots\}.$$

Clearly, C_{p^∞} is the union of its subgroups $1 \leqslant Z_p \leqslant Z_{p^2} \leqslant Z_{p^3} \leqslant \dots$, where

$$Z_{p^k} = \{z \in \mathbb{C} \mid z^{p^k} = 1\},$$

and any nontrivial proper subgroup of C_{p^∞} coincides with Z_{p^k} for some $k \in \{1, 2, \dots\}$. Thus, C_{p^∞} is quasi-finite. Note that C_{p^∞} is not finitely generated.

2) Any Tarski Monster group is quasi-finite.
Recall that a group G is called *Tarski Monster* if it is infinite, simple and all proper subgroups of G are finite cyclic. Note that any Tarski Monster is necessarily finitely generated and non-amenable. The existence of Tarski Monsters was first proved by Ol'shanskii in 1980, see [5]. Moreover, Ol'shanskii constructed there continuum non-isomorphic Tarski Monsters of exponent p for each prime $p > 10^{75}$.

3) The free Burnside group $B(m, n)$ contains the free Burnside group $B(\infty, n)$ for all $m \geqslant 2$ and all sufficiently large odd exponents n, see [4, Corollary 35.6]. Hence, such $B(m, n)$ is not quasi-finite.

Theorem 1.3. ([2], see also [4, Corollary 35.3]) There exists a quasi-finite group which contains any finite group of odd order.

Exercise 1.4. (difficult) S_3 cannot be a subgroup of a quasi-finite group.

Theorem 1.5. ([1], see also [4, Theorem 35.5]) A finite group K can be embedded into some quasi-finite group G if and only if $K = K_1 \times K_2$, where $|K_1|$ is odd and K_2 is an abelian 2-group.

Definition 1.6. We call a group G *quasi-countable* if it is uncountable and any proper subgroup of G is countable.

In [6], Shelah proved (assuming continuum hypothesis **CH**) that quasi-countable groups exist. Obraztsov improved this result as follows.

Theorem 1.7. ([3, Theorem D]) Assuming **CH**, there exists a simple uncountable group G such that

1) any proper subgroup of G is countable,
2) any countable group occurs as a proper subgroup of G (up to isomorphism).

Moreover, given an arbitrary group H with $1 \leqslant |H| \leqslant 2^{\aleph_0}$, one can construct such G with $Out(G) = H$.

Exercise 1.8. (easy) If we assume the negation of **CH**, then there does not exist an uncountable group satisfying conditions 1) and 2) from the theorem above.

References

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