One comment on a role of **CH** in group theory

O. Bogopolski

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The continuum hypothesis **CH** is the following statement: There does not exist a cardinal $\varkappa$ with

$$\aleph_0 < \varkappa < 2^{\aleph_0}.$$

In 1996, Obraztsov proved the following theorem, which is stated below in a simplified form:

Theorem 0.1. (see [1, Theorem D]) Assuming **CH**, there exists an uncountable group G such that any proper subgroup of G is countable and any countable group occurs as a proper subgroup of G (up to isomorphism).

He noticed, without any explanation, that if **CH** is not valid, then such G does not exist. Here is an explanation:

(1) There exist exactly $2^{\aleph_0}$ nonisomorphic 2-generated groups (B. Neumann; 1937).

(2) Suppose that **CH** is not valid and suppose that there exists a group G as in the above theorem. In particular, $|G| \geqslant \varkappa$. If $|G| = \varkappa$, then the number of all 2-generated subgroups of G is at most $\varkappa$ that contradicts (1). Therefore $|G| > \varkappa$. Then G contains a subset S of cardinality $\varkappa$. Then the subgroup $\langle S \rangle$ is of cardinality $\varkappa$ and therefore proper. A contradiction.

References

[1] V.N. Obraztsov, *Embedding into groups with well described lattices of subgroups*, Bull. Austral. Math. Soc., **54** (1996), 221-240.