A comment on classes of elementarily equivalent groups of infinite cardinality $\varkappa$: How many? How large can they be? Model theory and acylindrically hyperbolic groups

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The answer to these questions is well known: $2^{\aleph_0}$ and $2^{\varkappa}$. Interestingly, the cardinality $\aleph_0$ plays a special role. Therefore we start with the following well known lemma, which easily follows from some classical results (of course, it also follows from [6]).

Lemma. There exist exactly $2^{\aleph_0}$ nonisomorphic countable groups.

Proof. The proof follows from the following three statements.

(1) Each countable group can be embedded into a 2-generated group (G. Higman, B. Neumann, H. Neumann; 1949).

(2) There exist exactly $2^{\aleph_0}$ nonisomorphic 2-generated groups (B. Neumann; 1937).

(3) Any 2-generated group contains at most $2^{\aleph_0}$ subsets and hence subgroups.

□

• The paper [6] gives continuum (i.e. $2^{\aleph_0}$) countable nilpotent 2-groups of exponent 4 which are all countably categorical and admit elimination of quantifiers. In particular each pair of them is not elementarily equivalent.

By Löwenheim – Skolem theorem any group is elementarily equivalent to a countable one. Therefore, in any infinite cardinality κ a maximal family of groups which are pairwise non-equivalent is of size continuum.

The paper [3] gives a construction which assigns to any graph Γ a nilpotent group G which interprets Γ. It is well known that any first order structure of a finite language is interpretable in a graph. In particular there is a nilpotent group G such that its theory is not stable. By a theorem of S. Shelah this means that for any uncountable cardinal κ there exist 2^{κ} groups of cardinality κ which are elementarily equivalent to G. □

• In [5], Osin writes: "It is worth noting that finding examples of elementarily equivalent, non-isomorphic, finitely generated groups is rather non-trivial task since the standard tools for constructing models – ultrapowers, ommiting types, and the Löwenheim – Skolem theorem – are not available in this case." As a by-product he proves the following

Statement. (see [5, Corollaries 1.3, 1.4])

1) Suppose that $\mathcal{U}$ is a non-abelian and $\mathcal{V}$ is a non-locally-finite varieties of groups. Then the product variety $\mathcal{UV}$ contains a subset of cardinality $2^{\aleph_0}$ consisting of pairwise non-isomorphic, elementarily equivalent, finitely generated groups.

2) In particular, such a subset can be chosen in the variety $\mathcal{B}_n$ of all groups of exponent $n = n_1 n_2$, where $n_1 > 2$ and n_2 is any number for which $\mathcal{B}_{n_2}$ is not locally finite (e.g., we can take $n_2 = 1995$ as showed Adian in his book).

• There is a lot of interesting theorems and inspiring questions in the recent papers of André [1, 2] and Osin [4, 5] in the area *model theory – topological space of finitely generated groups – (not necessarily finitely generated) acylindrically hyperbolic groups.*

References

[1] S. André, *Acylindrical hyperbolicity and existential closedness*, ArXiv, 2020.

[2] S. André, *Hyperbolicity and cubulability are preserved under elementary equivalence*, ArXiv, 2018.

[3] A. Mekler, *Stability of nilpotent groups of class 2 and prime exponent*, The Journal of Symbolic Logic, **46** (4) (1981), 781-788.

[4] D. Osin, *A topological zero-one law and elementary equivalence of finitely generated groups*, ArXiv, 2020; to appear in Ann. Pure App. Logic.

[5] D. Osin, *Condensed groups in product varieties*, ArXiv, 2020.

[6] D. Saracino and C. Wood, *QE nil-2 groups of exponent 4*, J. Alg., **76** (1982), 337-352.