A question about the elementary equivalence and acylindricity

O. Bogopolski

December 6, 2020

We consider the following ascending chain of groups

$$\begin{array}{ll} G_1 = & \langle t_1, t_2 \,|\, \rangle \cong F(t_1, t_2) \\ \wedge & \\ G_2 = & \langle t_1, t_2, t_3 \,|\, t_3^{-1} t_2 t_3 = t_2 t_1 \rangle \cong F_2(t_2, t_3) \\ \wedge & \\ G_3 = & \langle t_2, t_3, t_4 \,|\, t_4^{-1} t_3 t_4 = t_3 t_2 \rangle \cong F_2(t_3, t_4) \\ \wedge & \\ \vdots & \\ \wedge & \\ G = & \bigcup\limits_{i=1}^{\infty} G_i \end{array}$$

Clearly G is locally free, but not free since $G = [G, G]$. Moreover, G is not finitely generated, the centralizer of each element of G is cyclic, and G is not simple since $\langle\langle G_i \rangle\rangle \neq G$. It seems that the following questions can be easily answered: Is $\langle\langle G_i \rangle\rangle \cong F_\omega$? What is the structure of $\langle\langle G_{i+1} \rangle\rangle / \langle\langle G_i \rangle\rangle$?

Conjecture. (O. Bogopolski)

1) F_2 and G are (existentially) elementarily equivalent.

2) G is acylindrically hyperbolic.

Remark 1. Theorem of Tarski-Vaught says that if G is an ascending union of G_i's, where every G_i is elementary embedded to G_{i+1}, then G_i is elementary embedded to G. In particular, G is elementarily equivalent to G_i. However this approach does not work, since in our case the natural embedding of G_i to G_{i+1} is not elementary: $t_i \in G_i$ is a commutator in G_{i+1}, but not in G_i.

Remark 2. Sela proved that if G is finitely generated and $G \equiv H$, where H is torsion-free hyperbolic, then G is also torsion-free hyperbolic. André omitted the torsion-free condition here. More precisely, he proved that if G is finitely generated and $G \equiv H$, where H is hyperbolic, then G is also hyperbolic.

Note that this cannot be applied to our conjecture since our G is not finitely generated.

Question. (Osin) Is acylindrical hyperbolicity preserved under elementary equivalence among finitely generated groups?

Section 5 of [1] gives an easy counterexample among countable groups. In this counterexample the centralizers of all nontrivial elements of G are noncyclic. Note that for our G these centralizers are cyclic.

References

[1] S. André, *Acylindrical hyperbolicity and existential closedness*, ArXiv, 2020.

[2] S. André, *Hyperbolicity and cubulability are preserved under elementary equivalence*, ArXiv, 2018.

[3] S. André, *Formal solutions and the first-order theory of acylindrically hyperbolic groups*, ArXiv, 2020.

[4] D. Osin, *A topological zero-one law and elementary equivalence of finitely generated groups*, ArXiv, 2020; to appear in Ann. Pure App. Logic.

[5] D. Osin, *Condensed groups in product varieties*, ArXiv, 2020.