A short survey on the elementary equivalence in the classes of finite-by-nilpotent and virtually free groups

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Prehistory: In [5], Oger proved that if two groups G and H satisfy $G \times \mathbb{Z} \cong H \times \mathbb{Z}$, then they are elementarily equivalent.

The following theorems of Oger and André show that the elementary equivalence becomes the same as the isomorphism after a kind of stabilization, depending on the class of groups under consideration.

Theorem 1. (Oger [6]) Let G and H be two finitely generated finite-by-nilpotent groups. Then $G \equiv H$ if and only if $G \times \mathbb{Z} \cong G \times \mathbb{Z}$.

Example 1.1 from [1]. Consider two virtually cyclic groups:

$$\begin{aligned} G &= \langle a, t \,|\, a^{25} = 1,\, t^{-1}at = a^6 \rangle, \\ H &= \langle a, t \,|\, a^{25} = 1,\, t^{-1}at = a^{11} \rangle. \end{aligned}$$

Then
1) $G \not\cong H$,
2) $G \times \mathbb{Z} \cong H \times \mathbb{Z}$,
3) $G \equiv H$.

André does not give a proof of claims 1) and 2). Indeed this is an easy exercise, and we give a solution of it here.

Proof. First we note that $\langle a \rangle \trianglelefteq G$. Moreover, $\langle a \rangle$ is characteristic in G, since it is the maximal (up to conjugacy) finite subgroup of G. In particular, any element of G can be written in the form $a^i t^j$. Furthermore, the center of G is $\langle t^5 \rangle$. The same is true for H.

1) Suppose that there is an isomorphism $\varphi : G \to H$. Then $\varphi(a) = a^i$ and $\varphi(t) = a^j t^{\pm 1}$ for some i, j. Then it is easy to check that $\varphi(t^{-1}at) \neq \varphi(a^6)$ in H. A contradiction.

2) Let z be a generator of $\mathbb{Z}$. Then the map $\varphi : G \times \mathbb{Z} \to H \times \mathbb{Z}$ defined by

$$\begin{aligned} a &\mapsto \quad a \\ t &\mapsto \quad t^3 z \\ z &\mapsto \quad t^5 z^2, \end{aligned}$$

is an isomorphism. 3) follows from the theorem of Oger.

Remark. The intersection of the classes of finite-by-nilpotent and virtually free groups is the class of virtually cyclic groups.

Theorem 2. (André [1, Theorem 1.17]) Two finitely generated virtually free groups G and H are $\forall\exists$-elementarily equivalent if and only if there exist two isomorphic overgroups G' and H', obtained respectively from G and H by performing a sequence of specific HNN extensions over finite groups and specific replacements of virtually cyclic subgroups by virtually cyclic overgroups. This can be algorithmically verified.

Conjecture 3. (André [1, Conjecture 1.25]) Two virtually free groups have the same $\forall\exists$-theory if and only if they are elementarily equivalent.

References

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[6] F. Oger, *Cancellation and elementarily equivalence of finitely generated finite-by-nilpotent groups*, J. London Math. Soc., (2) **44** (1991), 173-183.