(1)

Lecture of Oleg. Bogopolski, 26.11.2020

# Rips construction

(E. Rips, Subgroups of small cancellation groups, Bull. London M.S. 14 (1982), 45-47)

<u>Remark 1</u>. Let $1 \neq N \trianglelefteq F_n$. Then $N$ is finitely gen. $\iff |F_n : N| < \infty$
(This follows from Marshal Hall thm)

<u>Thm</u> (Rips). Let $G$ be a finitely presented group. Then there exists a short exact sequence of groups

$$1 \to K \to H \to G \to 1$$

s.t. (1) $H$ is a finitely presented group satisfying $C'(\lambda)$

(2) $K$ is finitely generated.

<u>Proof</u> $G = \langle a_1, \dots, a_m \mid R_1, \dots, R_n \rangle$. We define $H$:

<u>Generators of $H$</u>: $a_1, \dots, a_m, b_1, b_2$.

<u>Relations of $H$</u> ~~$R_i b_1 b_2^{r_i} b_1 b_2^{r_i+1}$~~ ~~$R_i \underbrace{b_1 b_2^{r_i} b_1 b_2^{r_i+1} \cdots b_1 b_2^{r_i'}}_{\text{B-word of type } [r_i, r_i']}$~~

~~$a_i^{-1} b_1 a_i = b_1 b_2^{p_i} \cdot b_1 b_2^{p_i+1} \cdots b_1 b_2$~~

$$R_i b_1 b_2^{p_i} b_1 b_2^{p_i+1} \cdots b_1 b_2^{p_i'} \qquad [p_i, p_i']$$

$$a_i^{-1} b_1 a_i = b_1 b_2^{q_i} b_1 b_2^{q_i+1} \cdots b_1 b_2^{q_i'} \qquad [q_i, q_i']$$

$$a_i^{-1} b_2 a_i = b_1 b_2^{r_i} b_1 b_2^{r_i+1} \cdots b_1 b_2^{r_i'} \qquad [r_i, r_i']$$

$$a_i b_1 a_i^{-1} = b_1 b_2^{s_i} b_1 b_2^{s_i+1} \cdots b_1 b_2^{s_i'} \qquad [s_i, s_i']$$

$$a_i b_2 a_i^{-1} = b_1 b_2^{t_i} b_1 b_2^{t_i+1} \cdots b_1 b_2^{t_i'} \qquad [t_i, t_i']$$

Set $K = \langle b_1, b_2 \rangle$

**Def** $G$ has a Howson property if for any two f. gen. subgroups $H_1, H_2 \leq G$, their intersection $H_1 \cap H_2$ is fin. gen.

**Ex** 1) $F$ is Howson

2) $\underset{\langle x, y\rangle}{\underset{\shortparallel}{F_2}} \times \underset{\langle t \rangle}{\underset{\shortparallel}{\mathbb{Z}}}$ is not Howson

$H_1 = \langle x, y \rangle$, $H_2 = \langle x, yt \rangle$

$g \in H_1 \cap H_2 \Rightarrow g = x^{k_1}(yt)^{l_1} \cdots x^{k_n}(yt)^{l_n} = x^{k_1}y^{l_1} \cdots x^{k_n}y^{l_n} t^{l}$

$[\text{where } l = l_1 + \dots + l_n = 0]$ $\qquad = x^{k_1}y^{l_1} \cdots x^{k_n}y^{l_n}$

$$= x^{k_1}\left(y^{l_1}x^{k_2}y^{-l_1}\right)\cdot\left(y^{l_1+l_2}\cdot x^{k_3}y^{-(l_1+l_2)}\right)\cdots\left(y^{l_1+\dots+l_{n-1}}x^{k_n}y^{-(l_1+\dots+l_{n-1})}\right)\cdot y^{\overbrace{l_1+\dots+l_n}^{=0}}$$

$$\in \langle y^{-i}xy^{i} \mid i \in \mathbb{Z} \rangle$$

Thus $H_1 \cap H_2 = \langle y^{-i}xy^{i} \mid i \in \mathbb{Z} \rangle$

$\supseteq$ is clear: $y^{-i}xy^{i} = (yt)^{-i}x(yt)^{i} \in H_2$.

3) $\underset{\langle cb^2, (cb)^2 \rangle}{\underset{\shortparallel}{F_2}} \times \mathbb{Z}_{= \langle c^2 \rangle} \leq \langle c, b \mid c^2 = b^3 \rangle \leq \langle a, b \mid a^{-1}b^2a = b^3 \rangle$

$a^{-1}ba = c$

stable letter

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**Corollary 1.** There exists a finitely ~~generated~~ presented $H$ satisfying $C'(\lambda)$ which is not Howson

**Proof**

$$1 \to \underset{\text{f.g.}}{K} \to H \xrightarrow{\varphi} G \to 1$$

Let $Q_1, Q_2 \leq G$ be fin. gen, but $Q_1 \cap Q_2$ not fin. gen

Then $P_1 = \varphi^{-1}(Q_1)$, $P_2 = \varphi^{-1}(Q_2)$ are fin. gen, but $P_1 \cap P_2$ not since $\varphi(P_1 \cap P_2) = Q_1 \cap Q_2$.

Remark $\exists\ F_1 < F_2 < F_3 < \cdots < F_\infty = [F,F] = F$ if $rk(F) \geq 2$

Thm (Takahasi, 1951) For any strictly ascending chain of subgroups $H_1 < H_2 < \cdots < \underset{\text{free}}{F}$ the set $\{rk(H_i)\}$ is unbounded.

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Now we construct a group $G$ - finitely presented and a chain

$$G_1 < G_2 < \cdots < G \quad \text{with } \{rk(G_i)\} = 2.$$

$$G_1 = \langle t_1, t_2 \rangle = F_2$$

$$\vee$$

$$\langle t_1, t_2, t_3 \mid t_3^{-1} t_2 t_3 = t_2 t_1 \rangle$$

$$\|$$

$$G_2 = \langle t_2, t_3 \rangle = F_2$$

$$\vee$$

$$G_3 = \langle t_3, t_4 \rangle = F_2$$

$$\vdots$$

$$\underset{\text{countable}}{\widetilde{G} = \bigcup G_i} = \underset{\text{rekursively enumer.}}{\langle t_1, t_2, \ldots \mid r_1, r_2, \ldots \rangle} \xrightarrow{\text{HNN, 1949}} \underset{\text{rekursively enum.}}{\langle a, b \mid s_1, s_2, \ldots \rangle} \xhookrightarrow{\text{Higman, 1961}} \underset{\text{f.p.}}{G}$$

Thm (G. Higman, 1961) Any finitely generated group $H$ can be embedded into a finitely presented group $G$ iff $H$ is rekursively presentable.

Corollary 2 There exists a finitely presented $H$ satisfying $C'(\lambda)$ with strictly ascending chain of subgroups

$$H_1 < H_2 < \cdots < H \quad \text{of } rk(H_i) \leq 4.$$

**Corollary 3** There exists a finitely presented $H$ with $C'(\lambda)$ which contains a finitely generated subgroup $H_1 \leq H$ which is not finitely presented.

**Proof**
$$G_1 = \langle a, b \rangle * \langle c, d \rangle \quad \begin{array}{l} a^2b^2 = c^2d^2 \\ a^4b^4 = c^4d^4 \\ \vdots \end{array}$$
is finitely generated but not finitely presented. (though rekursively presented)

By Higman embedding theorem

$$G_1 \hookrightarrow \underset{\text{finitely presented}}{G}$$

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r] & H \arrow[r, "\varphi"] \arrow[d, phantom, "\cup"] & G \arrow[r] \arrow[d, phantom, "\cup"] & 1 \\
1 \arrow[r] & K \arrow[r] & H_1 \arrow[r, "\varphi"] & G_1 \arrow[r] & 1
\end{tikzcd}$$

**Corollary 4** There exists a finitely presented $H$ with $C(\lambda)$ with unsolvable membership problem for finitely gener. subgroups

**Proof** Note that solvability of $MP(H)$ implies solv. of $MP(G)$

$$g \in \langle g_1, \cdots, g_n \rangle$$
$$\Updownarrow$$
$$\tilde{g} \in \langle \tilde{g}_1, \cdots, \tilde{g}_n, b_1, b_2 \rangle$$